The Collected Papers

of

Sir Thomas Havelock

on

HYDRODYNAMICS

C. Wigley
Editor

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FOREWORD

The interests of the Office of Naval Research in arranging for the publication of the collected hydrodynamical papers of Sir Thomas Havelock have a dual nature. First, we take great pleasure at this opportunity to express our respect and admiration for Sir Thomas, whose intellectual and scientific achievements in hydrodynamics have served as a source of inspiration and guidance for those researchers who are following him. Second, we feel that the increased accessibility of these important contributions of Sir Thomas which bear directly on many of today's urgent problems will prove to be of great value to the hydrodynamic research community.

Our sincerest appreciation is extended to the many persons who assisted in the preparation of this volume. We should like to express our thanks specifically to Sir Thomas himself, for his gracious permission to undertake this work; to the late CAPTAIN H. E. Saunders (USN, Ret.), for his continued interest and encouragement; to Mr. C. Wigley, for his efficient and careful task of editing; to Messrs. W. H. Ramey and I. S. Rudin and other staff members of the Technical Information Division, Naval Research Laboratory, for the preparation of the material for publication; and to Mr. J. W. Brennan, Navy European Patents Program, Office of Naval Research Branch Office, London, for obtaining copyright releases from the various publishers involved. Our thanks must also be extended to the following organizations for the copyright releases which made the publication of this volume possible:

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PREFACE

The editor was very honoured and delighted to receive the invitation of the Office of Naval Research to edit the collected edition of Sir Thomas Havelock's hydrodynamical papers. Since his first introduction to hydrodynamical research many years ago, the editor has always regarded Professor Havelock's work with the greatest admiration and respect. And, for nearly forty years, after making personal acquaintance with Professor Havelock, the editor has received much very kind advice and assistance from him, which he is very glad to acknowledge here.

Nearly all the mathematical analysis in these papers has been reworked, and a number of minor misprints have been found. In one or two papers more serious changes have been made, either by Professor Havelock himself, or with his agreement. The papers are arranged in chronological order, without reference to their content. The subject receiving the most attention in the papers is the development of the mathematical theory of wave resistance and wave formation for a moving body. The papers in the following list deal with this and show the development of the theory from elementary methods to a complete solution for any body, subject only to the assumption of small wave height, that is, of a linearised potential.

Paper Nos.	*Pages in this Collection*
1 to 4	1 to 80
6	94 to 104
7 & 8	105 to 131
10	146 to 157
15 to 27	192 to 329
29 to 36	347 to 428
44	500 to 511
46 to 52	520 to 582
59	615 to 616

Paper No. 20 (pages 249 to 264), Paper No. 32 (pages 377 to 389), and paper 51 (pages 563 to 574) give a summary of the practical results to be deduced from the theory at their respective dates 1925, 1934, and 1951).

All but five of the remaining papers deal with various motions of a ship by similar methods, i.e., with rolling, pitch and heave, motion in a seaway, etc., and their individual subjects are sufficiently specified in their titles in the List of Contents.

The remaining five papers, numbers 5, 9, 12, 13 and 28 deal with certain mathematical questions which arise in hydrodynamical analysis.

Finally the editor's thanks are due to Professor Lunde of Trondheim for his kind advice on one difficult question, and also to Dr. T. Francis Ogilvie and to Dr. J. N. Newman, both of the David Taylor Model Basin, for their kindness in verifying some of the references to American papers.

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11th March, 1963

CONTENTS

CONTENTS–Continued

CONTENTS–Continued

[*Reprinted from the* PROCEEDINGS OF THE ROYAL SOCIETY, A. Vol. 81]

The Propagation of Groups of Waves in Dispersive Media, with Application to Waves on Water produced by a Travelling Disturbance.

By T. H. HAVELOCK, M.A., D.Sc., Fellow of St. John's College, Cambridge; Lecturer in Applied Mathematics, Armstrong College, Newcastle-on-Tyne.

(Communicated by Prof. J. Larmor, Sec. R.S. Received August 26,—Read November 19, 1908.)

CONTENTS.

§ 1. *Introduction.*

The object of this paper is to illustrate the main features of wave propagation in dispersive media. In the case of surface waves on deep water it has been remarked that the earlier investigators considered the more difficult problem of the propagation of an arbitrary initial disturbance as expressed by a Fourier integral, ignoring the simpler theory developed subsequently by considering the propagation of a single element of their integrals, namely, an unending train of simple harmonic waves. The point of view on which stress is laid here consists of a return to the Fourier integral, with the idea that the element of disturbance is not a simple harmonic wave-train, but a simple group, an aggregate of simple wave-trains clustering around a given central period. In many cases it is then possible to select from the integral

the few simple groups that are important, and hence to isolate the chief regular features, if any, in the phenomena.

In certain of the following sections well-known results appear; the aim has been to develop these from the present point of view, and so illustrate the dependence of the phenomena upon the character of the velocity function. In the other sections it is hoped that progress has been made in the theory of the propagation of an arbitrary initial group of waves, and also of the character of the wave pattern diverging from a point impulse travelling on the surface.

§ 2. *Definition of Simple Group.*

We have to consider the transmission of disturbances in a medium for which the velocity of propagation of homogeneous simple harmonic wave-trains is a definite function of the wave-length. The kinematically simplest group of waves is composed of only two simple trains, of wave-lengths λ, λ', differing by an infinitesimal amount $d\lambda$; then with the usual approximation we have for the combined effect

$$y = \mathrm{A}\cos\frac{2\pi}{\lambda}(x-\mathrm{V}t)+\mathrm{A}\cos\frac{2\pi}{\lambda'}(x-\mathrm{V}'t)$$

$$= 2\mathrm{A}\cos\frac{\pi d\lambda}{\lambda^2}(x-\mathrm{U}t)\cos\frac{2\pi}{\lambda}(x-\mathrm{V}t), \tag{1}$$

where
$$\mathrm{U} = \mathrm{V}-\lambda\frac{d\mathrm{V}}{d\lambda}. \tag{2}$$

The expression (1) may be regarded as representing at any instant a train of wave-length λ, whose amplitude varies slowly with x according to the first cosine factor. Thus it does not represent a form which moves forward unchanged; but it has a certain periodic quality, for the form at any given instant is repeated after equal intervals of time $\lambda/(\mathrm{V}-\mathrm{U})$, being displaced forward through equal distances $\lambda\mathrm{U}/(\mathrm{V}-\mathrm{U})$. The ratio of these quantities, namely U, is called the group-velocity. It has also the following significance: in the neighbourhood of an observer travelling with velocity U the disturbance continues to be approximately a train of simple harmonic waves of length λ.

The most general simple, or elementary, group may be defined in the following manner. Let the central form be a simple harmonic wave of length $2\pi/\kappa_0$, and let the other members be similar waves whose amplitude, wave-length, and velocity differ but slightly from the central type; then, with similar approximation, we have

$$y = \Sigma\mathrm{A}\cos\{\kappa(x-\mathrm{V}t)+\alpha\}$$
$$= \Sigma\mathrm{A}_n\cos\{\kappa_0(x-\mathrm{V}_0t)+(x-\mathrm{U}_0t)\,\delta\kappa_n+\alpha\}. \tag{3}$$

The range of values of κ being infinitesimal, the group as a whole may be written, as in the previous case, in the form

$$y = \phi(x - U_0 t) \cos\{\kappa_0(x - V_0 t) + \beta\}, \tag{4}$$

where ϕ is a slowly varying function; and the group-velocity U_0 is given by

$$U_0 = \frac{d}{d\kappa_0}(\kappa_0 V_0). \tag{5}$$

The group, to an observer travelling with velocity U_0, appears as consisting of approximately simple waves of length $2\pi/\kappa_0$. The simple group is, in fact, propagated as an approximately homogeneous simple wave-train; the importance of the group-velocity lies in the fact that any slight departure from homogeneity on a simple wave-train, due to local variation of amplitude or phase, is propagated with the velocity U.

§ 3. *The Fourier Integral regarded as a Collection of Groups.*

An arbitrary disturbance can, in general, be analysed by Fourier's method into a collection of simple wave-trains ranging over all possible values of κ; thus after a time t the disturbance will be given by an expression of the type

$$\int_0^\infty \phi(\kappa) \cos \kappa (x - Vt + \alpha)\, d\kappa, \tag{6}$$

where V is a given function of κ.

The method adopted with these integrals is based on Lord Kelvin's* treatment of the case, in which the amplitude factor $\phi(\kappa)$ is a constant, so that

$$y = \int_0^\infty \cos \kappa (x - Vt)\, d\kappa.$$

An integral solution of this kind is constructed to represent the subsequent effect of an initial disturbance which is infinitely intense, and concentrated in a line through the origin; Lord Kelvin's process gives an approximate evaluation suitable for times and places such that $x - Vt$ is large, and the argument may be stated in the following manner:—

In the dispersive medium the wave-trains included in each differential element of the varying period are mutually destructive, except when they are in the same phase and so cumulative for the time under consideration, this being when the argument of the undulation is stationary in value. Thus each differential element as regards period, in the Fourier integral, represents a disturbance which is very slight except around a certain point which itself changes with the time.

Now if we apply this method to the more general integral (6), we obtain an

* Sir W. Thomson, 'Roy. Soc. Proc.,' vol. 42, p. 80 (1887).

expression for the total disturbance, attending only to its prominent features and neglecting the rest, provided we assume the change of the amplitude factor $\phi(\kappa)$ to be gradual. On this hypothesis the resulting expression contains the amplitude of the component trains simply as a factor; and in this way the trains for which it is a maximum show predominantly in the formula, which exhibits the main features of the disturbance as they arise from place to place through cumulation of synchronous component trains.

The argument shows that in some respects the integral (6) may be more conveniently regarded as a collection of travelling groups instead of simple wave-trains; when $\phi(\kappa)$ is a slowly varying function, the groups will be simple groups of the type (3). The limitations within which this is the case will appear from the subsequent analysis; one method of procedure would be graphical: to take a graph of the fluctuating factor and see that the other factor, which is taken constant, does not vary much within the range that is important for the integral.

In the cases we shall examine, the effect is due to a limited initial disturbance, and the salient features are due to the circumstance that $\phi(\kappa)$ has well-defined maxima; thus the prominent part of the effect can be expressed in the form of simple groups belonging to the neighbourhood of the maxima.

Before considering in detail special cases with assigned forms of the velocity function V, two illustrations of interest may be mentioned.

(a) *Damped harmonic wave-train.*—If $f(x)$ is a function satisfying the conditions for the Fourier transformation, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty d\kappa \int_{-\infty}^\infty f(\omega)\cos\kappa(\omega - x)\,d\omega.$$

For an even function of x, this gives

$$f(x) = \frac{2}{\pi}\int_0^\infty \phi(\kappa)\cos\kappa x\,d\kappa, \quad \text{where} \quad \phi(\kappa) = \int_0^\infty f(\omega)\cos\kappa\omega\,d\omega. \tag{7}$$

Now let $f(x)$ be an even function of x, defined for all values, and such that it is equal to $e^{-\mu x}\cos\kappa' x$ for x positive; then we find

$$2\phi(\kappa) = \frac{\mu}{\mu^2 + (\kappa - \kappa')^2} + \frac{\mu}{\mu^2 + (\kappa + \kappa')^2}. \tag{8}$$

Consider this function $f(x)$ as the initial value of a disturbance y which occurs in a dispersive medium; then the value of y at any time can be expressed, in general, by

$$y = \mathrm{A}\int_0^\infty \phi(\kappa)\cos\kappa(x - \mathrm{V}t)\,d\kappa + \mathrm{B}\int_0^\infty \phi(\kappa)\cos\kappa(x + \mathrm{V}t)\,d\kappa, \tag{9}$$

where A, B are constants which need not be specified further in the present connection.

These integrals are of the type (6), and represent infinite wave-trains travelling in the positive and negative directions respectively. We see from (8) that when μ is small, the amplitude factor $\phi(\kappa)$ consists practically of a single well-defined peak in the neighbourhood of the value κ'. Hence, when the damping coefficient μ is small, the wave-trains in question may be considered as travelling in the form of a group κ' of unchanging waves of this specified structure.

This example serves to illustrate the propagation of a very long train of simple harmonic waves subsiding as they travel owing to a small damping coefficient, and is of interest in connection with Lord Rayleigh's general proof that the group-velocity U is the velocity with which energy is being propagated.* A small damping coefficient μ is introduced by him, so that the energy transmitted is determined by the energy dissipated; the argument, which of course loses its meaning if μ is actually zero, shows that when μ is diminished indefinitely the rate of transmission of energy approaches U as a limiting value. Similarly, although the Fourier transformation is inapplicable when μ is actually zero, we infer from the above analysis that when μ is diminished indefinitely, the disturbance is representable as a simple group of unchanging waves of definite structure.

(b) *Interrupted simple wave-train.*—Consider an initial disturbance defined by

$$\begin{aligned} f(x) &= 0, \qquad (-d < x < d) \\ &= e^{-\mu x} \sin \kappa' (x-d), \qquad (x > d), \\ &= -e^{\mu x} \sin \kappa' (x+d), \qquad (x < -d). \end{aligned}$$

Then the disturbance is given by an expression of the form (9), in which

$$\begin{aligned} 2\phi(\kappa) &= 2\int_d^\infty e^{-\mu\omega} \sin \kappa'(\omega - d) \cos \kappa\omega \, d\omega \\ &= \frac{(\kappa+\kappa') \cos \kappa d - \mu \sin \kappa d}{\mu^2 + (\kappa+\kappa')^2} + \frac{(\kappa'-\kappa) \cos \kappa d - \mu \sin \kappa d}{\mu^2 + (\kappa'-\kappa)^2}. \qquad (10) \end{aligned}$$

Now suppose μ and d very small, so that the initial disturbance approximates to an infinite simple harmonic form with a narrow range of discontinuity; we see that the graph of the amplitude factor $\phi(\kappa)$ is then reduced to a single peak in the vicinity of the value κ'. We infer from this example that a very long simple harmonic wave-train which is interrupted for a short interval is kinematically equivalent to a group of unchanging waves, of definite structure ranging round the value $2\pi/\kappa'$ of the wave-length.

* Lord Rayleigh, 'Proc. Lond. Math. Soc.,' vol. 9, p. 24 (1877).

§ 4. *Features of the Integrals Involved.*

The integrals we have to consider in such problems are of the type

$$y = \int \phi(u) \cos\{f(u)\}\, du. \tag{11}$$

All such integrals we can treat in the same manner, adopting the method employed by Lord Kelvin for the particular case referred to above (§ 3). This method consists in supposing that $f(u)$ is large, so that the cosine factor is a rapidly varying quantity compared with the first factor; thus, much as in the Fresnel discussion of the diffraction of light waves, the prominen part of the graph of the integral is contained within a small range of u f.. which $f(u)$ is stationary in value, so that the elements are then cumulative. In other words, we select from (11) the group or groups of terms ranging round values u_0 of u which make

$$f'(u_0) = 0. \tag{12}$$

In such a group of terms we may put

$$f(u) = f(u_0) + \tfrac{1}{2}(u-u_0)^2 f''(u_0).$$

Then if we write σ^2 for $\frac{1}{2}(u-u_0) f''(u_0)$. the contribution of the group to the value of (11) is given by

$$y_0 = \left\{\frac{2}{f''(u_0)}\right\}^{\frac{1}{2}} \int_{-\infty}^{\infty} \phi(u_0) \cos\{f(u_0)+\sigma^2\}\, d\sigma, \tag{13}$$

where the limits of the integral may be in general extended, as in diffraction theory, to $\pm\infty$, provided u_0 does not coincide with either limit of the integral (11), and also provided that $f''(u_0)$ is not zero.

Thus we have, from (13),

$$\begin{aligned} y_0 &= \left\{\frac{\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0)\left[\cos\{f(u_0)\} - \sin\{f(u_0)\}\right] \\ &= \left\{\frac{2\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) + \tfrac{1}{4}\pi\}. \end{aligned} \tag{14}$$

This is the sum of the contributions of the constituents of each group around a central value u_0 given by (12), provided the value u_0 comes within the range of values of u in the integral (11).

If $f''(u_0)$ is negative, the corresponding result may be written

$$y_0 = \left\{\frac{2\pi}{-f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) - \tfrac{1}{4}\pi\}. \tag{15}$$

We write down for reference the similar pair of results for a group of terms from the integral

$$y = \int \phi(u) \sin\{f(u)\}\, du. \tag{16}$$

If $f''(u_0)$ is positive,

$$y_0 = \left\{\frac{2\pi}{f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) - \tfrac{1}{4}\pi\}; \tag{17}$$

and if $f''(u_0)$ is negative,

$$y_0 = \left\{\frac{2\pi}{-f''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{f(u_0) + \tfrac{1}{4}\pi\}. \tag{18}$$

The chief form in which such integrals occur is

$$y = \int \phi(\kappa) \cos\kappa(x - Vt)\, d\kappa, \quad \text{where} \quad V = f(\kappa). \tag{19}$$

The principal groups are given by the values κ_0 such that

$$\frac{d}{d\kappa}\{\kappa(x - Vt)\} = 0, \quad \text{or} \quad \frac{x}{t} = \frac{d}{d\kappa}(\kappa V) = U. \tag{20}$$

The aggregate value of the group can be written down from one of the previous forms; if $\partial U/\partial\kappa$ is negative, we should have

$$y_0 = \left\{\frac{2\pi}{-t\,\partial U/\partial\kappa}\right\}^{\frac{1}{2}} \phi(\kappa_0) \cos\{\kappa_0(x - Vt) + \tfrac{1}{4}\pi\}. \tag{21}$$

As an illustrative example we may suppose a disturbance y to be given at time t by the expression*

$$y = \int_0^\infty \cos\kappa(x - Vt)\, d\kappa. \tag{22}$$

When $x - Vt$ is large, the elementary waves given by (22) reinforce each other only for the simple groups given by values κ_0 for which the argument of the cosine is stationary, so that

$$x - Ut = 0. \tag{23}$$

This equation (23) defines a velocity U such that to an observer starting from the origin and travelling with this velocity the complex disturbance has the appearance of simple waves of length $2\pi/\kappa_0$. Or again, we may regard (23) as giving the predominant value of κ_0 at any position and time in terms of x and t. The features of the disturbance will depend on the form of the velocity function V; we proceed to consider some special forms.

§ 5. *Initial Line Displacement on Deep Water.*

We consider surface waves on an unlimited sheet of deep water, the only bodily forces being those due to gravity. Let the x-axis be in the undisturbed horizontal surface, and the y-axis be drawn vertically upwards. Let η be the elevation of surface waves of small amplitude with parallel crests and troughs perpendicular to the xy-plane. It can be shown that for an initial displace-

* Lord Kelvin, *loc. cit. ante.*

ment given by $\eta = \cos \kappa x$, without initial velocity, the surface form at any subsequent time is given by

$$\eta = \cos \kappa \mathrm{V} t \cos \kappa x = \tfrac{1}{2} \{\cos \kappa (x - \mathrm{V}t) + \cos \kappa (x + \mathrm{V}t)\},$$

where
$$\mathrm{V} = (g/\kappa)^{\frac{1}{2}}. \tag{24}$$

Let $f(x)$ be any even function of x which can be analysed by Fourier's integral theorem. Then, corresponding to an initial surface displacement $f(x)$, without initial velocity, there is a surface form given at any subsequent time by

$$\eta = \frac{1}{2\pi} \int_0^\infty \phi(\kappa) \cos \kappa (x - \mathrm{V}t)\, d\kappa + \frac{1}{2\pi} \int_0^\infty \phi(\kappa) \cos \kappa (x + \mathrm{V}t)\, d\kappa, \tag{25}$$

where
$$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega) \cos \kappa\omega \, d\omega. \tag{26}$$

If we suppose the initial elevation to be limited practically to a line through the origin and assume that $\int_{-\infty}^{\infty} f(x)\, dx = 1$, so that $\phi(\kappa) = 1$, we can use, as an illustration of the procedure, the form

$$\eta = \frac{1}{2\pi} \int_0^\infty \cos \kappa (x - \mathrm{V}t)\, d\kappa + \frac{1}{2\pi} \int_0^\infty \cos \kappa (x + \mathrm{V}t)\, d\kappa. \tag{27}$$

We select from these integrals the groups which give the chief regular features at large distances from the original disturbance. This cumulative group from the first integral is given for a given position and time by the value of κ for which $\kappa(x - \mathrm{V}t)$ is stationary, where $\mathrm{V} = \sqrt{(g/\kappa)}$, so that

$$\frac{x}{t} = \mathrm{U} = \tfrac{1}{2} \sqrt{\frac{g}{\kappa}};$$

and, similarly, from the second integral by

$$-\frac{x}{t} = \tfrac{1}{2} \sqrt{\frac{g}{\kappa}}.$$

Thus there are symmetrical groups of waves proceeding in the two directions from the origin; for x positive we need only consider the first integral in (27), and for x negative the second integral. Thus the predominant wave-length at a point x at time t is given by

$$\kappa = gt^2/4x^2. \tag{28}$$

Evaluating this predominant group by means of expression (21), we obtain the known result

$$\eta = \frac{g^{\frac{1}{2}} t}{2\pi^{\frac{1}{2}} x^{\frac{3}{2}}} \cos \left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \tag{29}$$

At a given position, far enough from the source for the train to be taken

as unlimited, this indicates oscillations succeeding each other with continually increasing frequency and amplitude; also if we follow a group of waves with given value of κ the amplitude varies inversely as $t^{\frac{1}{2}}$, or inversely as the square root of x.*

§ 6. *Initial Displacement of Finite Breadth.*

If l is the range within which the initial displacement is sensible, the previous results hold with l/x small; further, as Cauchy shewed, $gt^2l/4x^2$ must be small if the function $\phi(\kappa)$ of (26) is to be taken as constant. Prof. Burnside† has obtained approximate equations for the surface form due to certain limited initial displacements not confined to an indefinitely narrow strip. From the present point of view, such results can be recovered simply by selecting from the integrals the more important groups of waves.

(*a*) Let the initial displacement be given by

$$f(x) = c\alpha^2/(\alpha^2+x^2), \tag{30}$$

where α may be supposed small.

Then
$$\phi(\kappa) = \int_{-\infty}^{\infty} \frac{c\alpha^2 \cos \kappa\omega}{\alpha^2+\omega^2}\, d\omega = \pi c\alpha e^{-\alpha\kappa}.$$

Hence from (25) the surface form is

$$\eta = \tfrac{1}{2}c\alpha \int_0^{\infty} e^{-\alpha\kappa} \cos \kappa (x-\mathrm{V}t)\, d\kappa + \tfrac{1}{2}c\alpha \int_0^{\infty} e^{-\alpha\kappa} \cos \kappa (x+\mathrm{V}t)\, d\kappa. \tag{31}$$

For points at some distance from the range in which the original disturbance was sensible, $e^{-\alpha\kappa}$ varies slowly compared with the cosine term; thus we may consider the integrals as made up of simple groups. For x positive we need only consider the first integral.

The predominant value of κ is thus connected with x and t by the same equation (28) as before. Since the greater amplitudes are associated with the smaller values of κ and these have the greater values of U, it is clear that, at a particular point, the disturbance dies away from its maximum at a slower rate than its growth up to it. Using the previous results we can write down the disturbance involved in the main group form as

$$\eta = c\alpha\pi^{\frac{1}{2}} \frac{g^{\frac{1}{2}}t}{2x^{\frac{3}{2}}} e^{-\frac{gt^2\alpha}{4x^2}} \cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \tag{32}$$

The following results can be deduced. The cosine term varies rapidly compared with the other factors, hence we may obtain the maximum by considering the latter alone; it is easily seen that this occurs when

$$x/t = \sqrt{(\tfrac{1}{2}g\alpha)}.$$

* Lamb, 'Proc. Lond. Math. Soc.' (2), vol. 2, p. 371 (1904).

† W. Burnside, 'Proc. Lond. Math. Soc.,' vol. 20, p. 22 (1888).

Thus the maximum is propagated out with uniform velocity; and we see that in its neighbourhood the predominant wave-length is $4\pi\alpha$.

(b) Let the initial displacement have a constant value A over a range of breadth $2c$, and be zero at all other points; then we have

$$\phi(\kappa) = 2A\frac{\sin \kappa c}{\kappa}.$$

Hence the surface elevation is

$$\eta = \frac{A}{\pi}\int_0^\infty \frac{\sin \kappa c}{\kappa}\cos \kappa(x - Vt)\,d\kappa + \frac{A}{\pi}\int_0^\infty \frac{\sin \kappa c}{\kappa}\cos \kappa(x + Vt)\,d\kappa. \qquad (33)$$

With the same argument as before, we consider the value of η at a point as due to the most important of a succession of simple groups, that one, namely, for which the argument is there stationary so that the components reinforce over a considerable range of κ; and we can write down, from the previous results, an expression for this group which is valid at least in the vicinity of the travelling maxima of the disturbance. We have

$$\eta = \frac{4A}{\pi^{\frac{1}{2}}}\left(\frac{x}{gt^2}\right)^{\frac{1}{2}}\sin\frac{gt^2c}{4x^2}\cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right), \qquad (34)$$

corresponding to Burnside's result in the paper already cited.

Here we have a succession of maxima given by those of $(x/gt^2)^{\frac{1}{2}}\sin(gt^2/4x^2)$, that is, at times given by $\tan\theta = 2\theta$, where $\theta = gt^2c/4x^2$.

The period of the group that is thus cumulative is different for different localities, and for different times at the same locality; but the accumulation is very prominent only for those times and localities which give a maximum value to the amplitude, which has been graphed for the next example in fig. 1.

The maxima here diminish continually in value, and are propagated each with uniform velocity, namely, the group-velocity corresponding to the predominant wave-length in the neighbourhood.

§ 7. *Limited Train of Simple Oscillations.*

Another interesting example is the case of an initial displacement consisting of a limited length of simple harmonic oscillations. If $f(x)$ is symmetrical with respect to the origin, and is zero except for a range of $(2n + \frac{1}{2})$ wave-lengths within which it is $A\cos\kappa' x$, we have

$$\phi(\kappa) = 2\int_0^{(2n+\frac{1}{2})\pi/\kappa'} A\cos\kappa'\omega\cos\kappa\omega\,d\omega = 2\kappa' A\frac{\cos(2n+\frac{1}{2})\pi\kappa/\kappa'}{\kappa'^2 - \kappa^2}. \qquad (35)$$

Hence, from (25), we have the surface elevation η, of which we write down

only the integral necessary for propagation in the direction of x positive, that is,

$$\eta = \frac{\kappa' A}{\pi} \int_0^\infty \frac{\cos (2n+\frac{1}{2}) \pi\kappa/\kappa'}{\kappa'^2 - \kappa^2} \cos \kappa (x - Vt)\, d\kappa. \tag{36}$$

If n is very large, the main feature consists of the component waves round the value κ'; but in general it is to be noticed that a series of subsidiary components appears whose effects may be of sufficient magnitude to be appreciable. But the component waves are cumulative only for values of x and t such that

$$\kappa = gt^2/4x^2,$$

which is the value corresponding to a stationary argument of the cosine; thus the prominent effect at time t, of any group of parameter κ, will be at localities where κ has the value κ', or, else, a value belonging to one of the subsidiary maxima. The result may be evaluated in the same manner as before; we find

$$\eta = 16\kappa' A \left(\frac{g}{\pi}\right)^{\frac{1}{2}} \frac{x^{\frac{1}{2}} t}{16\kappa'^2 x^4 - g^2 t^4} \cos (2n+\tfrac{1}{2}) \frac{\pi g t^2}{4\kappa' x^2} \cos\left(\frac{gt^2}{4x} - \tfrac{1}{4}\pi\right). \tag{37}$$

We can obtain the prominent travelling groups above referred to, which this involves, by evaluating the maxima of the amplitude function

$$\frac{t}{16\kappa'^2 x^4 - g^2 t^4} \cos (2n+\tfrac{1}{2}) \frac{\pi g t^2}{4\kappa' x^2}. \tag{38}$$

The form of this function is shown by fig. 1; it is obtained by plotting the curve

$$y = \frac{\alpha}{1-\alpha^4} \cos \tfrac{9}{2}\pi\alpha^2, \tag{39}$$

where α is proportional to t, and, further, α equal to 1 corresponds to κ equal to κ'.

The curve represents the variation of the disturbance at a given point with the time, neglecting the local variations of the last cosine factor in (37); it shows the grouped propagation of an initial displacement consisting of $4\frac{1}{2}$ complete wave-lengths of a simple cosine wave of wave-length $2\pi/\kappa'$, or λ'.

The main undulatory disturbance appears as a simple group around the predominant wave-length λ', moving forward with the corresponding group-velocity $\frac{1}{2}\sqrt{(g/\kappa')}$ or $\frac{1}{2}V$. But *in advance* of this main group of undulations there are two or three subsidiary groups of sensible magnitude with wave-lengths in the neighbourhood of $9\lambda/2$, $9\lambda/4$, $9\lambda/6$, moving with corresponding group-velocities of $3V/2\sqrt{2}$, $3V/4$, $3V/2\sqrt{6}$. Thus in advance of the main group we have slighter groups of larger wave-lengths moving with group-velocities which may be larger than the wave-velocity of the original dis-

turbance if it were unlimited. In the *rear* of the main group we have also a series of alternating groups, following each other much more quickly and with their wave-lengths and velocities less separated out than in the front of the main group. Hence the disturbance in the rear, especially at distances from the origin not very great, may be expected to consist of small, more

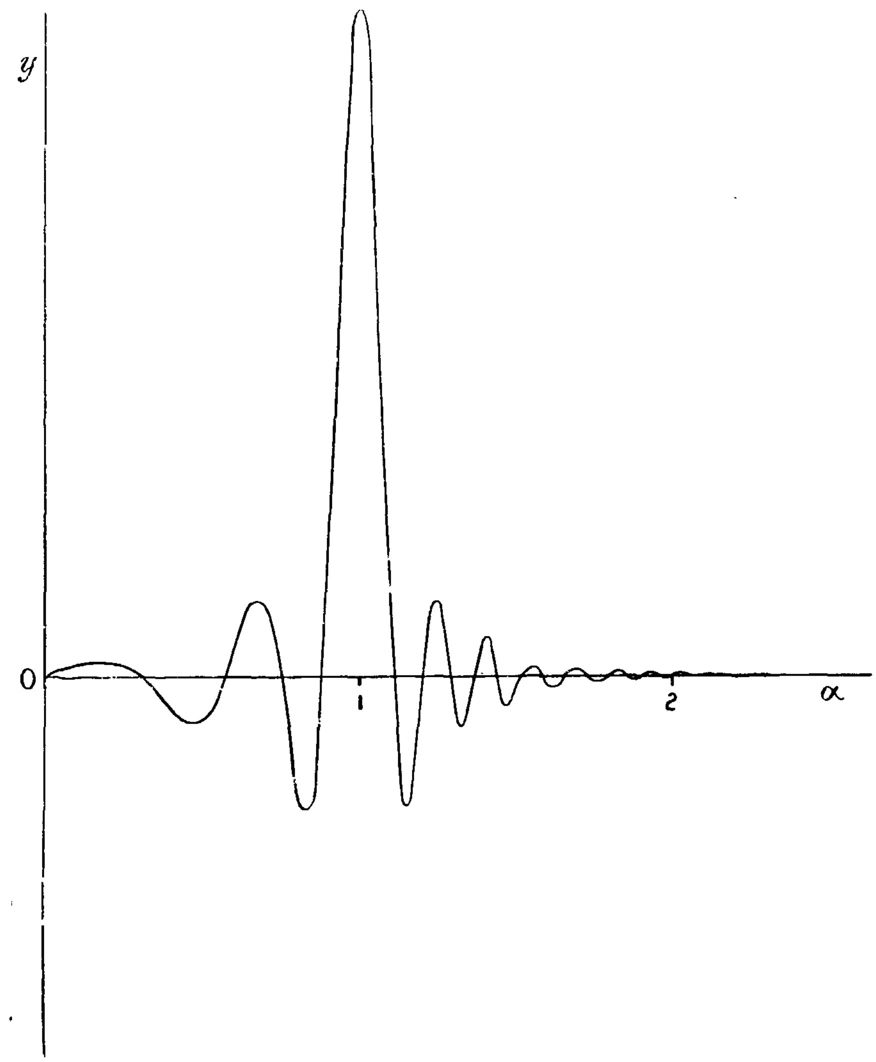


FIG. 1.

irregular, motion resulting from the superposition of this latter system of groups, thus there will be a more distinctive rear of disturbance moving forward with velocity $\frac{1}{2}$V. These inferences may be compared with some results given in Lord Kelvin's later papers. Starting from a solution of the equations for an initial elevation in the form of a single crest, the results were combined graphically so as to show in a series of figures the propagation of

an initial disturbance consisting of five crests and four hollows of approximately sinuscidal shape; the following remarks are made:—* "Immediately after the water is left free, the disturbance begins analysing itself into two groups of waves, seen travelling in contrary directions from the middle line of the diagram. The perceptible fronts of these two groups extend rightwards and leftwards from the end of the initial static group far beyond the 'hypothetical fronts,' supposed to travel at half the wave-velocity, which (according to the dynamics of Osborne Reynolds and Rayleigh, in their important and interesting consideration of the work required to feed a uniform procession of water-waves) would be the actual fronts *if* the free groups remained uniform. How far this *if* is from being realised is illustrated by the diagrams of fig. 35, which show a great extension outwards in each direction far beyond distances travelled at half the 'wave-velocity.' While there is this great extension of the fronts outward from the middle, we see that the two groups, after emergence from coexistence in the middle, travel with their rears leaving a widening space between them of water not perceptibly disturbed, but with very minute wavelets in ever augmenting number following slower and slower in the rear of each group. The extreme perceptible rear travels at a speed closely corresponding to the 'half wave-velocity.' . . . . Thus the perceptible front travels at speed actually higher than the wave-velocity, and this perceptible front becomes more and more important relatively to the whole group with the advance of time . . . ."

This extract will serve to emphasise the importance of strict definition and use of the word "group." A simple group, of whatever structure, has associated with it a definite velocity depending only on the wave-length, but not so an arbitrary limited displacement. In various cases we have found it convenient to analyse such into its important elementary groups, each with definite velocity; in special cases the disturbance may be equivalent practically to one simple group.

§ 8. *Initial Impulse on Deep Water.*

Suppose that initially the surface is horizontal, but that given impulses are applied to it. Then for any given symmetrical distribution of impulse $f(x)$, suitable for Fourier analysis, with no initial elevation, the surface elevation at any subsequent time is given by

$$\pi g\rho\eta = \tfrac{1}{2}\int_0^\infty \kappa \mathrm{V}\phi(\kappa)\sin\kappa(x-\mathrm{V}t)\,d\kappa - \tfrac{1}{2}\int_0^\infty \kappa\mathrm{V}\phi(\kappa)\sin\kappa(x+\mathrm{V}t)\,d\kappa, \quad (40)$$

where
$$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega)\cos\kappa\omega\,d\omega.$$

* Lord Kelvin, 'Phil. Mag.,' vol. 13, p. 11 (1907).

If we assume $\phi(\kappa)$ equal to 1, so that it is confined to an indefinitely narrow strip of impulse (*cf.* § 5), we obtain the result corresponding to (29) for initial displacement by multiplying that expression by the value of κV; thus we find

$$\eta = \frac{g^{\frac{1}{2}}t^2}{4\pi^{\frac{1}{2}}\rho x^{\frac{3}{2}}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right). \tag{41}$$

For comparison with the previous results, suppose that

$$f(x) = \frac{c\alpha^2}{\alpha^2+x^2}, \qquad \phi(\kappa) = \pi c\alpha e^{-\alpha\kappa}.$$

Then we find the surface form as an aggregate of groups, each of them cumulative and so prominent only in a limited region, given by

$$\eta = \frac{\pi^{\frac{1}{2}}c\alpha g^{\frac{1}{2}}t^2}{4\rho x^{\frac{3}{2}}}e^{-\frac{gt^2\alpha}{4x^2}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right). \tag{42}$$

For a given place the maxima are given by

$$\frac{d}{dt}\left(t^2e^{-g\alpha t^2/4x^2}\right) = 0, \quad \text{that is, by} \quad \frac{x}{t} = \tfrac{1}{2}\sqrt{(g\alpha)}.$$

Thus the maximum moves with velocity $\frac{1}{2}\sqrt{(g\alpha)}$, and consists of nearly simple waves of wave-length $2\pi\alpha$. Comparing with the result in § 6 for an initial displacement of the same character, we see that the maximum is propagated outwards with slower velocity, the wave-length at the maximum being one-half the corresponding value in the former case.

§ 9. *Moving Line Impulse on Deep Water.*

Suppose that the line impulse of the previous section is moving over the surface of deep water at right angles to its length with uniform velocity c, having started at some time practically infinitely remote. Then we may regard the effect at (x, t) as the summation of the effects due to all the consecutive elements of impulse, and we can obtain an expression by modifying (40) and integrating with respect to the time. We measure x from a fixed origin which the line impulse passes at zero time; then we substitute $x-ct_0$ for x and $t-t_0$ for t in (40), and integrate with respect to t_0 for all the time the impulse has been moving. Thus we obtain

$$\begin{aligned}\pi g\rho\eta &= \tfrac{1}{2}\int_{-\infty}^{t}dt_0\int_0^\infty \kappa V\sin\kappa\{x-ct_0-V(t-t_0)\}\,d\kappa \\ &\qquad -\tfrac{1}{2}\int_{-\infty}^{t}dt_0\int_0^\infty \kappa V\sin\kappa\{x-ct_0+V(t-t_0)\}\,d\kappa \\ &= \tfrac{1}{2}\int_0^\infty du\int_0^\infty \kappa V\sin\kappa\{\varpi+(c-V)u\}\,d\kappa \\ &\qquad -\tfrac{1}{2}\int_0^\infty du\int_0^\infty \kappa V\sin\kappa\{\varpi+(c+V)u\}\,d\kappa, \end{aligned} \tag{43}$$

where $\varpi = x - ct$, and represents distance in advance of the present position of the impulse. We proceed to obtain now the important regular features of the disturbance represented by these integrals.

With the notation of (19) and (20) we have in the first integral

$$f(\kappa) = c - V = c - \sqrt{(g/\kappa)},$$

$$\frac{d}{d\kappa}\{\kappa f(\kappa)\} = c - \tfrac{1}{2}\sqrt{(g/\kappa)}.$$

Hence the required predominant value of κ, which corresponds to a stationary argument, is given by

$$c - \tfrac{1}{2}\sqrt{\frac{g}{\kappa}} = -\frac{\varpi}{u}, \quad \text{or} \quad \kappa = \frac{gu^2}{4(cu+\varpi)^2}. \tag{44}$$

Thus the first integral in (43) gives

$$\tfrac{1}{4}\pi^{\frac{1}{2}}g^{\frac{1}{2}}\int_0^\infty \frac{u^2}{(\varpi+cu)^{\frac{3}{2}}}\cos\left\{\frac{gu^2}{4(\varpi+cu)}+\tfrac{1}{4}\pi\right\}du. \tag{45}$$

We choose again the principal groups of oscillations by the condition

$$\frac{d}{du}\left\{\frac{gu^2}{4(\varpi+cu)}+\tfrac{1}{4}\pi\right\} = 0, \quad \text{or} \quad cu = -2\varpi.$$

Now u must be positive to come within the range of the integral (45); hence if ϖ is positive we obtain no contribution towards a regular undulatory disturbance. If ϖ is negative we obtain a series of travelling waves which we can evaluate from (45).

We have

$$\frac{d^2}{du^2}\left\{\frac{gu^2}{4(\varpi+cu)}+\tfrac{1}{4}\pi\right\} = -\frac{g}{2\varpi}, \quad \text{when} \quad cu = -2\varpi.$$

Hence, using expression (18), we obtain the value of the chief group from (45), namely,

$$\frac{2\pi g}{c^3}\sin\frac{g\varpi}{c^2}, \tag{46}$$

which holds when ϖ is negative.

As regards the second integral in (43), we easily see by taking the principal group in κ that $\varpi + cu$ must be negative: thus ϖ must be negative and cu between zero and ϖ numerically. Then taking the chief group in u, we have cu equal to 2ϖ numerically. Hence there is no resulting group of waves falling in the range, and the second integral contributes nothing to the regular disturbance.

We have then the well-known result that in front of the travelling impulse there is no regular disturbance, while in the rear there is a train of regular waves, proportional to (46), with wave-length suitable to the velocity c.

The same method can be used for waves on water of depth h, due to a travelling impulse system. For in the integrals (43) we should have

$$f(\kappa) = c - \mathrm{V} = c - \sqrt{\left(\frac{g}{\kappa}\tanh\kappa h\right)}. \tag{47}$$

The group with respect to κ would give a term proportional to

$$\cos\{u\kappa^2 f'(\kappa) + \tfrac{1}{4}\pi\}, \tag{48}$$

where κ has the value given by

$$-\varpi/u = f(\kappa) + \kappa f'(\kappa). \tag{49}$$

We then select the group with respect to u by

$$\frac{d}{du}\{u\kappa^2 f'(\kappa)\} = 0. \tag{50}$$

Using (49) we find this leads to*

$$f(\kappa) = 0, \quad \text{or} \quad \mathrm{V} = c = \sqrt{(gh)}\sqrt{\left(\frac{\tanh\kappa h}{\kappa h}\right)}. \tag{51}$$

Since $\tanh\kappa h/\kappa h$ diminishes continually from 1 to 0 as κh increases from 0 to ∞, there is only a real solution of (51) when c^2 is less than gh. In this case we have regular waves of length suitable to velocity c following in the rear of the impulse; when c is greater than the maximum wave-velocity there is no regular wave form.

§ 10. *Capillary Surface Waves.*

In order to illustrate the propagation of an element of the Fourier expression as a limited travelling group of undulations, we consider another form of velocity function. If waves are propagated over the surface of a liquid of density ρ under the action of the surface tension T, it can be shown that the velocity of simple waves of length $2\pi/\kappa$ is

$$\mathrm{V} = \sqrt{(\mathrm{T}\kappa/\rho)}. \tag{52}$$

Hence in this case the group-velocity is

$$\mathrm{U} = \tfrac{3}{2}\sqrt{(\mathrm{T}\kappa/\rho)} = \tfrac{3}{2}\mathrm{V};$$

thus the group-velocity is greater than the wave-velocity, and we shall see how this affects some of the previous results.

(a) *Initial elevation consisting of* $(2n+\frac{1}{2})$ *simple oscillations of wave-length* $2\pi/\kappa'$.—If we consider the same problem as in § 7 we have

$$\eta = \mathrm{A}\int_0^\infty \frac{\cos(2n+\frac{1}{2})\pi\kappa/\kappa'}{\kappa'^2 - \kappa^2}\cos\kappa(x - \mathrm{V}t)\,d\kappa. \tag{53}$$

* *Cf.* Lord Rayleigh, 'Phil. Mag.,' vol. 10, p. 407 (1905).

The predominant value of κ, for given time and place, is given by

$$\kappa = 4\rho x^2/9\mathrm{T}t^2. \tag{54}$$

The chief groups, each with approximately constant amplitude, are given by

$$\eta = \mathrm{A}' \frac{x^{\frac{1}{2}}t^3}{81\mathrm{T}^2\kappa'^2t^4 - 16\rho^2x^4} \cos\,(2n+\tfrac{1}{2})\,\frac{4\pi\rho x^2}{9\kappa'\mathrm{T}t^2} \cos\left(\frac{4\rho x^3}{27\mathrm{T}t^3} - \tfrac{1}{4}\pi\right). \tag{55}$$

At a given place the maxima of amplitude are those of

$$\frac{t^3}{81\mathrm{T}^2\kappa'^2t^4 - 16\rho^2x^4} \cos\,(2n+\tfrac{1}{2})\,\frac{4\pi\rho x^2}{9\kappa'\mathrm{T}t^2}. \tag{56}$$

Fig. 2 represents the curve

$$y = \frac{\alpha^3}{\alpha^4 - 1} \cos \frac{9\pi}{2\alpha^2}, \tag{57}$$

where α is proportional to the time and α equal to 1 corresponds to κ equal to κ'.

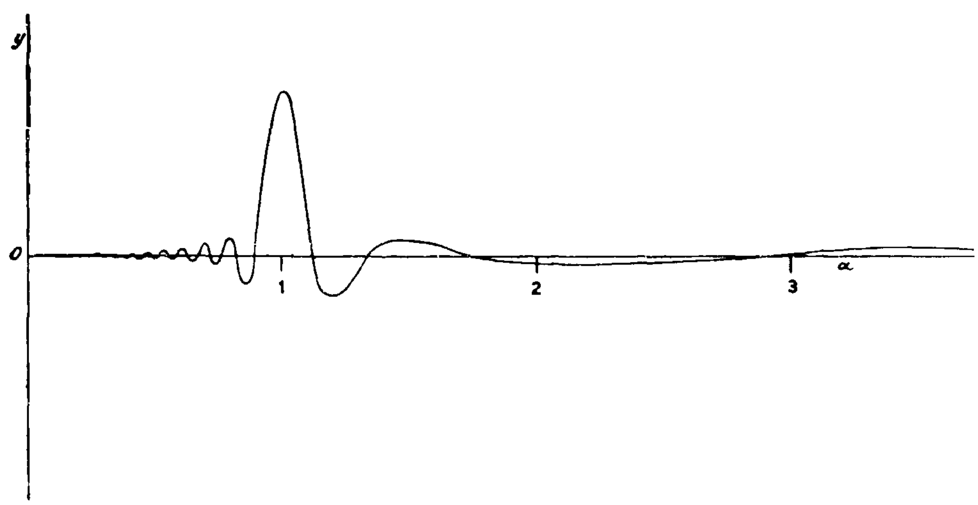


Fig. 2.

Comparing this with § 7 we draw the inference that in this case the perceptible front of the advancing train is more clearly marked than the rear and advances with the half-wave-velocity corresponding to κ', in agreement with simple observation.

(b) *Moving line impulse.*—A line impulse at rest leads to

$$\eta = \mathrm{C}x^{\frac{1}{2}}t^{-4} \cos\left(\frac{4\rho x^3}{27\mathrm{T}t^2} + \tfrac{1}{4}\pi\right).$$

Consequently a moving line impulse will give

$$\eta = \mathrm{A}\int_0^\infty \frac{(\varpi + cu)^{\frac{1}{2}}}{u^4} \cos\left\{\frac{4\rho}{27\mathrm{T}}\,\frac{(\varpi + cu)^3}{u^2} + \tfrac{1}{4}\pi\right\} du. \tag{58}$$

Then we choose u so that

$$\frac{d}{du}\left\{\frac{(\varpi+cu)^3}{u^2}\right\} = 0, \quad \text{or} \quad u(cu-2\varpi)(\varpi+cu)^2 = 0.$$

The value giving a regular wave pattern is the positive root

$$cu = 2\varpi, \text{ for } \varpi \text{ positive.}$$

Hence in this case we have a regular train of waves of length suitable to the velocity c in *advance* of the moving pressure system, with no regular pattern in the rear.

§ 11. *Water Waves due to Gravity and Capillarity.*

If we take account of gravity and the surface tension together, we have the velocity function

$$V = (T\kappa + g/\kappa)^{\frac{1}{2}}. \tag{59}$$

Hence
$$U = \frac{d}{d\kappa}(\kappa V) = \frac{3T\kappa^2+g}{2(T\kappa^3+g\kappa)^{\frac{1}{2}}}. \tag{60}$$

We have not here a simple ratio U/V, independent of κ. The velocity V has a minimum c_m for a certain value κ_m, equal to $(g/T)^{\frac{1}{2}}$, and for this value U is equal to V—as in fact follows from the definition of U. For $\kappa < \kappa_m$, U is less than V, tending ultimately to $\frac{1}{2}$V; while for $\kappa > \kappa_m$, U is greater than V and approaches as a limit $\frac{3}{2}$V.

If we consider a travelling line impulse, the whole problem of finding the principal groups is contained in the equations

$$\left.\begin{aligned} \frac{\varpi+cu}{u} = U &= \frac{3T\kappa^2+g}{2(T\kappa^3+g\kappa)^{\frac{1}{2}}} \\ c = V &= (T\kappa+g/\kappa)^{\frac{1}{2}} \end{aligned}\right\}. \tag{61}$$

Hence
$$c^2u^2 = \frac{4c^4\varpi^2}{c^4-c_m{}^4}, \qquad \kappa = \frac{c^2 \pm (c^4-c_m{}^4)^{\frac{1}{2}}}{2T},$$

where the positive sign is taken for ϖ positive (in advance of the impulse), and the negative sign for ϖ negative (in the rear). Thus there is no wave pattern unless c is greater than the minimum wave-velocity c_m; and if so there are regular trains both in advance and in the rear, the smaller wave-lengths being in advance. With the ratio c/c_m large, the results approximate to very small waves in front and waves in the rear with κ equal to g/c^2.

§ 12. *Surface Waves in two Dimensions.*

Suppose that the initial data instead of being symmetrical about a transverse straight line are symmetrical around the origin. Let the axes of

x, y be in the undisturbed surface and the axis of z vertically upwards; we write ϖ for $\sqrt{(x^2+y^2)}$. Then, corresponding to (25), the surface elevation ζ due to an initial displacement $f(\varpi)$, set free without initial velocity, is given by

$$\zeta = \int_0^\infty \mathrm{J}_0(\kappa\varpi)\cos(\kappa \mathrm{V}t)\,\phi(\kappa)\,\kappa d\kappa, \tag{62}$$

where

$$\phi(\kappa) = \int_0^\infty f(\alpha)\,\mathrm{J}_0(\kappa\alpha)\,\alpha d\alpha. \tag{63}$$

For an initial point-elevation we may take for simplicity $\phi(\kappa)$ equal to $1/2\pi$; then we have

$$\begin{aligned}\zeta &= \frac{1}{2\pi}\int_0^\infty \mathrm{J}_0(\kappa\varpi)\cos(\kappa\mathrm{V}t)\,\kappa d\kappa \\ &= \frac{1}{\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos(\kappa\varpi\cos\beta)\cos(\kappa\mathrm{V}t)\,\kappa d\kappa \\ &= \frac{1}{2\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos\kappa(\varpi\cos\beta - \mathrm{V}t)\,\kappa d\kappa \\ &\qquad + \frac{1}{2\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \cos\kappa(\varpi\cos\beta + \mathrm{V}t)\,\kappa d\kappa. \end{aligned} \tag{64}$$

For deep water we separate a real principal group from the first integral, with respect to κ, around the value of κ given by

$$\frac{\varpi\cos\beta}{t} = \tfrac{1}{2}\sqrt{\frac{g}{\kappa}}.$$

This is replaced by the equivalent form

$$\zeta = \frac{g^{\frac{1}{2}}t^3}{8\pi^{\frac{1}{2}}\varpi^{\frac{1}{2}}}\int_0^{\pi/2}\frac{d\beta}{\cos^{\frac{1}{2}}\beta}\cos\left(\frac{gt^2}{4\varpi\cos\beta} - \tfrac{1}{4}\pi\right). \tag{65}$$

Considering now the range for β, we can again select the principal group of oscillations from (65); it occurs at β equal to zero, so we take one-half the result given by the expression (14) and obtain the known result

$$\zeta = \frac{gt^2}{2^{\frac{3}{2}}\pi\varpi^3}\cos\frac{gt^2}{4\varpi}. \tag{66}$$

Similarly, for an initial point impulse we have, instead of (64), the expression

$$\zeta = \frac{1}{2g\rho\pi^2}\int_0^{\pi/2} d\beta \int_0^\infty \kappa\mathrm{V}\{\sin\kappa(\varpi\cos\beta - \mathrm{V}t) - \sin\kappa(\varpi\cos\beta + \mathrm{V}t)\}\,\kappa d\kappa, \tag{67}$$

leading in the same way to the result

$$\zeta = -\frac{gt^3}{2^{\frac{5}{2}}\pi\rho\varpi^4}\sin\frac{gt^2}{4\varpi}. \tag{68}$$

§ 13. *Point Impulse Travelling over Deep Water.*

Let the impulse be moving along Ox with constant velocity c; let B be the position at time t, A at any previous time t_0, and suppose the system to have been moving for an indefinitely long time.

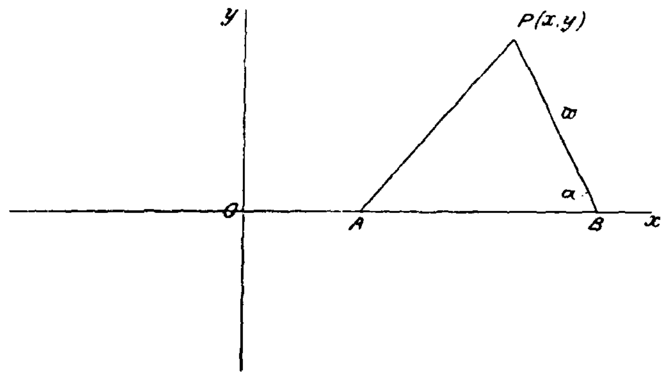


Fig. 3.

We have
$$\mathrm{OA} = ct_0\,;\ \mathrm{OB} = ct\,;$$
$$\mathrm{PB} = \varpi = \{(ct-x)^2+y^2\}^{\frac{1}{2}}\,;$$
$$\cos\alpha = (ct-x)/\varpi.$$

Then in (67) we have to substitute $\{\bar{\omega}^2 - 2c\,\omega(t-t_0)\cos\alpha + c^2(t-t_0)^2\}^{\frac{1}{2}}$ for $\bar{\omega}$, $t-t_0$ for t, and integrate with respect to t_0 from $-\infty$ to t; we obtain

$$\zeta = \frac{1}{2g\rho\pi^2}\int_0^{\pi/2} d\beta \int_0^{\infty} du \int_0^{\infty} \kappa \mathrm{V}\,[\sin\kappa\,(\cos\beta\,\{\varpi^2-2cu\varpi\cos\alpha+c^2u^2\}^{\frac{1}{2}}-\mathrm{V}u)$$
$$-\sin\kappa\,(\cos\beta\,\{\varpi^2-2cu\varpi\cos\alpha+c^2u^2\}^{\frac{1}{2}}+\mathrm{V}u)]\,\kappa\,d\kappa. \quad (69)$$

With $\mathrm{V} = (g/\kappa)^{\frac{1}{2}}$, we select the group around the value of κ given by
$$\kappa^{-1} = 4\cos^2\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)/gu^2. \quad (70)$$

By using the formula (17) we find
$$\zeta = \frac{g^{\frac{1}{2}}}{16\rho\pi^{\frac{3}{2}}}\int_0^{\infty} du \int_0^{\pi/2} \frac{u^4\,d\beta}{\cos^{\frac{3}{2}}\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{3}{4}}}$$
$$\cos\left\{\frac{gu^2}{4\cos\beta\,(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{1}{2}}}+\tfrac{1}{4}\pi\right\}. \quad (71)$$

Selecting from this the chief group which occurs near β equal to zero, we find
$$\zeta = -\frac{g}{2^{\frac{1}{2}}\pi\rho}\int_0^{\infty}\frac{u^3du}{(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^2}\sin\frac{gu^2}{4(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{\frac{1}{2}}}. \quad (72)$$

Finally we choose the chief groups of terms in u from the condition
$$\frac{d}{du}\tfrac{1}{4}gu^2(\varpi^2-2cu\varpi\cos\alpha+c^2u^2)^{-\frac{1}{2}} = 0\,; \quad (73)$$

that is, from $$c^2u^2 - 3cu\varpi \cos\alpha + 2\varpi^2 = 0, \tag{74}$$

or $$cu = \tfrac{1}{2}\varpi\,\{3\cos\alpha \pm (9\cos^2\alpha - 8)^{\frac{1}{2}}\}, \tag{75}$$

We have then different cases to consider according to the nature of these values for cu, remembering that cu gives a position of the moving impulse, at time u previously, for which the waves sent out reinforce each other at the point (ϖ, α) at time t.

(i) In the region where $9\cos^2\alpha < 8$, both roots are imaginary; thus the previous position is non-existent, and there is no principal group in the integral (72). Hence all the regular wave pattern is contained within two straight lines radiating from the point impulse, each making with the line of motion an angle $\cos^{-1} 2\sqrt{2}/3$, or approximately $19° \, 28'$.

(ii) When $9\cos^2\alpha < 8$, there are two different real roots for cu. Thus we have two chief groups in the integral (72), corresponding to two regular wave systems superposed on each other.

At any point P within the two bounding radii the disturbance consists of two parts: one part sent out from A at time u_1 previously, where

$$\mathrm{OA} = \tfrac{1}{2}\varpi\,\{3\cos\alpha + (9\cos^2\alpha - 8)^{\frac{1}{2}}\} \text{ and } u_1 = \mathrm{OA}/c; \tag{76}$$

and another part sent out from B at time u_2 before, where

$$\mathrm{OB} = \tfrac{1}{2}\varpi\,\{3\cos\alpha - (9\cos^2\alpha - 8)^{\frac{1}{2}}\} \text{ and } u_2 = \mathrm{OB}/c. \tag{77}$$

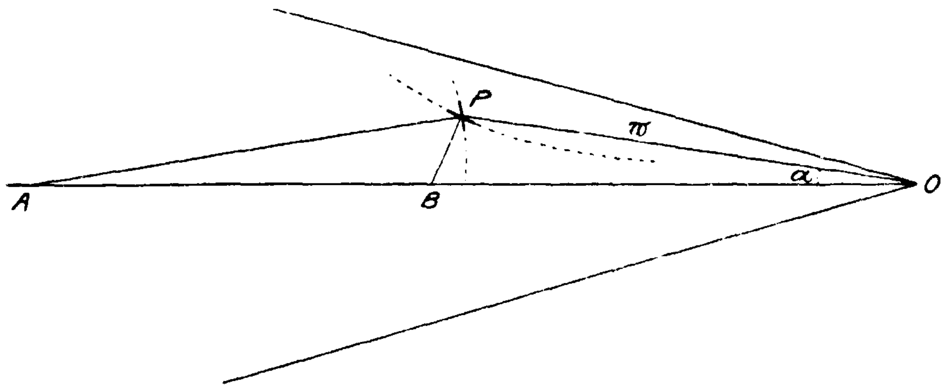


FIG. 4.

We have then two wave systems, which may be called the transverse waves and the diverging waves; we shall examine them separately.

(a) *The transverse wave system.*—Taking the larger value of cu in (76) we find

$$\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2 = \tfrac{1}{2}\varpi^2\,\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\},$$

$$f(u) = \frac{gu^2}{4\,(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}} = \frac{g\varpi\sqrt{2}}{16c^2}\,\frac{18\cos^2\alpha - 8 + 6\cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}}{\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{1}{2}}}. \tag{78}$$

Further, when $f'(u)$ is zero, we have

$$f''(u) = \tfrac{1}{4} gcu\,(2cu - 3\varpi \cos\alpha)/(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{3}{2}}. \tag{79}$$

Using the formula (17) we obtain the particular group of terms from the integral (72) as

$$\zeta = -\frac{g}{2^{\frac{1}{2}}\pi\rho}\,\frac{u^3}{(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^2}\left\{\frac{2\pi}{f''(u)}\right\}^{\frac{1}{2}}\cos\{f(u) - \tfrac{1}{4}\pi\}, \tag{80}$$

in which the special value of u must be substituted.

Evaluating this expression we obtain

$$\zeta = -\frac{g^{\frac{1}{2}}}{2^{\frac{13}{4}}\pi^{\frac{1}{2}}\rho c^3}\,\frac{\{3\cos\alpha + (9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{3}{2}}}{\varpi^{\frac{1}{2}}(9\cos^2\alpha - 8)^{\frac{1}{4}}\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{3}{4}}} \times$$

$$\cos\left\{\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha + (9\cos^2\alpha - 8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{1}{2}}} - \tfrac{1}{4}\pi\right\}. \tag{81}$$

This represents a system of transverse waves travelling with the originating impulse; the amplitude for a given azimuth α diminishes as $\varpi^{-\frac{1}{2}}$. On the central line, where α is zero, this reduces to

$$\zeta = -\frac{2^{\frac{1}{2}} g^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\rho c^3 \varpi^{\frac{1}{2}}}\cos\left(\frac{g\varpi}{c^2} - \tfrac{1}{4}\pi\right), \tag{82}$$

corresponding to simple line waves of length suitable to velocity c on deep water, but with the amplitude factor $\varpi^{-\frac{1}{2}}$.

Following the crest of a transverse wave we have

$$\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha + (9\cos^2\alpha - 8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{1}{2}}} - \tfrac{1}{4}\pi = (2n+1)\pi, \tag{83}$$

where n is a positive integer. The crests cut the axis in points given by

$$\varpi = c^2(2n + \tfrac{5}{4})\pi/g, \tag{84}$$

and cut the radial boundaries given by $\alpha = \pm\cos 2\sqrt{2}/3$, in the points

$$\varpi = 2c^2(2n + \tfrac{5}{4})\,\pi/g\sqrt{3}. \tag{85}$$

Consider the variation of amplitude following a crest; we substitute for ϖ from (83) in (82) and obtain

$$\zeta = \frac{\text{const.}}{(2n + \frac{5}{4})^{\frac{1}{2}}}\,\frac{\{3\cos\alpha + (9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{1}{2}}}{(9\cos^2\alpha - 8)^{\frac{1}{4}}\{3\cos^2\alpha - 2 + \cos\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}\}^{\frac{1}{2}}}. \tag{86}$$

This becomes infinite at the outer boundary, when α is approximately $19^\circ\ 28'$; this is due to the failure of the method of approximation and we shall consider it later. For the present the following table of values and curve show that the approximation holds up to angles very near the limit.

Table I.

ζ = relative amplitude, along the same crest at different azimuths.

α.	ζ.
0	1
6	1·03
12	1·18
18	2
19	2·9
19 15	3·5
19 27	7·5
19 28	∞

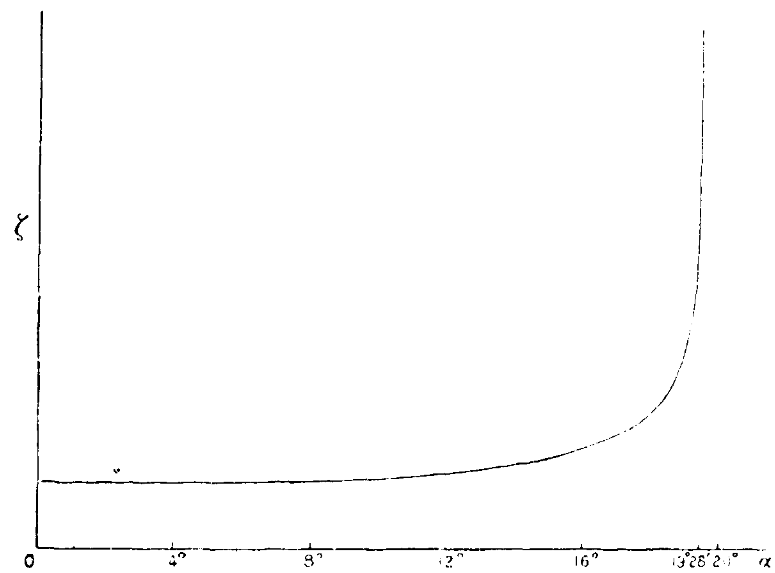


Fig. 5.

(b) *The diverging wave system.*—By taking the smaller root for *cu* given by (77), we obtain the system of diverging waves: we need only change the sign of the radicle in order to write down the corresponding results in this case.

The crests of the waves are given by

$$\frac{g\varpi\sqrt{2}}{16c^2}\,\frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{1}{2}}}=(2n+\tfrac{5}{4})\,\pi, \tag{87}$$

When α is zero, ϖ is also zero: thus all the crests diverge from the point of impulse. Further, we have

$$\alpha=\cos^{-1}2\sqrt{2}/3\,; \qquad \varpi=2c^2(2n+\tfrac{5}{4})\,\pi/g\sqrt{3}. \tag{88}$$

The law of amplitude along the same crest is given by

$$\zeta = \frac{\text{const.}}{(2n+\frac{5}{4})^{\frac{1}{2}}} \frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{7}{2}}}{(9\cos^2\alpha-8)^{\frac{1}{4}}\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{3}{2}}}. \tag{89}$$

In this case, and for the same reason as for the transverse waves, the expression for the amplitude tends to infinity at the outer end of each diverging crest; we shall find an approximation in the next section. But (89) becomes infinitely large for small values of α. From (88) we see that ϖ also becomes small, so that the approximation fails; further, we should expect the expression to become infinite near the impulse on account of its special character. We can show how the infinity disappears if we remove this cause. Consider, as an example, a finite impulse, of constant intensity over a circular area of radius d round the origin, and of zero value outside this circle. Then, as we see from (63), we shall have the same expressions as before, with a new factor given by

$$\phi(\kappa) = \int_0^\infty f(\alpha)\,J_0(\kappa\alpha)\,\alpha\,d\alpha$$

$$= C\int_0^d J_0(\kappa\alpha)\,\alpha\,d\alpha = C\,d\kappa^{-1}J_1(\kappa d).$$

Now in the final group for the diverging system we have

$$\kappa = \frac{g}{8c^2}\frac{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2}{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}}.$$

Hence the additional factor due to $\phi(\kappa)$ is proportional to

$$\frac{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}}{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^2} J_1\left\{\frac{gd\,\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}}{8c^2\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}}\right\}. \tag{90}$$

When α approaches zero, the argument of the Bessel's function increases indefinitely and we may use the asymptotic expansion: then (90) is proportional to

$$\frac{\{3\cos^2\alpha-2-\cos\alpha\,(9\cos^2\alpha-8)^{\frac{1}{2}}\}^{\frac{3}{2}}}{\{3\cos\alpha-(9\cos^2\alpha-8)^{\frac{1}{2}}\}^3}. \tag{91}$$

If now we multiply (89) by (91) we obtain a limiting value of the amplitude of the diverging system near the axis: it is proportional to

$$(2n+\tfrac{1}{4})^{-\frac{1}{2}}\{3\cos\alpha-\surd(9\cos^2\alpha-8)\}^{\frac{1}{2}},$$

and the infinity near the axis has disappeared.

(c) *The line of cusps.*—We shall consider now the infinity which occurs at the outer boundary of the two wave systems, when α is $\cos^{-1} 2\surd 2/3$. At

any point P the lines of constant phase in the two wave patterns cross at an angle ϕ, which is easily seen to be given by

$$\tan\phi = \tfrac{1}{3}\operatorname{cosec}\alpha\,(9\cos^2\alpha - 8)^{\frac{1}{2}}. \tag{92}$$

As P approaches either radial boundary the two waves ultimately have the same direction, and they will also have the same phase when they meet; consequently an abnormal elevation is to be expected along the two outer boundaries, where the two systems unite in lines of cusps. As we see from (75), the two points A, B coincide for a point P on the line of cusps; and it is on account of this fact that the previous approximations fail for both systems. We have in fact a double root of the equation for finding the chief groups of the integral (72).

Consider the integral

$$y = \int \phi(u)\sin\{f(u)\}\,du, \tag{93}$$

when u_0 is such that

$$f'(u_0) = 0; \qquad f''(u_0) = 0.$$

Following the previous method, we have

$$f(u) = f(u_0) + \tfrac{1}{6}(u-u_0)^3 f'''(u_0);$$

and provided $f'''(u_0)$ is not small, we can write the value of the group for the double root as

$$y = \left\{\frac{6}{f'''(u_0)}\right\}^{\frac{1}{3}}\int_{-\infty}^{\infty}\phi(u_0)\sin\{f(u_0)+\sigma^3\}\,d\sigma \tag{94}$$

$$= \left\{\frac{6}{f'''(u_0)}\right\}^{\frac{1}{3}}\phi(u_0)\sin f(u_0)\int_{-\infty}^{\infty}\cos\sigma^3\,d\sigma.$$

Now at the line of cusps the integral (72) becomes

$$\zeta = -\frac{g}{2^{\frac{7}{2}}\pi\rho}\int_0^{\infty}\frac{u^3\,du}{(\varpi^2 - \frac{4}{3}cu\varpi\sqrt{2}+c^2u^2)^2}\sin\frac{gu^2}{4(\varpi^2-\frac{4}{3}cu\varpi\sqrt{2}+c^2u^2)^{\frac{1}{2}}}. \tag{95}$$

And we find that $$cu_0 = \varpi\sqrt{2}$$

makes $$f'(u_0) = 0; \qquad f''(u_0) = 0;$$

$$f(u_0) = g\varpi\sqrt{3}/2c^2; \qquad f'''(u_0) = 3gc\sqrt{6}/2\varpi^2.$$

Also we have $$\int_{-\infty}^{\infty}\cos\sigma^3\,d\sigma = \tfrac{2}{3}\pi/\Gamma(\tfrac{2}{3}).$$

Hence, substituting these values, we have

$$\zeta = -\frac{3^{\frac{5}{6}}g^{\frac{2}{3}}}{2^{\frac{1}{2}}c^{\frac{10}{3}}\rho\Gamma(\frac{2}{3})\varpi^{\frac{1}{3}}}\sin\frac{g\varpi\sqrt{3}}{2c^2}. \tag{96}$$

We notice first the difference of phase of $\frac{1}{4}\pi$ between this and the expressions for the separate systems where they cut the outer boundaries; this is analogous to the change of phase along an optical ray in passing a focus. We saw that the separate transverse and diverging crests converged towards points of equal phase on the outer boundaries given by

$$\varpi = 2c^2(2n+\tfrac{5}{4})\,\pi/g\sqrt{3},$$

but with the result given in (96) we see that the actual crests on the line of cusps are given by

$$\varpi = 2c^2(2n+\tfrac{3}{2})\,\pi/g\sqrt{3}. \tag{97}$$

The amplitude of the cusped waves diminishes at a slower rate than the transverse waves, so that their size becomes relatively more marked towards the rear of the disturbance. The amplitude of successive crests is given by (96) and (97) as

$$\zeta_{mc} = \frac{3}{2^{\frac{1}{3}}\Gamma(\frac{2}{3})(2n+\frac{3}{2})^{\frac{1}{3}}\pi^{\frac{1}{3}}}\frac{g}{c^4\rho}. \tag{98}$$

The amplitude of successive crests of the transverse waves where they cut the axis are given by (82) and (84), and we find

$$\zeta_{ma} = \frac{2^{\frac{1}{2}}}{(2n+\frac{5}{4})^{\frac{1}{2}}\pi}\frac{g}{c^4\rho}. \tag{99}$$

Taking the ratio of these two quantities we have an expression for the magnitude of the crests at the cusps compared with the transverse crests on the axis; approximately

$$\frac{\zeta_{mc}}{\zeta_{ma}} = 1{\cdot}9\,\frac{(2n+\frac{5}{4})}{(2n+\frac{3}{2})^{\frac{1}{3}}}. \tag{100}$$

The following table and curve shew how the successive crests at the axis and outer line diminish, and exhibit their relative magnitudes for different values of n.*

* On August 3, 1887, Lord Kelvin delivered a lecture "On Ship Waves" before the Institution of Mechanical Engineers at Edinburgh, in which he appears to have shown a model to scale of the theoretical wave pattern produced by a ship. Only a diagram of the crest curves has been published ('Popular Lectures,' vol. 3, p. 482); the form of the crests agrees with that deduced above, except of course near the disturbance or the radial boundaries. It has, in fact, been verified that on substituting his expressions for x, y in terms of a parameter w in the present equations, the latter are satisfied identically. The law of amplitude along the waves is not stated by Lord Kelvin: as Prof. Lamb conjectures, his result seems to have been obtained by an application of the idea of group-velocity (H. Lamb, 'Hydrodynamics,' 1932 edn. pp. 406/7.)

Table II.

n.	ζ_{ma}.	ζ_{mc}.	ζ_{mc}/ζ_{ma}.
5	15	35	2·3
10	10·8	28·6	2·6
15	9	25	2·7
20	7·7	23	3
50	5	17	3·4
100	3·6	13·6	3·8

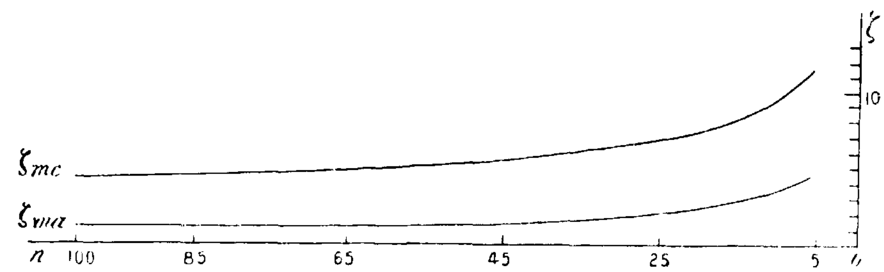


FIG. 6.

§ 14. *Point Impulse for Different Media.*

Consider a point impulse moving with uniform velocity c over the surface of a dispersive medium for which U and V are respectively the group- and wave-velocity for a value κ of $2\pi/\lambda$.

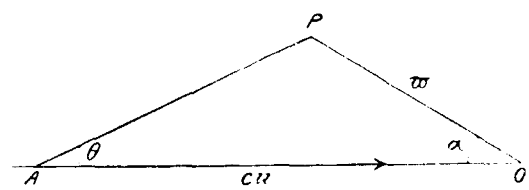


FIG. 7.

Let the disturbance from the impulse when in the neighbourhood of a point A combine so as to produce waves κ at P at the present moment when the impulse is at O. Then the problem of finding the possible persistent wave systems is contained in the equations

$$\frac{\mathrm{AP}}{\mathrm{AO}} = \frac{\mathrm{U}}{c}, \qquad c\cos\theta = \mathrm{V}; \tag{101}$$

that is, in

$$(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}/u = \mathrm{U}, \tag{102}$$

$$c\,(cu - \varpi\cos\alpha)/(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}} = \mathrm{V}. \tag{103}$$

The wave pattern depends upon the character of the positive roots of these equations for cu and κ; each such value of cu defines a wave system

with wave front through P at right angles to AP, and each system can be expressed in the form

$$\zeta = \mathrm{F}(\varpi, \alpha) \cos \{\kappa (\varpi^2 - 2cu\varpi \cos \alpha + c^2u^2)^{\frac{1}{2}} + \epsilon\},$$

with cu and κ as functions of ϖ and α.

Suppose the medium is such that the group-velocity bears a constant ratio to the wave-velocity, that is, suppose

$$\mathrm{U} = \tfrac{1}{2}(n+1)\,\mathrm{V}, \tag{104}$$

where n is independent of κ.

Then the equations (102) and (103) lead to a quadratic for cu, namely,

$$(1-n)\,c^2u^2 + (n-3)\,cu\varpi \cos \alpha + 2\varpi^2 = 0. \tag{105}$$

Hence we have the roots

$$cu = \frac{\varpi}{2(1-n)}[(3-n)\cos\alpha \pm \sqrt{\{(3-n)^2\cos^2\alpha - 8(1-n)\}}]. \tag{106}$$

We shall examine some special cases.

(*a*) $0<n<1$.—There are two positive values of cu which are real, provided

$$\cos^2\alpha > 8(1-n)/(3-n)^2.$$

Thus there are two wave systems, transverse and diverging, with a line of cusps corresponding to the double roots, and the whole wave pattern is included within an angle

$$2\cos^{-1}\{8(1-n)\}^{\frac{1}{2}}/(3-n), \tag{107}$$

which increases with n.

The previous section on deep-water waves is the case of n zero.

(*b*) $n = 1$. This is a critical case, implying coincidence of wave-velocity with group-velocity, and consequently no dispersion.

(*c*) $n = 2$. This is the case of capillary surface waves. We see that there is only one positive root of the quadratic, and it is real for all values of α; the root is

$$cu = \tfrac{1}{2}\varpi\,\{(\cos^2\alpha + 8)^{\frac{1}{2}} - \cos\alpha\}. \tag{108}$$

There is only one wave system, but it extends over the whole surface, along the line of motion κ is zero in the rear, while in advance of the impulse it is of value suitable to simple waves moving with velocity c.

(*d*) $n = 3$. This holds for flexural waves on a plate; there is one system of waves extending over the surface, corresponding to the root $cu = \varpi$.

The crests, and other lines of equal phase, are given by the curves

$$\varpi \sin^3 \tfrac{1}{2}\alpha = \text{constant}.$$

(*e*) *Gravity and capillarity combined.*—The relation between U and V is

not a constant ratio in this case; we had in §11 the expressions for the two velocities as functions of κ. It can be shown that in certain cases the equations for cu lead to four possible roots, giving four wave-branches through the point.

§ 15. *Point Impulse moving on Water of Finite Depth.*

With the same problem we have now, if the water is of depth h,

$$V = \left(\frac{g}{\kappa}\tanh \kappa h\right)^{\frac{1}{2}},$$

$$U = \tfrac{1}{2}\left(\frac{g}{\kappa}\tanh \kappa h\right)^{\frac{1}{2}}.\left(1+\frac{2\kappa h}{\sinh 2\kappa h}\right). \tag{109}$$

If we write

$$U = \tfrac{1}{2}(n+1)\,V,$$

n varies between 0 and 1, being dependent upon the value of κ. We use the notation

$$p = \frac{gh}{c^2}, \qquad m = \frac{\tanh \kappa h}{\kappa h}, \qquad n = \frac{2\kappa h}{\sinh 2\kappa h}. \tag{110}$$

Then m and n are monotonic functions of κ with the following limiting values:

$$\kappa = 0; \quad m = 1; \quad n = 1.$$
$$\kappa = \infty; \quad m = 0; \quad n = 0.$$

The two equations for cu and κ become

$$\frac{(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}}{cu} = \tfrac{1}{2}(pm)^{\frac{1}{2}}(1+n), \tag{111}$$

$$\frac{cu - \varpi\cos\alpha}{(\varpi^2 - 2cu\varpi\cos\alpha + c^2u^2)^{\frac{1}{2}}} = (pm)^{\frac{1}{2}}. \tag{112}$$

From these we obtain

$$\cos^2\alpha = \frac{\{1-\frac{1}{2}pm(1+n)\}^2}{1-\frac{1}{4}pm(1+n)(3-n)}; \tag{113}$$

$$cu = \frac{\varpi}{2(1-n)}\left[(3-n)\cos\alpha \pm \{(3-n)^2\cos^2\alpha - 8(1-n)\}^{\frac{1}{2}}\right]. \tag{114}$$

Combining the last two we have the values of cu as

$$cu = \frac{2\varpi}{1-n}\{1-\tfrac{1}{4}pm(1+n)(3-n)\}^{\frac{1}{2}}, \tag{115}$$

or

$$cu = \varpi/\{1-\tfrac{1}{4}pm(1+n)(3-n)\}^{\frac{1}{2}}. \tag{116}$$

We have two cases to consider according as $p >$ or < 1.

(*a*) $c < \sqrt{(gh)}$; $p > 1$.—From (114) we see that the equal values of cu,

defining the lines of cusps within which the wave pattern lies, are given by such values of κ that

$$\cos^2\alpha = \frac{8(1-n)}{(3-n)^2}. \tag{117}$$

Whatever the value of κ, n can only lie between 1 and 0; hence α can only lie between $\cos^{-1}2\sqrt{2}/3$ and $\pi/2$, or between 19° 28′ and 90°. The smaller value is the limiting angle for deep water, when n is considered zero for all values of κ.

We see from (115) and (116) that the equal values of cu occur when

$$1-\tfrac{1}{4}pm(1+n)(3-n) = \tfrac{1}{2}(1-n),$$

or when

$$m(3-n) = 2/p.$$

The greatest possible value of $m(3-n)$ is 2; hence we have the limitation $p>1$. Only in this case is there a double wave system with a line of cusps.

As p decreases to 1, that is as the velocity c approaches the critical value $\sqrt{(gh)}$, m and n at the line of cusps both approach their limiting value 1; and at the same time the cusp angle widens out, approaching a right angle. Further, along the axis we have

$$pm = 1, \qquad m = 1/p = c^2/gh.$$

Hence on the axis the transverse waves are the simple waves travelling with velocity c on water of depth h. As p decreases to 1, the wave-length increases indefinitely; m, and consequently n, approach unity on the axis.

Now if n is 1, the group-velocity U equals the wave-velocity V, and the medium is non-dispersive. Thus at the critical velocity c, equal to $\sqrt{(gh)}$, we have a source emitting disturbances and travelling at the rate of propagation of the disturbances; we see that the whole effect is practically concentrated into a line through the source at right angles to the direction of motion. This agrees with observations of ship waves when approaching shallow water at the critical velocity.*

(*b*) $c>\sqrt{(gh)}$; $p<1$.—We may now have the greatest value, unity, of m; it is easily seen that for less values of m and n the values of α given by (113) become smaller.

At the outer limit we have

$$\cos^2\alpha = 1-p, \qquad \sin^2\alpha = p = gh/c^2. \tag{118}$$

Consequently the wave pattern is contained within two lines making with the axis an angle which diminishes as c increases.

* 'Trans. Inst. Nav. Arch.,' vol. 47, p. 353 (1905). Compare also the motion of an electron with the velocity of radiation.

Further, since equal values of cu are given by

$$m\,(3-n) = 2/p,$$

we see that there are no cusps, for the left-hand side cannot be greater than 2.

The values of cu given in (115) and (116) correspond to the transverse and diverging waves respectively. If we substitute (116) in equations (111) and (112) we find that they are satisfied identically; hence there is always a diverging wave system. On the other hand, if we substitute (115) we find we must have

$$pm = \frac{1-pm}{1-\frac{1}{4}pm\,(1+n)\,(3-n)}, \quad \text{or} \quad m\,(2-n) = \frac{1}{p}.$$

But the greatest possible value of the left-hand side is unity.

Hence there can be a transverse wave system only so long as p is greater than 1; when c exceeds $\sqrt{(gh)}$, the transverse waves disappear.

At the outer line given by

$$\sin^2\alpha = p, \qquad m = n = 1,$$

we have, for the diverging waves,

$$cu = \varpi\,(1-p)^{-\frac{1}{2}} = \varpi \sec\alpha.$$

Hence the outer line forms a wave front of the diverging wave system. We see also that the other wave fronts (lines of equal phase) are now concave† to the axis, instead of being convex as when $p > 1$. There is no definite inner limit to the system: as the axis is approached, the wave fronts become more nearly parallel to the axis, and the wave-length diminishes indefinitely. Finally, as the velocity c is increased, the angle α diminishes, and the regular waves are contained within a narrower angle radiating from the centre of disturbance.

The following tables (III) and (IV) and the curve in fig. 8 show how the angle α varies as the velocity c is increased up to and beyond the critical velocity.

Table III.

κh at cusps.	p.	α.	$c\ \sqrt{(gh)}$.
10	7	19° 28′	0·38
8	5·4	19 28	0·42
6	4	19 29	0·5
5	3·3	19 30	0·55
4	2·7	19 37	0·6
3	2	20 18	0·7
2	1·5	23 42	0·82
1	1·18	39 19	0·92
0·5	1·08	59 27	0·96
0·2	1·01	78	0·99
0	1	90	1

$\alpha = \cos^{-1}\sqrt{8\,(1-n)\ (3-n)}$.

†See Editorial Note on page 33.

Table IV.

p.	$\alpha = \sin^{-1}\sqrt{(p)}$.	$c/\sqrt{(gh)}$.
	°	
0·99	84	1·005
0·5	45	1·41
0·33	35	1·73
0·25	30	2
0·11	19 28	3

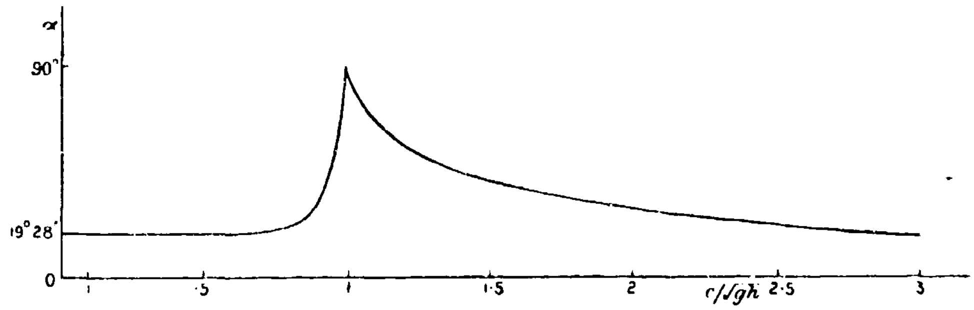
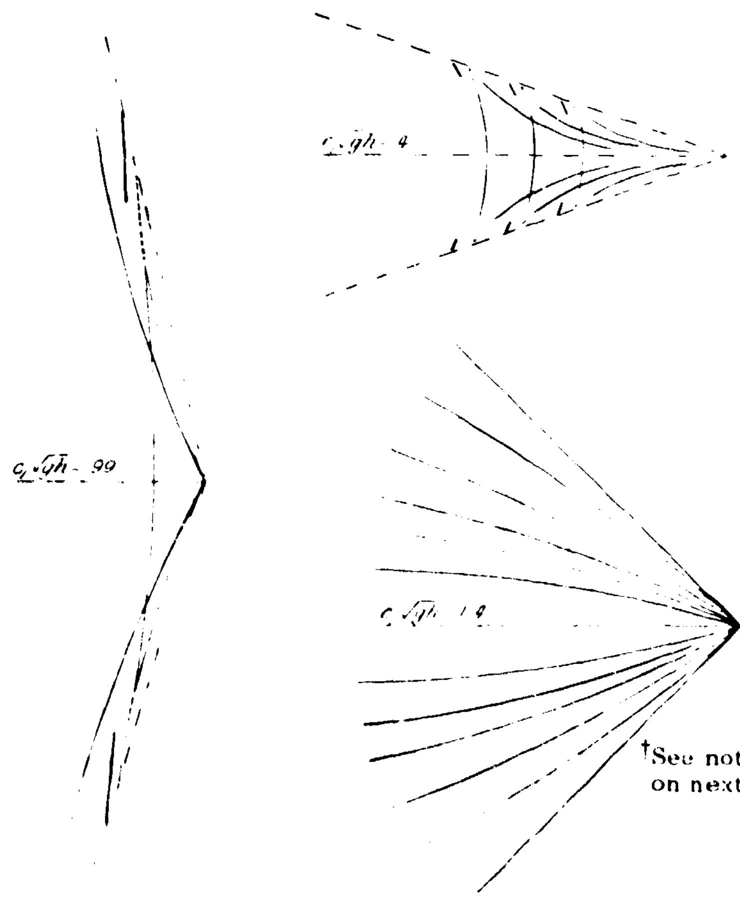


Fig. 8.



†See note by editor on next page.

Fig. 9.

With the help of these results, sketches are given in fig. 9 to represent the change in the wave pattern, as the critical velocity is approached and passed.

†Editorial Note [The shapes of these wave-fronts have been recalculated by Inui (Physico Mathematical Society of Japan, Vol. 18, pt. 2 [1936]) who does not agree that they are concave to the axis.]

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The Wave-making Resistance of Ships: a Theoretical and Practical Analysis.

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(Communicated by Prof. J. Larmor, Sec. R.S. Received April 1,—Read April 29, 1909.)

CONTENTS.

§ 1. *Introduction and Summary.*

The theoretical investigation of the total resistance to the forward motion of a ship is usually simplified by regarding it as the sum of certain independent terms such as the frictional, wave-making, and eddy-making resistances. The experimental study of frictional resistance leads to a formula of the type

$$R_s = fSV^m, \tag{1}$$

where S is the wetted surface, V the speed, f a frictional coefficient, and m an index whose value is about 1·83.

After deducting from the total resistance the frictional part calculated from a suitable formula of this kind, the remainder is called the residuary resistance. Of this the wave-making resistance is the most important part; the present paper is limited to the study of wave-making resistance, and chiefly its variation with the speed of the ship. The hydrodynamical theory as it stands at present may be stated briefly.

Simplify the problem first by having no diverging waves; that is, suppose the motion to be "in two dimensions in space," the crests and troughs being in infinite parallel lines at right angles to the direction of motion. Further, suppose that the motion was started at some remote period and has been maintained uniform. We know that, except very near to the travelling disturbance, the surface motion in the rear consists practically of simple periodic waves of length suitable to the velocity c of the disturbance. Let

a be the amplitude of the waves, and w the weight of unit volume of water; then the mean energy of the wave motion per unit area of the water surface is $\frac{1}{2}wa^2$. Imagine a fixed vertical plane in the rear of the disturbance; the space in front of this plane is gaining energy at the rate $\frac{1}{2}wa^2v$ per unit time. But on account of the fluid motion, energy is supplied through the imaginary fixed plane to the space in front, and it can be shown that the rate of supply is $\frac{1}{2}wa^2u$, where u is the group-velocity corresponding to the wave-velocity v. The nett rate of gain of energy is $\frac{1}{2}wa^2(v-u)$, and this represents the part of the power of the ship which is needed, at uniform velocity, to feed the procession of regular waves in its rear. An equivalent method of stating this argument is to regard the whole procession of regular waves from the beginning of the motion as a simple group; then the rear moves forward with velocity u while the head advances with velocity v, and the whole procession lengthens at the rate $v-u$. If we write $\mathrm{R}v$ for the rate at which energy must be supplied by the ship, we call R the wave-making resistance, and we have

$$\mathrm{R} = \tfrac{1}{2}wa^2(v-u)/v. \tag{2}$$

We notice that R is the wave-making resistance in *uniform* motion; it is only different from zero because u differs from v, that is, because the velocity of propagation depends upon the wave-length.

In deep water, u is $\frac{1}{2}v$, so that R is $\frac{1}{4}wa^2$. In the application of this to a ship at sea, it is assumed that the transverse waves have a certain average uniform breadth and height, and, further, that the diverging waves may be considered separately and as having crests of uniform height inclined at a certain angle to the line of motion; if the amplitude is taken to vary as the square of the velocity, it follows that R varies as v^4. Several formulæ of the type $\mathrm{R} = \mathrm{A}v^4$, or $\mathrm{R} = \mathrm{A}v^4+\mathrm{B}v^6$, have been proposed; although these may be of use practically by embodying the results of sets of experiments, they are not successful from a theoretical point of view. Recently many such cases have been analysed graphically by Prof. Hovgaard;* the general result is that a fair agreement may be made for lower velocities with an average experimental curve neglecting the humps and hollows due to the interference of bow and stern wave systems, but at higher velocities the experimental curve falls away very considerably from the empirical curve.

The method used here consists in considering the ship, in regard to its wave-making properties, as equivalent to a transverse linear pressure distribution travelling uniformly over the surface of the water. Taking a simple form of diffused pressure system and making some necessary

* W. Hovgaard, 'Inst. Nav. Arch. Trans.,' vol. 50, p. 205, 1908.

35

assumptions, we obtain an expression for the amplitude of the transverse waves thus originated, and for the resistance R, in which the velocity enters in the form e^{-a/v^2}; this function is seen to have the general character of the experimental curves. Adding on a similar term for the waves diverging from bow and stern, and, finally, in the manner of W. Froude, an oscillating factor for the interference of these bow and stern waves, we find a formula for the wave-making resistance of the type

$$R = \alpha e^{-l/v^2} + \beta \{1 - \gamma \cos (m/v^2)\} e^{-n/v^2}.$$

In this expression there are six adjustable constants; we proceed to reduce the number of these after transforming into units which utilise Froude's law of comparison. We use the quantity c, defined as

(speed in knots)/$\surd$(length of ship in feet),

and we express the resistance in lbs. per ton displacement of the ship. An inspection of experimental curves, and other considerations suggest that the quantities l, m, n may be treated as universal constants; with this assumption, a three-constant formula is obtained, viz.,

$$R = \alpha e^{-2 \cdot 53/9c^2} + \beta \{1 - \gamma \cos (10 \cdot 2/c^2)\} e^{-2 \cdot 53/c^2}, \tag{3}$$

where the constants α, β, γ depend upon the form of the ship.

We then treat (3) as a semi-empirical formula of which the form has been suggested by the preceding theoretical considerations; several experimental model curves are examined, and numerical calculations are given which show that these can be expressed very well by a formula of the above type.

Since the constant α is found to be small compared with β, it is not allowable to press too closely the theoretical interpretation of the first term, especially as the experimental curves include certain small elements in addition to wave-making resistance. If we limit the comparison to values of c from about 0·9 upwards, it is possible to fit the curves with an alternative formula of the type

$$R = \beta \{1 - \gamma \cos (10 \cdot 2/c^2)\} e^{-n/c^2},$$

and some examples of this are given.

The effect of finite depth of water is considered, and a modification of the formula is obtained to express this effect as far as possible. Starting from an experimental curve for deep water, curves are drawn, from the formula, for the transverse wave resistance of the same model with different depths; although certain simplifications have to be made, the curves show the character of the effect, and allow an estimate of the stage at which it becomes appreciable.

In the last section the question of other types of pressure distribution is

discussed, and one is given in illustration of the wave-making resistance of an entirely submerged vessel.

§2. *Pressure System travelling over Deep Water.*

It is known that a line pressure-disturbance travelling over the surface of water with uniform velocity c at right angles to its length gives rise to a regular wave-train in its rear of equal wave-velocity.* Take the axis of x in the direction of motion and let the pressure system be symmetrical with respect to the origin and given by $p = f(x)$; suppose that $f(x)$ vanishes for all but small values of x, for which it becomes infinite so that $\int_{-\infty}^{\infty} f(x)\, dx = \mathrm{P}$. The regular part of the surface depression η due to this integral pressure P practically concentrated on a line is given by

$$\eta = \frac{2g\mathrm{P}}{wc^2} \sin \frac{gx}{c^2}. \tag{4}$$

The part of the surface effect which is neglected in this expression consists of a local disturbance symmetrical with respect to the origin and practically confined to its neighbourhood.

If we suppose P constant, the amplitude in the regular wave-train and the consequent drain of energy due to its maintenance diminish with the velocity.

To obtain results in any way comparable with practical conditions it is necessary to suppose the pressure system diffused over a strip which is not infinitely narrow.

An illustration is afforded by taking

$$p = f(x) = \frac{\mathrm{P}}{\pi} \frac{\alpha}{\alpha^2 + x^2}, \tag{5}$$

where α is small compared with the distances at which the regular surface effects are estimated. This type of pressure distribution is shown in fig. 1.

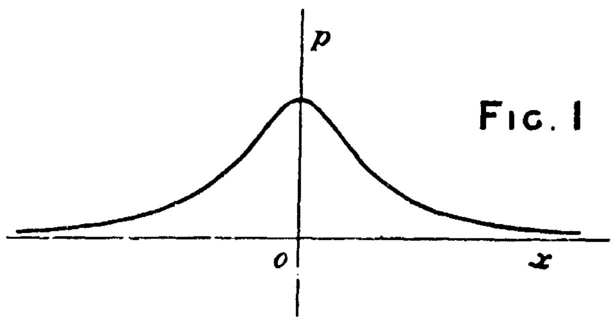


FIG. 1

* For a discussion of the wave pattern, see Lamb, 'Hydrodynamics,' § 241 *et seq.*; or Havelock, 'Roy. Soc. Proc.,' A, vol. 81, p. 398, 1908.

The effect of thus diffusing the pressure system is expressed by the introduction of a factor $\phi(\kappa)$ into the amplitude of the regular waves, where $2\pi/\kappa$ is the wave-length and

$$\phi(\kappa) = \int_{-\infty}^{\infty} f(\omega)\cos\kappa\omega\, d\omega. \tag{6}$$

Using (5) in (6), we find

$$\phi(\kappa) = \mathrm{P}e^{-\alpha\kappa} = \mathrm{P}e^{-\alpha g/v^2}.$$

Hence the amplitude of the waves is given by

$$a = \frac{2g\mathrm{P}}{wv^2}e^{-\alpha g/v^2}. \tag{7}$$

Further, since $\kappa = v^2/g$, the group velocity $u = d(\kappa v)/d\kappa = \frac{1}{2}v$. Hence the wave-making resistance R is given by

$$\mathrm{R} = \frac{g^2\mathrm{P}^2}{wv^4}e^{-2\alpha g/v^2}. \tag{8}$$

We have to examine the variation of these quantities with the velocity v under the supposition that the pressure system is due to the motion of a body either floating on the surface or wholly immersed in the water. The pressures concerned being the vertical components of the excess or defect due to the motion, it seems possible to assume as a first approximation that P varies as v^2; this is the case in the ordinary hydrodynamical theory of a solid in an infinite perfect fluid, and a similar assumption is also made in the theory of Froude's law of comparison. This being assumed, we find

$$a = \mathrm{A}e^{-\alpha g/v^2}, \qquad \mathrm{R} = \mathrm{B}e^{-2\alpha g/v^2}. \tag{9}$$

We see that both the amplitude and the resistance increase steadily from zero up to limiting values.

If we draw the curve representing this relation between R and v, there is a point of inflection when

$$\frac{d^2\mathrm{R}}{dv^2} = 0, \qquad \text{or} \qquad v^2 = \tfrac{4}{3}g\alpha. \tag{10}$$

Writing v' for this velocity, we see that $d\mathrm{R}/dv$ increases as the velocity rises to v' and then falls off in value as the velocity is further increased.

We can write the relation now in the form

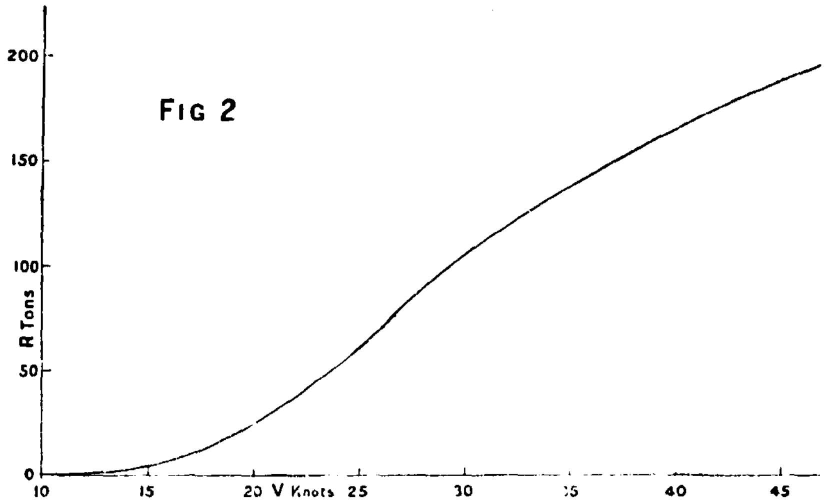
$$\mathrm{R} = \mathrm{B}e^{-\frac{3}{2}(v'/v)^2}. \tag{11}$$

The character of this relation is shown by the curve in fig. 2, which represents the case

$$\mathrm{R} = 315e^{-\frac{3}{2}(26/\mathrm{V})^2}, \tag{12}$$

R being in tons, and V in knots.

The values of the constants in (12) have been chosen for comparison with an experimental curve of residuary resistance given by R. E. Froude;* it was obtained from model experiments and by means of the law of



corresponding speeds and dimensions the results were given for a ship (model A) of 4090 tons displacement and 400 feet length. The actual curve is given in fig. 4 and is discussed more fully later; we neglect for the present the undulations which are known to be due to the interference of the bow and stern wave systems, and we consider a fairly drawn mean experimental curve denoted by R′. Table I shows a comparison of the values of R′ with those of R calculated from the formula (12).

Table I.

V.	R.	R′.
10	0·02	1·8
14	2	4
18	14	16
22	38	39·5
26	70	70
30	106	107
34	132	136
38	157	156
42	176	175
46	195	192

From this comparison we see that the point of inflection given by V′ corresponds to the point at which the slope of the mean experimental curve

* R. E. Froude, 'Inst. Nav. Arch. Trans.,' vol. 22, p. 220, 1881.

begins to fall off. This effect is general in residuary resistance curves; we see that it is really an interference effect, the character of the curve being due to the mutual interference of the wave-making elements of the pressure system. Superposed on the mean curve we have a further interference effect due to the combination of two systems, the bow and stern systems.

From Table I we infer that the mean curve agrees well with the calculated values R from about 18 knots upwards, but at the lower speeds the values of R are much too small; this suggests the addition of a term to represent the effect of the diverging waves.

§ 3. *Diverging Wave System.*

In the example considered above, the calculated values of R are much too small at the lower velocities. This might have been expected; for we obtained (12) by the consideration of line-waves on the surface, that is waves with crests of uniform height along parallel infinite lines. But the model experiments correspond more to a point disturbance travelling over the surface, with the formation of diverging waves as well as transverse waves. In fact, W. Froude* infers from his experimental curves that the residuary resistance at the lower velocities is chiefly due to the diverging wave system, on account of the absence of undulations; for the latter signify interference of the transverse systems initiated by the bow and stern, and these become very important at the higher velocities.

We have to add to (12) a term representing the diverging waves; the comparison in Table I suggests for this a term of the same type, $e^{-\frac{1}{2}(V''/V)^2}$, with V″ much smaller than the corresponding velocity V′ for the transverse waves. With the data at our disposal we might then determine the various constants so as to obtain the closest fit possible; however, we can make the process appear less artificial by the following considerations. We know that the wave pattern produced by a travelling point source consists of a system of transverse waves and a system of diverging waves, the whole pattern being contained with two radial lines making angles of about 19° 28′ with the direction of motion; a fuller investigation of the effects produced by a diffused source must be left over at present. In applying energy considerations as in the previous sections, the usual method is to suppose that the transverse waves form on the average a regular wave-train of uniform amplitude and uniform breadth; using the same approximation for the diverging waves we suppose that these form on the average a regular wave-train on each side, with the crests inclined at some angle θ to the direction

* W. Froude, 'Inst. Nav. Arch. Trans.,' vol. 18, p. 86, 1877.

of motion of the disturbance. Then the velocity of the diverging wave-trains normally to their crests is $V \sin \theta$. Now the same features of the ship are responsible for the character of both transverse and diverging waves; then if V' is the velocity at which there is a point of inflection in the resistance curve for the transverse waves, the suggestion is that $V' \sin \theta$ is the corresponding velocity for the diverging waves. Taking as a first approximation the angle given above, viz., 19° 28′ or $\sin^{-1}\frac{1}{3}$, we test now a formula of the type

$$R = Ae^{-\frac{3}{4}(V'/3V)^2} + Be^{-\frac{3}{4}(V'/V)^2}. \tag{13}$$

For the particular example already used (Froude, Ship A) we take V' equal to 26 knots, and determine A, B from two values of V. We obtain thus

$$R = 4{\cdot}5e^{-\frac{3}{4}(26/3V)^2} + 297e^{-\frac{3}{4}(26/V)^2}. \tag{14}$$

With this formula we find as good an agreement as before at the higher velocities, and we have now at lower velocities the comparison in Table II:—

Table II.

V.	R.	R′.
10	1·6	1·8
14	4·1	4
18	16·5	16
22	40	39·5

In calculating from (14) we find that the two terms both increase continually; at low velocities the second term is practically negligible, then at about 15 knots the two terms are of equal value, and after that the transverse wave term becomes all important.

It must be remembered that the experimental curve was obtained from tank experiments, and it is possible that the width of the tank may have an effect on the relative values of the transverse and diverging waves. It would be of interest if experiments were possible with the same model in tanks of different widths; if the methods used in obtaining (14) form a legitimate approximation, the effect might be shown in the relative proportions of the two terms—provided always that one can make a suitable deduction first for the frictional resistance, and can then separate out the relatively small effects of the diverging waves, the eddy-making and other similar elements.

§ 4. *Interference of Bow and Stern Wave-trains.*

The cause of the undulations in the resistance curves was shown by W. Froude to be interference of the wave system produced by the bow (or entrance) with that arising at the stern (or run). His experiments on the effect of introducing a parallel middle body between entrance and run confirmed his theory, which may be stated briefly. Let the wave-making features of the bow produce transverse waves which would have at a breadth b an amplitude a; owing to the spreading out of the transverse waves they will be equivalent to simple waves at the stern of smaller amplitude ka, at the same breadth b. Let a' be the amplitude there of the waves produced by the stern. Then in the rear of the ship we suppose there are simple waves of amplitude ka superposed upon others of equal wave-length of amplitude a'. At certain velocities the crests of the two systems coincide in position, giving rise to a hump on the resistance curve; and at intermediate velocities there are hollows on the curve owing to the crests of one system coinciding with the troughs of the other.

In developing a form for the resistance, subsequent writers have generally taken R proportional to an expression of the form $a^2+a'^2+2kaa' \cos(mg\mathrm{L}/v^2)$, where L is the length of the ship. This means that the bow is supposed to initiate a system of waves with a first crest at a short distance behind the bow, and that similarly the stern waves have their first crest shortly after the stern; the length mL is the distance between these two crests, and is called the wave-making length of the ship. The determination of a value for m appears to be doubtful, but from interference effects it is said to vary for different ships between the values 1 and 1·2.

It has seemed desirable here to follow more closely the point of view in W. Froude's original paper already quoted.* We regard the entrance of the ship as forming transverse waves with their first crest shortly aft of the bow, and the run of the ship as forming waves with their first trough in the vicinity of the middle of the run. It is suggested that this distance between first crest and first trough, in practice found to be about 0·9L, should be taken as the "wave-making distance"; the cosine term in the formula is then prefixed by a minus sign instead of a positive sign. We return to this point later; we first work out a definite simple illustration in "two-dimensional waves," and then build up a more complete formula for comparison with experiment. With the same notation as in § 1, let the pressure system be given by

$$p=f(x)=\frac{1}{\pi}\left\{\frac{\mathrm{P}_1\alpha^2}{\alpha^2+(x-\frac{1}{2}l)^2}-\frac{\mathrm{P}_2\alpha^2}{\alpha^2+(x+\frac{1}{2}l)^2}\right\}. \qquad (15)$$

* W. Froude, *loc. cit. ante*, p. 83.

This indicates two pressure systems, one of excess and the other of defect of pressure; each distribution is of the type already used, and their centres are separated by a distance l. Fig. 3 shows the character of the disturbance.

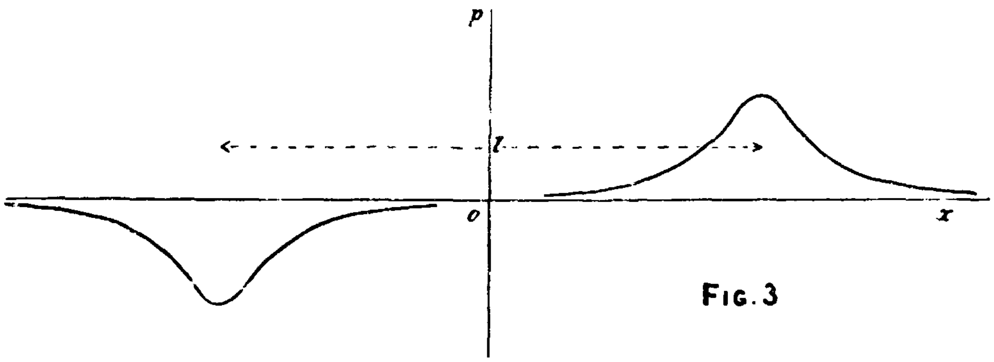


FIG. 3

In the rear of the whole disturbance there is interference between the regular wave-trains due to the two parts. With the same methods as before we find that the resulting waves are given by

$$\eta = \frac{2gP_1}{wv^2}e^{-ag/v^2}\sin\frac{g(x-\frac{1}{2}l)}{v^2} - \frac{2gP_2}{wv^2}e^{-ag/v^2}\sin\frac{g(x+\frac{1}{2}l)}{v^2}$$

$$= \frac{2g}{wv^2}e^{-ag/v^2}\left\{(P_1-P_2)\cos\frac{gl}{2v^2}\sin\frac{gx}{v^2}-(P_1+P_2)\sin\frac{gl}{2v^2}\cos\frac{gx}{v^2}\right\}. \quad (16)$$

Hence the average energy per unit area is proportional to

$$v^{-4}e^{-2ag/v^2}\{P_1^2+P_2^2-2P_1P_2\cos(gl/v^2)\}.$$

Now, assuming as before that P_1 and P_2 vary as v^2, we find that as regards variation with the velocity the effective resistance R, which is the expression of the energy required to feed the wave-trains, is given in the form

$$R = \{A^2+B^2-2AB\cos(gl/v^2)\}e^{-2ag/v^2}. \quad (17)$$

A more general expression might have been obtained by taking two quantities α_1 and α_2 in (15), corresponding to some difference in wave-making properties of entrance and run; this would have led to different exponential factors being attached to the bow and stern waves. However, we find (17), with a common exponential factor, sufficiently adjustable for present purposes.

In Froude's experiments in 1877 the effect of inserting different lengths of parallel middle body between the same entrance and run was examined; it was found that a hump in the residuary resistance curve corresponded to a trough of the bow waves being in the vicinity of the middle of the run and a hollow to a crest being in that position.

For the model, Ship A, we have: Length = L = 400 feet; entrance = run = 80 feet.

Hence, in this case we may take, in formula (17), l as approximately 360 feet. We notice that this gives $l = 0{\cdot}9\mathrm{L}$; and in subsequent comparisons, instead of leaving l to be adjusted to fit the experimental curve, we find there is sufficient agreement if we fix it beforehand as 0·9 of the length of the ship on the water-line.

Compare, now, the length l with the ordinary "wave-making length" of the ship; the latter is written as $m\mathrm{L}$ and is defined as the distance between the first regular bow crest and the first regular stern crest. From the present point of view (17) gives

$$m\mathrm{L} = l + \tfrac{1}{2}\lambda \quad \text{or} \quad m = 0{\cdot}9 + \tfrac{1}{2}\lambda/\mathrm{L}, \tag{18}$$

where λ is the wave-length in feet of deep-sea waves of velocity v ft./sec.

Calculating from this formula for Ship A, and writing V for velocity in knots (6080 feet per hour), we obtain Table III.

We see that the statement that m lies between 1 and about 1·2 would hold for this ship if it were measured for ordinary speeds between about 14 and 22 knots.

Table III.

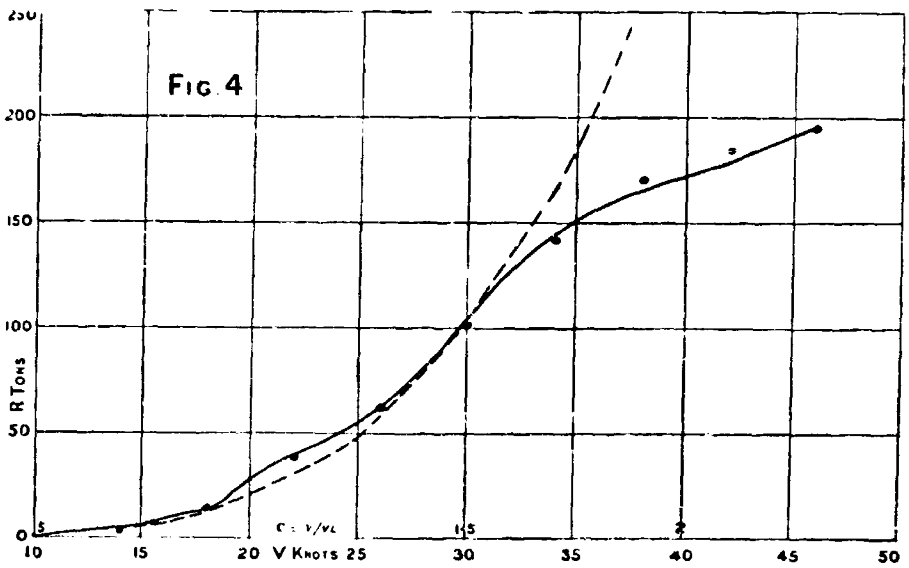
V.	λ.	m.
10	55·5	0·97
14	110	1·03
18	180	1·12
22	270	1·24
26	362	1·35
30	500	1·5

We proceed now to modify (14) by introducing into the second term a factor $1 - \gamma \cos(gl/v^2)$. With $l = 360$, we find gl/v^2 is approximately $4080/\mathrm{V}^2$, with V in knots; further, from one value from the experimental curve we obtain $\gamma = 0{\cdot}12$. Thus for Ship A we have R in tons given by

$$\mathrm{R} = 4{\cdot}5e^{-\frac{1}{2}(26/3\mathrm{V})^2} + 297\,\{1 - 0{\cdot}12\cos(4080/\mathrm{V}^2)\}\,e^{-\frac{1}{2}(26/\mathrm{V})^2}. \tag{19}$$

Table IV shows some calculated values for R, and these are represented in fig. 4 by dots; the continuous curve is the experimental residuary resistance curve given by Froude, that is, the total resistance less the calculated frictional part.

It is the custom to give the results of model experiments in the form of a fair curve, so that the positions of actual readings and the possible



error are not known. The interrupted curve is a curve $R = AV^4$ sketched in for comparison.

Table IV.

V.	R.	V.	R.
10	1·5	30	102
14	4·2	34	142
18	15	38	171
22	44	42	185
26	62	46	195

§ 5. *Comparison with Experimental Results.*

Before examining further model curves we must express the previous formula in a form more suitable for calculation; we use the system of units in which model results are now generally expressed. R is given in lbs. per ton displacement of the ship, while instead of the speed V we use the ratio $V/\sqrt{L}$, V being in knots and L in feet; this is called the speed-length ratio, and we shall denote it by c. The advantage of these units is that they utilise Froude's law of comparison; from the experimental curve between R and c we can write down at once the residuary resistance for a ship of any length and displacement at the corresponding velocity, provided the ship has the same lines and form as the model. Thus the constants which are left in the relation between R and c depend only upon the lines of the model, not upon its absolute size. At present we make no attempt to connect these constants with the form of the model, as expressed by the usual coefficients

of fineness or the curve of sectional areas, or in other ways; we are concerned with the form of R as a function of c, and the constants are chosen in each case to make the best fit possible.

First, as regards the exponential factor, we had $e^{-\frac{1}{3}(V'/V)^2}$, with V′ giving a point of inflection on the resistance curve; in the case of Ship A we had $V' = 26$, $L = 400$, so that $c' = 1·3$. Now, it is just about this value of c that there is a falling off in most experimental curves, so that we try first $c' = 1·3$ for the point of inflection on the R, c curve. Then the exponential factor becomes $e^{-\frac{1}{3}(c'/c)^2}$, or $e^{-2·53/c^2}$.

Secondly, as regards the cosine term which gives the undulations, we had $\cos(gl/v^2)$; we have decided to put $l = 0·9L$, so that we have

$$\frac{gl}{v^2} = 0·9gL \bigg/ \left(\frac{6080}{3600}V\right)^2 = \frac{10·2}{c^2}, \text{ approximately.}$$

Hence the previous relation for R reduces to the following general form:

$$R = \alpha e^{-2·53/9c^2} + \beta(1 - \gamma \cos 10·2/c^2)\, e^{-2·53/c^2}, \tag{20}$$

where R is in lbs. per ton displacement, and α, β, γ depend upon the form of the model.

There are humps on the curve when $10·2c^{-2}$ is an odd multiple of π, hollows when it is an even multiple, and mean values when it is an odd multiple of $\frac{1}{2}\pi$. For facilitating calculation, some of these positions are given in Table V; and, for the same reason, values of the exponentials and the cosine factor are given in Table VI.

Table V.

Humps	—	—	1·8	—	—	—	1·04	-	—	—	0·8		
Means...	---	2·54	—	1·47	—	1·13	--	0·96	—	0·85	—	0·76	
Hollows	∞	—	—	—	1·27	—	—	—	0·9	—	—	—	0·73

Values of c.

Table VI.

c.	$e^{-2·53/9c^2}$.	$e^{-2·53/c^2}$.	$\cos(10·2/c^2)$.
0·6	0·460	0·0009	+0·75
0·8	0·644	0·019	−0·97
1·0	0·756	0·080	−0·71
1·2	0·821	0·172	+0·70
1·4	0·866	0·275	+0·47
1·6	0·896	0·372	−0·65
1·8	0·916	0·458	−1·0
2·0	0·932	0·532	−0·83
2·2	0·943	0·592	−0·51
2·4	0·951	0·644	−0·20
3	0·970	0·756	+0·43

We examine, now, some examples of experimental curves, comparing them with the formula (20); several of the curves and other data, in particular for II, III, and V, have been taken from the collection in Prof. Hovgaard's paper already referred to, in which he essays to fit formulæ involving V^4 or V^6 with the experimental curves.

I. *R. E. Froude*, 1881, *Ship A*.

Displacement = 4090 tons; length = 400 feet; cylindrical coefficient = 0·694.

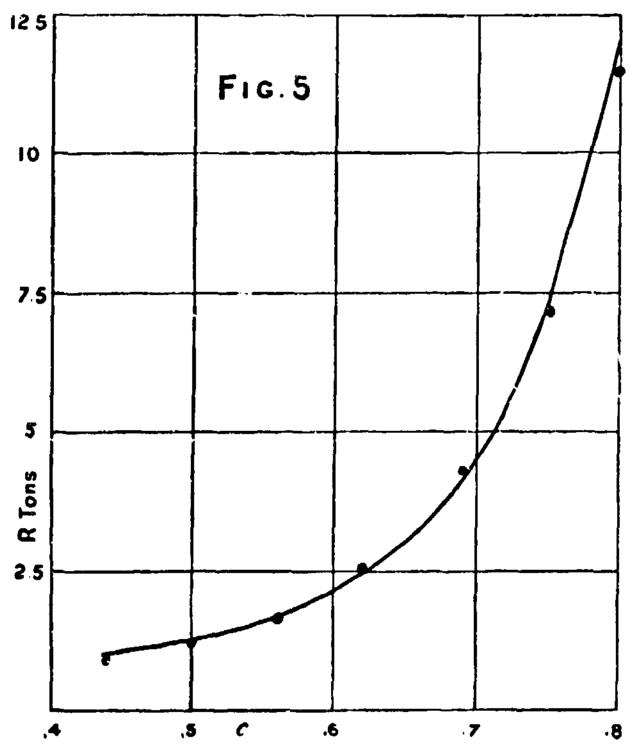
This is the case we have examined in the previous sections, so that we have only to change the numerical factors in (19) to cause R to be given in lbs. per ton displacement. We find the result is formula (20) with

$$\alpha = 2{\cdot}46; \quad \beta = 162{\cdot}6; \quad \gamma = 0{\cdot}12.$$

II. *W. Froude*, 1877.

Displacement = 3804 tons; length = 340 feet; cylindrical coefficient = 0·787.

The last two data include the cylindrical middle body. The curve is given in fig. 5; it was constructed by Hovgaard from the data of Froude's



experiments, and these were such that it was possible to make a mean residuary resistance curve, the effects of bow and stern interference being eliminated. The curve is given as total residuary resistance in tons on a base of V in knots. If we work in lbs. per ton, we find there is a very fair agreement with formula (20) if we take

$$\alpha = 2{\cdot}24; \quad \beta = 279{\cdot}7; \quad \gamma = 0.$$

Probably a closer agreement could be obtained by further slight adjustment of α and β. Fig. 5 shows a comparison of values of the total residuary resistance for the ship (in tons); the calculated values are indicated by small circles.

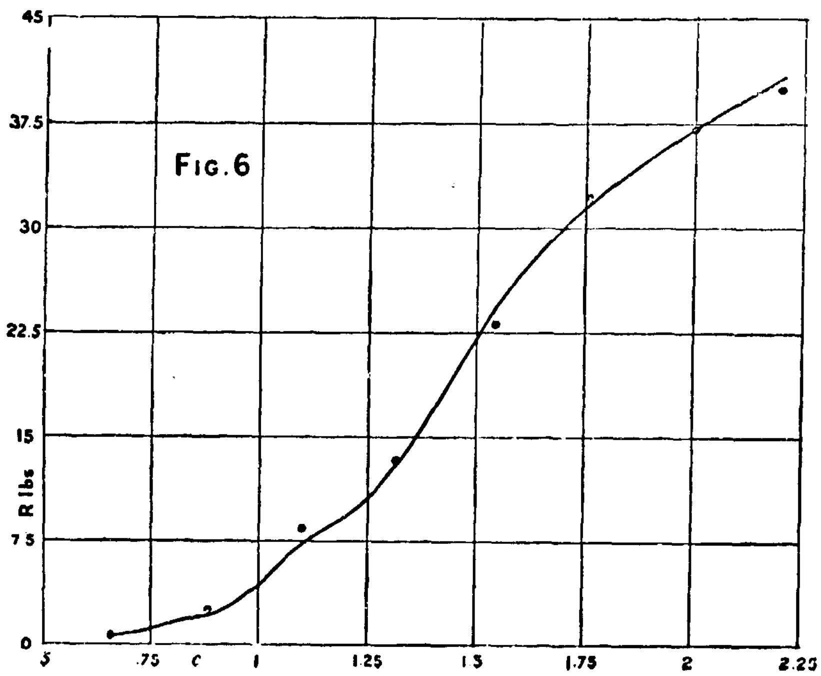
III. *D. W. Taylor*, 1000 *lbs. Model.*

Length on water line = 20·51 feet; cyl. coeff. = 0·680.

The experimental curve in this case is given as residuary resistance for the model in lbs. on a base of V in knots. With the same notation as before we find

$$\alpha = 2; \quad \beta = 136{\cdot}6; \quad \gamma = 0{\cdot}14.$$

Putting these values in (20), we can calculate R in lbs. per ton, and hence R_1 in lbs. for the model; fig. 6 shows the comparison between R_1 and the corresponding values on the curve; the calculated values R_1 are indicated by dots.



IV. *D. W. Taylor, Model No.* 892.*

Displacement = 500 lbs.; length on water line = 20·512 feet; longitudinal coeff. = 0·68; midship section coeff. = 0·70.

In this case the experimental curve is given as lbs. per ton displacement (R′) on a base of speed-length ratio (c). In the same manner as before, fig. 7 shows the comparison with the formula (20) when we take

$$\alpha = 2; \quad \beta = 82{\cdot}5; \quad \gamma = 0{\cdot}14.$$

Since the constant α is small compared with β, one is not able to lay much stress on the meaning of the first term. For as the velocity functions

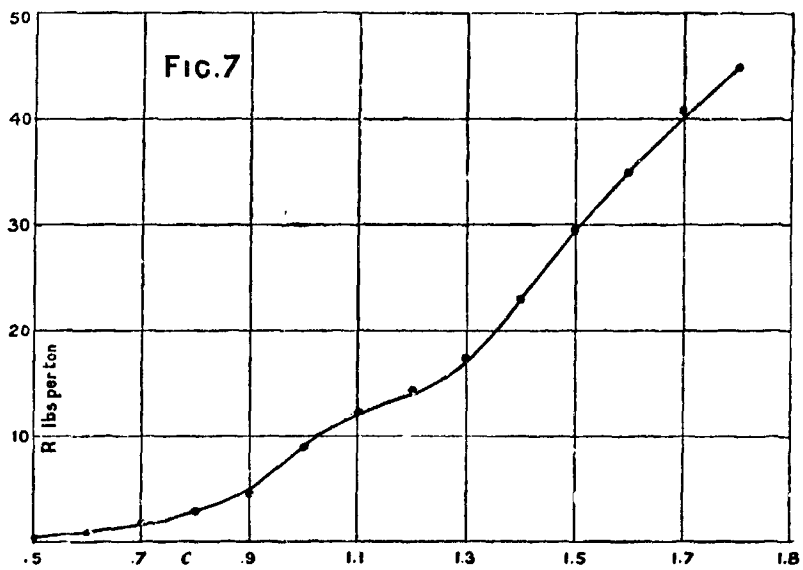


FIG. 7

are of a suitable type, the constants possess considerable elasticity as regards fitting an experimental curve. For instance, if we omit values of c below about 0·9, it is possible to represent the previous curves fairly well by a formula

$$R = \beta\,\{1 - \gamma \cos(10{\cdot}2/c^2)\}\, e^{-\frac{1}{4}(c'/c)^2}.$$

In the previous examples we took the value 1·3 for c'. In Case IV above we find now the values

$$\beta = 87; \quad \gamma = 0{\cdot}14; \quad c' = 1{\cdot}3.$$

For a similar curve taken from the same paper, viz., Model No. 891, displacement 1000 lbs., we find a good correspondence, except for slightly higher values near $c = 1{\cdot}1$, with the values

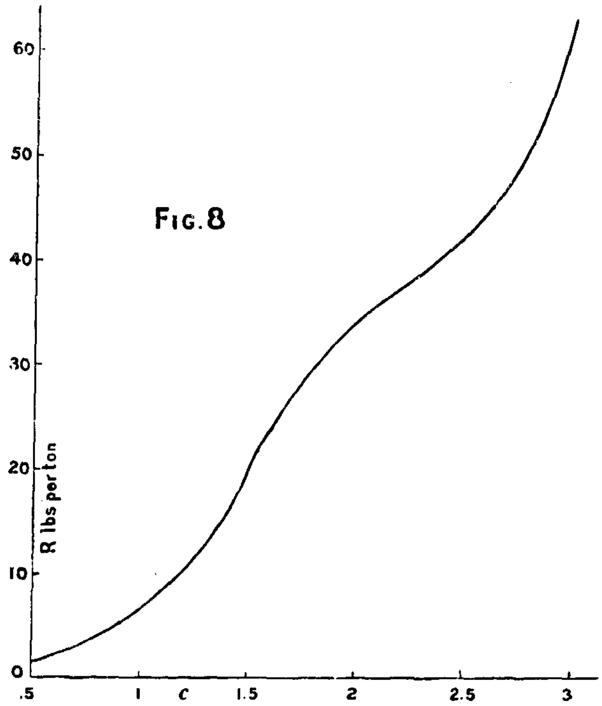
$$\beta = 174; \quad \gamma = 0{\cdot}14; \quad c' = 1{\cdot}4.$$

*D. W. Taylor, Trans. S.N.A.M.E., vol. 16, p. 13 (1908).

V. *I. I. Yates, Destroyer Model C.**

Displacement = 575 lbs.; length = 20 feet; cyl. coeff. = 0·529.

The experimental curve is given in lbs. for the model on a base of V in knots, and is a total resistance curve, that is, it includes the frictional resistance. The curve is reproduced in fig. 8.



This curve is not analysed here so as to compare the residuary resistance with the formula (20), but it is included in order to draw attention to certain possible complications. It may be noticed that the curve is carried to a high value of the speed-length ratio c, and that it continues to rise more rapidly after about $c = 2{\cdot}5$ than might be expected on the present theory. Now in the first place it is possible that the frictional resistance may account partly for this rise. The ordinary estimation of the frictional resistance assumes that it can be calculated separately from some expression like $fSV^{1\cdot85}$; now the legitimacy of this is beyond doubt in all ordinary cases, but at high speeds it is possible that the form of the expression may change, or even that it may not be a fair simplification to divide the total resistance into simple additive components.

In the second place a more important consideration must be taken into account, and that is the depth of the tank. For the experiments now under

* I. I. Yates, Thesis, 1907, Mass. Inst. Tech. U.S.A. See Hovgaard, *loc. cit. ante.*

consideration the depth of water in the tank is not known. The deepest experimental tank appears to be the U.S. Government tank at Washington, which has a maximum depth of about 14·7 feet. Now in that tank, with a 20-foot model, there would be a "critical" condition near the value $c = 2{\cdot}9$; before and up to that point the residuary resistance curve would rise sharply and abnormally. This effect is discussed more fully in the next section, and curves are given in fig. 11, with which fig. 8 may be compared. It appears, then, as far as one is able to judge, that it is possible the resistance curve in fig. 8 is complicated by the effect of finite depth of the tank.

§ 6. *The Effect of Shallow Water.*

We saw in the first section that the wave-making resistance R can be written in the form

$$\mathrm{R} = \tfrac{1}{2} w a^2 (v - u)/v,$$

where u is the group-velocity corresponding to wave-velocity v. For deep water $u = \frac{1}{2}v$, and the formulæ are comparatively simple. But for water of finite depth h the relation between u and v depends upon the wave-length $(2\pi/\kappa)$. We have

$$v = \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)},$$

$$u = \frac{d}{d\kappa}(\kappa v) = \tfrac{1}{2} v\,(1 + 2\kappa h/\sinh 2\kappa h).$$

Consequently we find

$$\mathrm{R} = \tfrac{1}{4} w a^2 \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right). \tag{21}$$

As v increases from zero to $\sqrt{(gh)}$, R diminishes from $\frac{1}{4}wa^2$ to 0, provided the amplitude remains constant. But as Prof. Lamb remarks,* the amplitude due to a disturbance of given character will also vary with the velocity. It is the variation of this factor that we have to examine in the manner used in the previous sections for deep water.

If a symmetrical line-pressure system $\mathrm{F}(x)$, suitable for Fourier analysis, is moving uniformly with velocity v over the surface of water, the surface disturbance η is given by

$$\pi w \eta = \tfrac{1}{2}\int_0^\infty dt \int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa\, \{x + (v - \mathrm{V})\, t\}\, d\kappa$$

$$- \tfrac{1}{2}\int_0^\infty dt \int_0^\infty \kappa \mathrm{V} \phi(\kappa) \sin \kappa\, \{x + (v + \mathrm{V})\, t\}\, d\kappa, \tag{22}$$

where $\phi(\kappa) = \int_{-\infty}^{\infty} \mathrm{F}(\omega) \cos \kappa\omega\, d\omega$.

*H. Lamb, 'Hydrodynamics,' (1932 edn. p. 415).

The method of evaluating these integrals approximately so as to give the regular wave-trains has been discussed in a previous paper and it is followed now in the case of finite depth.* We take, under certain limitations, the value of an integral such as

$$y = \int \phi(u) \sin \{g(u)\}\, du$$

to be the value of its principal group, viz.,

$$y_0 = \left\{\frac{2\pi}{g''(u_0)}\right\}^{\frac{1}{2}} \phi(u_0) \cos\{g(u_0) - \tfrac{1}{4}\pi\}, \tag{22A}$$

where u_0 is such that $g'(u_0) = 0$.

Now in the integrals in (22) we have to find successively two principal groups, first with regard to κ and then in the variable t; and thus we may evaluate the amplitude factor in the resulting regular wave-trains.

For water of depth h we may write

$$f(\kappa) = v - \mathrm{V} = v - \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)}.$$

The group with respect to κ gives a term proportional to

$$\cos\{t\kappa^2 f'(\kappa) + \tfrac{1}{4}\pi\},$$

where κ has the value given by

$$f(\kappa) + \kappa f'(\kappa) = -\frac{x}{t}. \tag{23}$$

From (22A), this introduces into the amplitude a factor

$$1/\sqrt{[t\{2f'(\kappa) + \kappa f''(\kappa)\}]}. \tag{24}$$

Further, the group with respect to t occurs for

$$\frac{d}{dt}\{t\kappa^2 f'(\kappa)\} = 0 \quad \text{or} \quad f(\kappa) = 0.$$

Also we have in these circumstances

$$\frac{d^2}{dt^2}\{t\kappa^2 f'(\kappa)\} = \frac{d}{dt}\left\{\kappa^2 f'(\kappa) + \frac{\kappa x}{t}\right\} = \frac{d}{dt}\{-\kappa f(\kappa)\}$$

$$= -\frac{x}{t^2}\frac{f+\kappa f'}{2f'+\kappa f''} = \frac{1}{t}\frac{(f+\kappa f')^2}{2f'+\kappa f''} = \frac{(\kappa f')^2}{t(2f'+\kappa f'')}. \tag{25}$$

Hence from (22A), (24), and (25) the selection of the two groups adds to the amplitude a factor $1/\kappa f'(\kappa)$, where

$$f(\kappa) = 0 = v - \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)}.$$

* Havelock, 'Roy. Soc. Proc.,' A, vol. 81, p. 411, 1908.

Also if u is the group-velocity for wave-length $2\pi/\kappa$ and wave-velocity V, we have, in this case,

$$u = \frac{d}{d\kappa}(\kappa \mathrm{V}) = \frac{d}{d\kappa}\{\kappa v - \kappa f(\kappa)\} = v - \{f(\kappa) + \kappa f'(\kappa)\}.$$

Hence, since in the final value $f(\kappa) = 0$, we have $\kappa f'(\kappa)$ equal to $v-u$. Thus if κ is the wave-length of the regular wave-trains in the rear of the disturbance, we find that they are given by

$$\eta = \text{const.} \times \frac{\kappa v \phi(\kappa)}{v-u} \sin \kappa x, \tag{26}$$

where $v = \sqrt{\left(\frac{g}{\kappa}\tanh \kappa h\right)}$, $\quad u = \frac{1}{2} v\left(1 + \frac{2\kappa h}{\sinh 2\kappa h}\right)$.

Hence for the amplitude a we have

$$a = \mathrm{C}\kappa\phi(\kappa) \Big/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right).$$

Substituting now in (21) we obtain for the wave-making resistance, R proportional to

$$\kappa^2 \{\phi(\kappa)\}^2 \Big/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right).$$

If we take the same distribution of pressure in the travelling disturbance, namely, $\mathrm{F}(x) = \mathrm{P}\alpha/\pi(\alpha^2 + x^2)$, we have $\phi(\kappa) = \mathrm{P}e^{-\alpha\kappa}$; further, we may again assume that the pressure P varies as v^2, so that we have the resistance in the form

$$\mathrm{R} = \mathrm{A}\,\kappa^2 v^4 e^{-\beta\kappa} \Big/ \left(1 - \frac{2\kappa h}{\sinh 2\kappa h}\right),$$

with

$$\frac{\tanh \kappa h}{\kappa h} = \frac{v^2}{gh}. \tag{27}$$

Considering R given as a function of v by these two equations, we see that R increases slowly at first and then rapidly up to a limiting value at the critical velocity $\sqrt{(gh)}$; after this point R is zero, for there is no value of κ satisfying the second equation with $v^2/gh > 1$.

Further, the limiting value of R at the critical velocity is finite, for we have

$$\lim_{\kappa=0} \frac{\kappa^2 h^2}{(1 - 2\kappa h/\sinh 2\kappa h)} = 1{\cdot}5.$$

We see that the R, v curve given by (27) is of the type sketched in fig. 9. We may compare this with some of the curves given by Scott Russell for canal boats. The continuous curve in fig. 10 is an experimental curve of

total resistance,* and the dotted curve is a parabolic curve inserted here to represent approximately the frictional resistance; the difference between the two curves represents the residuary resistance, and is clearly of the same type as the theoretical curve in fig. 9.

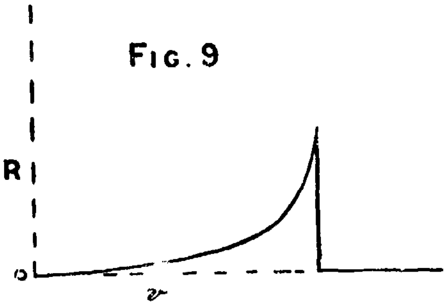


FIG. 9

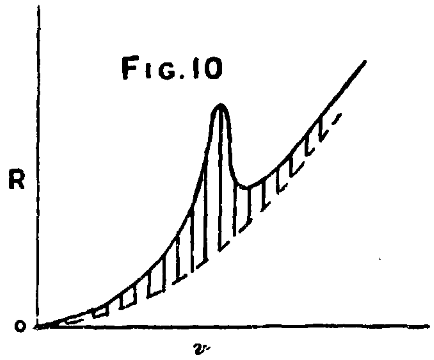


FIG. 10

We can obtain a better estimate of equation (27) by taking an experimental curve for a model in deep water, and then building up curves for different depths. We must first put (27) into a form suitable for comparison with deep water results.

Limiting the problem to one of transverse waves only, the formula (27) must reduce to $R = Ae^{-2 \cdot 53/c^2}$, for h infinite and $c =$ (speed in knots)/$\surd$(length in feet).

Writing v' for $v/\surd(gh)$ we find $c^2 = 11 \cdot 3 v'^2 h/L$; thus although the actual critical velocity does not depend upon the length of the ship but only on the depth of water, the speed-length ratio (c) has a critical value which is proportional to the square root of the ratio (depth of water)/(length of ship).

In (27) we cannot fix any value of v or c and then calculate R directly; we must work through the intermediate variable κh. The equations may now be written as

$$R = A(\kappa h)^2 v'^4 e^{-\beta' \kappa h}/(1 - 2\kappa h/\sinh 2\kappa h), \quad (28)$$

$$v'^2 = (\tanh \kappa h)/\kappa h; \quad \beta' = 0 \cdot 218 L/h; \quad c^2 = 11 \cdot 3 v'^2 h/L.$$

With h infinite this reduces to the previous form for deep water with the same constant A, so that a direct comparison is possible. As the velocity v increases from 0 to $\surd(gh)$, κ diminishes from ∞ to 0; we select certain values of κh, calculate the values from tables of hyperbolic functions, and thus obtain the set of values in Table VII, writing m for

$$(\kappa h)^2 v'^4/(1 - 2\kappa h/\sinh 2\kappa h).$$

* J. Scott Russell, 'Edin. Phil. Trans.,' vol. 14, p. 48, 1840.

Table VII.

κh.	$v/\sqrt{(gh)}$.	c^2L/h.	m.	$-\beta\kappa c^2$.
∞	0	0	1·0	2·53
10	0·316	1·13	1·0	2·53
6	0·41	1·87	1·0	2·53
4	0·5	2·82	1·005	2·53
2	0·69	5·42	1·077	2·43
1	0·87	8·57	1·287	1·92
0	1·0	11·3	1·5	0

We consider now the experimental curve analysed in Case IV in the previous section, a model of 20·5 feet taken up to a value $c = 1·8$. Assuming that the influence of finite depth was inappreciable in this range, we have for deep water

$$R = 2e^{-2·53/9c^2} + 82·5\,\{1 - 0·14\cos(10·2/c^2)\}\,e^{-2·53/c^2}. \tag{29}$$

We leave out of consideration at present the first term, which is supposed to represent the diverging waves, and we extend the calculations for R (transverse) from the rest of the formula up to $c = 3·3$ taken at intervals of 0·1 for c; we obtain thus the lowest curve given in fig. 11. With the help of Table VII, we calculate values of R for depths of about 5, 10, 12, 15, and 20 feet, taking in the formula (28) A equal to

$$82·5\,\{1 - 0·14\cos(10·2/c^2)\}$$

so that the results apply to the same model at different depths. An example of the calculations for one case may be sufficient; Table VIII shows the intermediate steps for $h = 12·3$ ft., $L = 20·5$.

Table VIII.

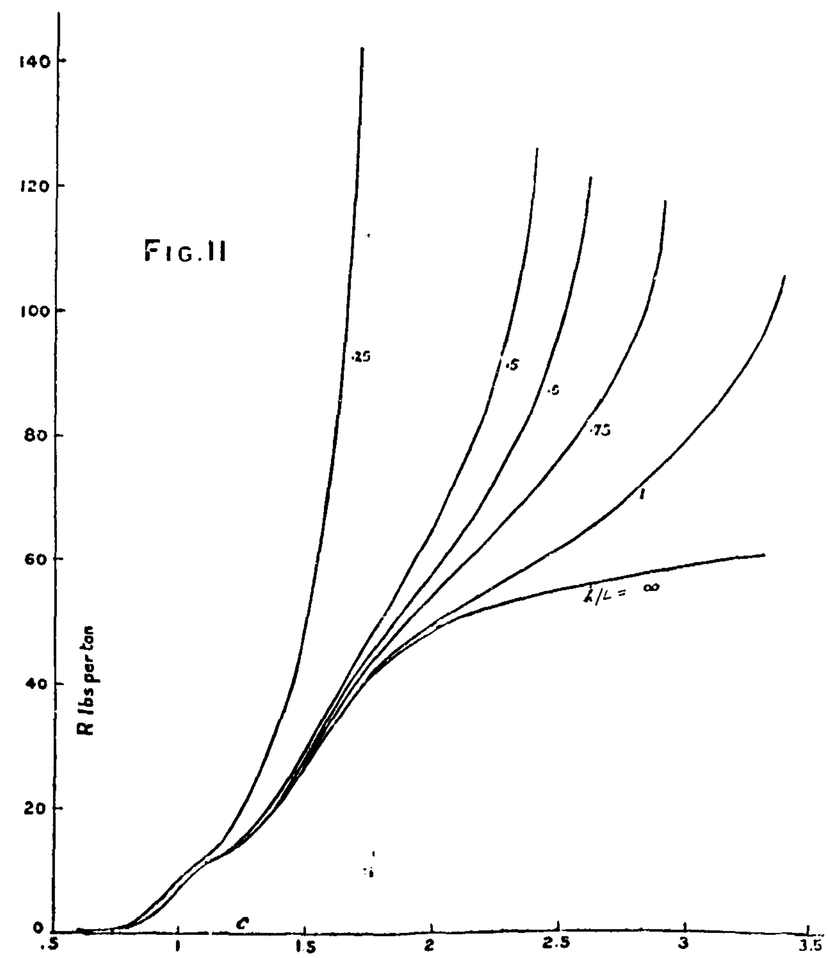
c^2.	c.	$-\beta\kappa$.	R/A.	$e^{-2·53/c^2}$.
0·68	0·825	3·73	0·024	0·024
1·12	1·06	2·26	0·106	0·106
1·69	1·3	1·5	0·224	0·223
3·25	1·8	0·75	0·508	0·472
5·14	2·27	0·374	0·385	0·687
6·8	2·61	0	1·5	1

The results for the five values of h are given in Table IX, and from these the curves in fig. 11 have been drawn.

The general character of the effect of finite depth is clear on inspection of the set of curves in fig. 11. If it is required to go to high values of the speed-length ratio in a given tank, the ratio of the depth of water to the length of the model must be adjusted so that there is no appreciable effect in

Table IX.

h/L.								
1	c R	0·7 0·7	1·0 7·5	1·5 27·5	1·7 39·2	2·33 59·2	3 80·5
0·75	c R	0·7 0·7	1·0 7·5	1·2 13	1·4 21·4	2 54·5	2·54 79·2
0·6	c R	0·7 0·7	1·0 7·5	1·3 17·9	1·8 47·7	2·3 78	2·6 122
0·5	c R	0·7 0·7	1·0 7·5	1·2 13·1	1·65 40	2·1 74	2·38 127
0·25	c R	0·7 0·7	0·8 1·7	0·84 3·3	1·16 14·1	1·46 43·6	1·68 142·5



the range of the experiments. Since the curves given here are theoretical curves for transverse waves only, each of them ends abruptly at the critical

velocity [illegible] that point. In practice we know that there are no such discontinuities in the resistance curves, and there are certain considerations which go to account for this difference. First, as regards the transverse waves alone, the preceding formulæ show that the amplitude tends to become infinite at the critical velocity, [illegible]ugh the corresponding resistance at uniform velocity remains finite; but, even apart from the effects of viscosity, there is a highest possible wave with a velocity depending partly upon the amplitude. Secondly, we have left out of consideration the diverging waves; but these must become more important in the neighbourhood of the critical velocity, for we may regard the two systems as coalescing into one solitary wave in the limit as the critical velocity is reached. After this point the diverging waves persist, so that the effect of these would be of the order of halving the drop in the resistance as the critical velocity is passed.

Finally, we must consider the frictional resistance, which increases steadily with the velocity; so that the fall is finally a smaller percentage of the total resistance than might appear at first. The curves given in fig. 11 give an estimate of a maximum effect of this kind, considering only the transverse wave system.

§7. *Further Types of Pressure Distribution.*

The preceding formulæ have been built up on the effect of a travelling pressure disturbance of simple type; we consider now another type which we may use as an illustration.

Let the pressure system be given by

$$p = f(x) = \mathrm{A}\,(h^2 - x^2)/(x^2 + h^2)^2.$$

The type of distribution is graphed in fig. 12.

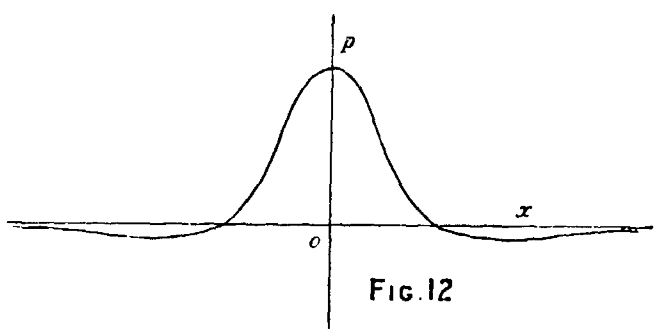


FIG. 12

Proceeding as in §2, we have

$$\phi(\kappa) = 2\mathrm{A}\int_0^\infty \frac{h^2 - \omega^2}{(\omega^2 + h^2)^2} \cos \kappa\omega \, d\omega = \pi \mathrm{A} \kappa e^{-\kappa h}. \tag{30}$$

Hence the amplitude of the regular wave-trains formed on deep water in the rear of this disturbance is proportional to $\kappa^2 A e^{-\kappa h}$, and the effective wave-making resistance is proportional to $\kappa^4 A^2 e^{-2\kappa h}$. We make the same assumption as before, viz., A proportional to v^2, and write $\kappa = g/v^2$; then the resistance is given by

$$R = Cv^{-4}e^{-2gh/v^2}. \tag{31}$$

We use this expression to show how R varies with the constant h of the pressure system. Let $v = 10$ ft./sec., and let $R = 1$ for $h = 0$; then we find the following relative values:

h.	R.
0	1·0
1	0·52
5	0·04
10	0·0016

R decreases very rapidly as h is increased. We have chosen this example for the following reason. Consider the motion of a thin infinite cylinder in an infinite perfect fluid; if we consider a plane parallel to the direction of motion and to the cylinder and at a distance h from it, we find that the distribution of excess or defect of pressure due to the motion is of the above type. Now, this is not the same as a cylinder moving in deep water at a depth h below the free surface, but it is suggested that as a first approximation the wave-forming effect is that of an equivalent diffused pressure system. The illustration shows how rapidly the wave-making resistance diminishes with the amount of diffusion, that is, with the depth h; this, of course, agrees with the experiments on the resistance to motion of submerged bodies, and, in fact, with the resistance of submarine vessels.

In the preceding work no attempt has been made to connect theoretically the constants in the pressure formula with those of the model; since the theory rests chiefly on the consideration of transverse waves only, this would presumably bring into question the length of entrance, run, and so forth. The consideration of any "transverse" constants, such as the beam, would need a fuller treatment of a diffused pressure system in two dimensions on the surface so as to give a more detailed investigation of both transverse and diverging wave systems.

HARRISON AND SONS, Printers in Ordinary to His Majesty, St. Martin's Lane

(Excerpt from the Proceedings of the
University of Durham Philosophical Society,
Vol. III., Part 4.)

Ship-Resistance: A Numerical Analysis of the Distribution of Effective Horse Power.

By T. H. Havelock, M.A., D.Sc.

[Read January 24th, 1910.]

Introduction.—The following paper contains, in its second part, a numerical study of the distribution of effective horse-power at different speeds. The data are taken from some recent experiments on models by D. W. Taylor, and are expressed for a ship of 400 tons displacement and 250 feet length. A theoretical formula is found to fit the experimental results, and from it the different terms in the E.H.P. are calculated for much higher speeds. In addition to the general analysis, attention is directed to the changes in the proportion of power which goes in wave-making, and also to the variation of the ratio E.H.P./(speed)^3 with the speed; in the latter case a curve is drawn and may be compared with the type of curve obtained from high-speed motor boats.

In the first part, an outline is given of the general theory of ship resistance; it is developed so as to lead to the introduction of a type of expression which exhibits the variation of the wave-making resistance with the speed, but for details of the mathematical analysis reference is made to previous papers.* One obtains a general formula which is based on theory in so far as it depends upon the speed, and with co-efficients which should depend upon the form of the ship but whose values are at present empirical. No attempt has been made to tabulate values suitable for different types of vessels, for without further information, it is uncertain whether the results would repay the labour; meantime, as already indicated, the formula has been used to analyse experimental

* *Proceedings of the Royal Society*, A, vol. 81, p. 398 (1908); A, vol. 82, p. 276 (1909).

results and to extend them to the region where more accurate data are needed.

General Theory.—We obtain a clearer view of the problem to be solved if we pass over the idea of resistance and fix our attention directly upon the transformations of energy which accompany the motion of a ship. Imagine a ship which is moving at constant speed and whose engines are developing energy effectively at a certain rate. None of the energy supplied goes into the motion of the ship, for its speed remains the same; clearly all the energy goes into the water. If we could calculate completely the motion of the water we should know the rate at which energy must be supplied from the ship and consequently the effective horsepower necessary to maintain a given speed. Naturally the problem has proved too difficult to solve as a whole. All that can be done is to classify the motions of the water into groups which seem more or less independent; the results of the separate calculations are then added together and the sum compared with the total effect in actual experiments. For a first attempt we consider the following groups of motions:—surface waves: wake and large eddies; smaller eddies of turbulent motion; rotations and heat-motions of the particles of water. Since the rate of supply of energy is equivalent to some resistance multiplied by the speed of the ship, we may express the results of calculations in terms of effective resistance obtained in this way. The latter groups in the above scheme are usually taken together, and their effect is expressed as a frictional resistance calculated from an empirical formula based on experiment. It has been found that a suitable expression is fSV^n, where V is the speed and S is the area of the wetted surface of the ship; the numerical values of f and n are taken from tables of experimental results. After this part has been deducted from the total effective resistance, the remainder is called the residuary resistance; following the usual custom, we assume that this is associated almost entirely with the surface waves, and we proceed to estimate the rate at which energy goes into the wave motion.

The well-known wave pattern which accompanies a moving ship is complicated, as it consists of both diverging waves and transverse waves. We begin with the simpler wave formation which is obtained by drawing a long rod over the surface at a steady speed in a direction at right angles to the rod: we observe that the water surface behind the rod is undulating, with parallel ridges and hollows succeeding each other regularly. The distance between consecutive ridges is called the wave-length; it is found that the waves have definite wave length and a definite height (a feet) above the mean water level for a given speed (v) of the rod. It can be proved that over the range where there are regular waves the mean energy of the wave motion is $\frac{1}{2}wa^2$ foot-pounds per square foot of the surface, where w pounds is the weight of a cubic foot of water.

What is the length of the train of regular waves behind the rod at any time? Its front is at the rod, and so moves forward with velocity v; its rear depends upon how and when the rod was started. Suppose the motion has been steady for a considerable time, so that the range of regular waves is large compared with the initial disturbances in getting up speed; it can be shown that the rear of the train of waves moves forward at a certain speed (u) less than v. This velocity of the rear is called the "group velocity"; if we observe a group of waves advancing into still water we may notice the crests moving forward relatively to the group, so that the wave velocity is greater than the group velocity. The result in the present connection is that the wave-train is increasing constantly in length at a certain speed (v-u); hence the energy in the wave motion is increasing at the rate $\frac{1}{2}wa^2$ (v-u) per foot-length of the rod. Energy must be supplied at this rate in order to maintain the constant speed v of the rod. If we write the rate of supply as Rv, then R is a force per foot of the rod and is called the wave-making resistance. We have then—

$$R=\frac{1}{2}\,wa^2\,(v-u)/v \quad . \quad . \quad . \quad (1)$$

Some interesting conclusions may be taken from this equation. R is zero if u equals v, and this is in fact approximately the case in shallow-water; the whole group of transverse waves consists then of a hump which accompanies the ship at its own speed (v), and in consequence, once the disturbance is formed, no further supply of energy is needed.

In deep water, it can be shown that u is $\frac{1}{2}v$: we limit our consideration here to this case.

We must examine now the variation of the height (a) of the waves with the speed. The motion of the ship makes differences of fluid pressure in its neighbourhood, so we may consider the problem as equivalent dynamically to a pressure disturbance moving over the surface of the water; the effects will depend both upon the speed and upon the character of the disturbance.

As regards the velocity, the differences of pressure increase with the speed, and probably they are proportional to its square.

The distribution of pressure in the disturbance depends upon the form of the ship. To take an extreme case, if the ship were an infinite raft moving over an infinite sea, the pressure would be constant over the surface and there would be no waves; on the other hand, if the lines of the ship are abrupt the pressure changes may be sudden and concentrated and the height of the waves greater. A detailed analysis confirms the impression that in general the height of the waves is diminished by diffusing the pressure system. But an increase of speed is equivalent to a diffusion of the pressure; hence we have a two-fold effect, increase of speed increases the magnitude of the pressures, and is at the same time equivalent to diffusing them over a greater area. Thus, there appear to be two opposing tendencies, and we infer that the height of the waves should not increase indefinitely with the speed of the ship, for the two effects may tend to balance at high speeds.

For a simple type of pressure distribution, a mathematical analysis shows that the wave-making resistance varies

with the velocity according to a certain exponential function, namely,

$$R=Ae^{-\frac{3}{2}\left(\frac{v^1}{v}\right)^2}, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

where A and v^1 are constants as regards v.

Before examining this relation we change the variables in a certain manner. Instead of the speed v, we use the speed-length ratio c defined as (speed of ship in knots)/√(length in feet); further, the resistance R is expressed in lbs. per ton displacement of the ship. The advantages of these changes are found in calculating from the results of model experiments similar quantities for a ship of any dimensions on the same lines as the model; R is the same for equal values of c in the two cases.

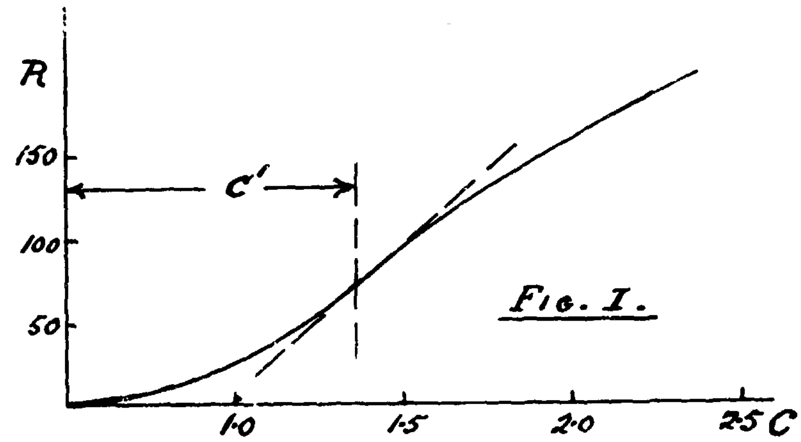


Fig. 1 represents the type of curve given by $R=Ae^{-\frac{3}{2}\left(\frac{c^1}{c}\right)^2}$.

The slope of the curve increases up to the value c^1, and then falls off for higher values of c. R increases continually with c, and approaches a limiting value equal to the coefficient A. If we compare Fig. 1 with any experimental curve of residuary resistance we find that the general features are the same, but if we wish to obtain a close agreement over a large range of values of c there are two ways in which the formula must be extended.

In the first place, in an actual ship there are two chief pressure systems, one associated with the entrance at the bow and the other with the run at the stern. The undulations caused by these are superposed upon each other, and the result is that the resistance curve is sometimes above and sometimes below a mean curve, according as the crests of one group coincide with the crests or with the troughs of the other group of waves. It appears probable that the variations from a mean value R can be represented by an additional term $bR \cos(n/c^2)$, where b and n do not involve c but depend upon the form and dimensions of the ship.

In the second place, in addition to the transverse waves which we have considered alone so far, there are waves diverging from the bow and stern. Regarding these as wave-trains inclined to the direction of motion, certain considerations suggest a similar form for the resistance as before, but with a value of c^1 one-third its value for the transverse waves. This term is found to have a small value compared with the others, and is only of importance relatively at lower speeds.

Summing up the various terms we obtain a general formula of the type

$$R = \alpha e^{-\frac{3}{2}\left(\frac{c^1}{3c}\right)^2} + \beta\left(1 - \gamma \cos\frac{n}{c^2}\right) e^{-\frac{3}{2}\left(\frac{c^1}{c}\right)^2} \quad . \quad . \quad (3)$$

An inspection of experimental results shows that some of the coefficients in this formula may be given fixed values provisionally, that is, they are practically the same for ordinary types of vessels; thus we find $c^1 = 1{\cdot}3$, $n = 10{\cdot}2$, and $\gamma = 0{\cdot}14$ approximately. A good agreement can be obtained at values of c greater than 1 by using only the second term in (3), but if we wish to cover the whole range by one formula we must include all the terms.* For present purposes we use (3) in the form

$$R = \alpha e^{-\frac{2{\cdot}53}{9c^2}} + \beta\left(1 - 0{\cdot}14 \cos\frac{10{\cdot}2}{c^2}\right) e^{-\frac{2{\cdot}53}{c^2}} \quad . \quad . \quad (4)$$

* *Loc. cit. ante*, p. 215.

A Numerical Analysis.—In a paper by D. W. Taylor† a residuary resistance curve is given for a certain model (No. 892) from the result of tank experiments up to a value of c of 1·8. This is given almost exactly by (4) with $\alpha=2$ and $\beta=82{\cdot}5$. All the results are now calculated in terms of effective horsepower for a ship on the same lines as the model. In the following table the column headed Experimental E.H.P. was obtained in this way from the residuary resistance curve, while the calculated E.H.P. was found from (4) with the above values of α and β; in both cases the total E.H.P. was obtained by adding a suitable frictional part $0{\cdot}00307 f S V^{2{\cdot}83}$.

The data for the ship on the lines of the model are: Displacement = 400 tons; length = 250 feet; wetted surface = 5,000 square feet; longitudinal coefficient = 0·68; midship section coefficient = 0·70; frictional coefficient = ·00897.

TABLE I.—EFFECTIVE HORSE POWER.

C.	V.	Experimental.	Calculated.	% Wave.	E.H.P./V^3.
0·5	7·9	53	54	11	·109
0·8	12·6	227	228	20	·112
1·0	15·8	514	510	33	·129
1·2	18·9	888	898	37	·132
1·4	22·1	1,495	1,495	42	·144
1·6	25·3	2,370	2,370	46	·147
1·8	28·4	3,360	3,360	47	·147
2·2	34·8	—	6,400	36	·152
2·6	41·1	—	8,010	36	·116
3·0	47·4	—	11,160	31	·105
4·0	63·2	—	22,250	23	·088

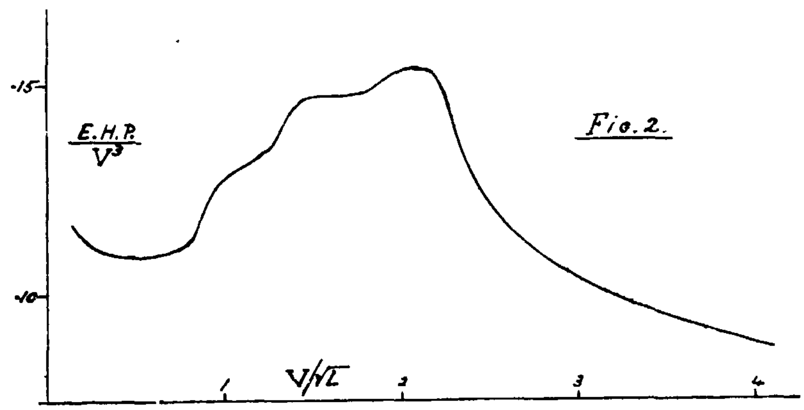
The agreement between columns 3 and 4 shows that the formula (4) is capable of representing the experimental results for this model. In regard to the calculated values at high values of c, it is of interest to compare these with other calculations. Cotterill* states that for certain types of torpedo boats the total resistance is approximately 80 c^2 for values of c between 1·5 and 2·5; for the ship under discussion this would give the total E.H.P. as 582 c^3. This formula gives

† D. W. Taylor, Trans. S.N.A.M.E. (New York, Vol. 16, p. 17 (1908).
* Cotterill, *Applied Mathematics*, p. 621.

2,380 at $c=1{\cdot}6$ and 3,400 at $c=1{\cdot}8$: but for higher values it increases very rapidly, and gives 15,700 at $c=3$ and 37,250 at $c=4$.

In the last column in Table I. the values of E.H.P./V^3 are given; these were obtained from the calculated total power in each case, and the results are graphed in Fig. 2 on a base of $V/\sqrt{L}$. The curve is of the same type as one which has been deduced from the performances of high-speed motor boats.*

Column 5 in Table I. gives the percentage which the wave horsepower is of the total horsepower. It attains a maximum in this case of nearly 50 per cent at about



$c=1{\cdot}8$. After this value the ratio diminishes, for the wave resistance begins to approach its limiting value while the frictional resistance continues increasing as $V^{1\cdot 83}$. If we suppose the total power to vary as V^n in the neighbourhood of any given value of V, we can find how the index n varies with the percentage (p) which the wave power is of the total power (E). For we have

$$E=F+\frac{p}{100}E=\frac{100}{100-p}CV^{2\cdot 83},$$

where C is some constant.
Since $n = V(dE/dV)./E$, we find

$$n=2{\cdot}83+\frac{V}{100-p}\frac{dp}{dV}.$$

* R. E. Froude, *Trans. Nav. Arch.*, vol. 48, p. 102 (1906).

The index n begins at the value 2·83, increases with increasing values of p, then decreases to a minimum, and finally increases again to a limiting value of 2·83; the position of the minimum index is not the place where the wave power is a maximum proportion ($dp/dV = 0$) but at some velocity

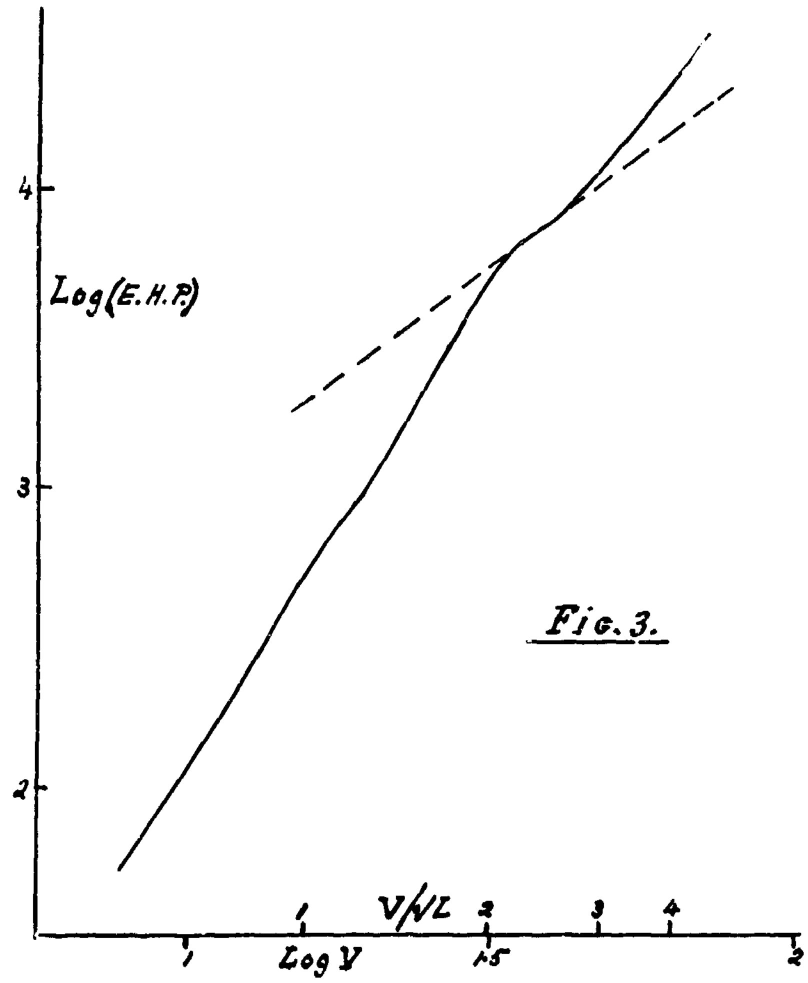


FIG. 3.

greater than this value. Since the equivalent index n is given by $d\,(\log E)/d\,(\log V)$, its changes may be exhibited by graphing log E upon a base log V; the slope of this curve gives the corresponding value of n. This has been done for the ship under discussion, and the result is shown in Fig. 3.

The scale is not sufficient to show the variations in the first part of the curve, but one may notice the short interval between $c=2$ and $c=3$ for which n drops to values in the vicinity of 2; after this the index rises again. The region of low values of n depends upon the relation of wave power to frictional horsepower; hence it will vary not only with the lines of the ship but also with its absolute dimensions. This, together with other points mentioned in the previous study, must be left to a more detailed analysis and comparison of experimental results for models of different types.

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The Wave-making Resistance of Ships: a Study of Certain Series of Model Experiments.

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(Communicated by Prof. Sir Joseph Larmor, Sec. R.S. Received June 7,—Read June 23, 1910.)

1. In a previous communication* I proposed a formula for the wave-making resistance of ships, and showed that it expressed certain general qualities of experimental results; further, notwithstanding the limitations of theory and the difficulty of interpretation of experimental data, a good numerical agreement was found in several cases with the published results of tank experiments on models when suitable numerical values were given to the coefficients in the formula.

This paper records the results of a more systematic study of the numerical values of some of the coefficients, the data being taken from certain recent series of experiments; for the present the discussion is limited to those types of model whose resistance-speed curves show clearly the humps and hollows which are attributed to interference of wave-systems originating at the bow and stern. It has been remarked that although the mode of disturbance is different, the action of the bows of a ship may be roughly compared to that of a travelling pressure-point, and further, that the stern may be regarded in the same way as a negative pressure-point.† This point of view originated in the well-known paper of W. Froude‡ on the effect of the length of parallel middle body, and the theory was developed in a later paper by R. E. Froude§; from an inspection of experimental results it was seen that the variations in magnitude and position of the oscillations were in directions which agreed with the above interpretation. On account of the lack of an adequate formula, the available data have not yet been examined numerically in any detail; the present investigation aims at supplying this in some measure. Section 2 is theoretical, with some necessary repetition of previous work; Sections 3 and 4 contain a numerical analysis of some available experimental curves. In Section 5 an attempt is made to estimate the effective horse-power of the "Turbinia," in order to illustrate certain points; while in Section 6 the limitations of the interference theory, in the

*'Roy. Soc. Proc.,' 1909, A, vol. 82, p. 276.
†H. Lamb, 'Hydrodynamics,' 1932 edn. p. 438.
‡W. Froude, 'Trans. Inst. Nav. Arch.,' 1877, vol. 18.
§R. E. Froude, 'Trans. Inst. Nav. Arch.,' 1881, vol. 22.

conventional use of the term, are discussed in connection with the residuary resistance curves of finer-ended models.

2. A transverse pressure disturbance travelling over the surface of water at right angles to its axis leaves in its rear a procession of regular waves; on account of the supply of energy needed to maintain this system, there is an effective resistance which may be called the wave-making resistance of the given disturbance. An illustration of a simple type of disturbance, symmetrical fore and aft with respect to its axis, is afforded by the function $P/(p^2+x^2)$, where Ox is in the direction of motion. The length p may be used to define the distribution of pressure to this extent: when p is decreased, the changes are more concentrated and abrupt, and conversely; we may, as a convention, call $2p$ the effective width b of the disturbance. If the disturbance is made to move with uniform velocity v at right angles to its axis, the height of the waves can be calculated, and thence, from considerations of energy, the corresponding wave-resistance R. If the quantity P which defines the magnitude of the disturbance is supposed an absolute constant, the calculation of R as a function of v gives an expression which rises to a maximum and then diminishes ultimately to zero with increasing values of the velocity.* But if the pressure disturbance is associated with a moving ship, it seems reasonable to suppose that P depends upon the velocity, and in fact the assumption is that P varies as the square of the speed.

In this way we obtain the result

$$R = Be^{-bg/v^2}. \tag{1}$$

where B is independent of v. According to this expression, R increases from zero up to a limiting value B; at any given speed R is a certain fraction of the value B, and if the quantity b were increased the same value of R would only be reached at some higher speed. Further if we have a second expression R_1 with constants B_1, b_1, greater than B, b, respectively, the curve for R_1 will intersect the curve for R at a certain velocity; at lower speeds $R_1 < R$, while at higher values $R_1 > R$.

Suppose now that a similar negative pressure system, with a different coefficient P, but with the same width b, is situated behind the first system, with a fixed distance l between the two axes. The wave-making resistance of the combined system is given by an expression $\beta(1 - \gamma \cos gl/v^2)e^{-bg/v^2}$, where β and γ are independent of v. In applying this result to the case of a ship, we can of course only expect agreement if the type of model is such that we may imagine distinct, but mutually interfering, wave-systems originating at the bow and stern; it is in fact, an attempt to describe the wave-making

* *Cf.* Lord Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 396.

properties of a ship in terms of the coefficients of a simple equivalent pressure distribution of the type specified. Another point which must be noted is that the previous expression is obtained by considering two-dimensional motion only; but the bow and stern of a ship act more like point disturbances than as transverse line systems, hence there are diverging, as well as transverse, waves. In default of a fuller analysis, I have suggested for certain reasons the addition to R of a term $\alpha e^{-bg/9v^2}$; it is retained for the present, because it indicates the necessity for some expression of the diverging waves and it agrees with certain general properties of them, and also in several cases it allows us to obtain better numerical agreement at lower speeds.

We suppose that R is expressed in pounds per ton displacement of the ship, also V is the speed in knots, L the length of the ship on the water line in feet, and c is equal to $V/\sqrt{L}$; then we have

$$R = \alpha e^{-\frac{m}{9c^2}} + \beta\left(1-\gamma \cos n/c^2\right)e^{-\frac{m}{c^2}} \text{ lbs. per ton,} \tag{2}$$

where $m = 11{\cdot}3b/L$ and $n = 11{\cdot}3l/L$.

In the following examples attention is directed chiefly to the variations of β and m, and incidentally to those of γ and n. The length b cannot be taken directly as the length of the entrance or run of the ship, for it will depend also on the lines of the model; but one may expect the ratio b/L to decrease as the slope of the model at the bow is increased, and conversely; similarly the number n will vary in a direction which may be predicted. In the previous paper sufficient agreement was found when m and n were assigned fixed values; in many cases the mean curve of residuary resistance appeared to have a point of inflection near $c = 1{\cdot}3$, and for this we had $m = 2{\cdot}53$; further, the humps and hollows agreed with $n = 10{\cdot}2$ for the angle n/c^2 in radians, or $n = 584$ for the angle in degrees. With none of the coefficients fixed beforehand, it is necessary to adopt some method of approximation. Drawing the experimental curve of residuary resistance on a suitable scale, a fair mean curve was sketched in and an equation $R = Ae^{-m/c^2}$ was found, generally by graphical methods, to fit this as closely as possible; in fact it was the original intention to limit the study to the two leading coefficients A and m so determined. The value of m is now fixed, and from the intersections of the mean curve with the actual oscillating curve one could assign a value to n with sufficient accuracy. Finally the three remaining quantities were found from three points on the actual curve, for example, at $c = 0{\cdot}6$, $1{\cdot}2$, $1{\cdot}8$, if the curve extended so far. In practice the lowest point determines α, for the term in β is negligible there; for the same reason the values of β and γ are more satisfactory when fairly high values of c are available. In all cases the

approximation was not carried further than the circumstances seemed to warrant; the values of the coefficients are given generally in round numbers, and the theoretical curves were calculated throughout their range from the formulæ so obtained.

3. The first series is taken from a paper by D. W. Taylor* on the influence of the shape of midship-section upon the resistance of ships; from the curves in that paper I have taken four, which form a series having the same midship-section coefficient, but with different displacements. The data and the results are given in the following table:—

Table I.—Models I to IV.

No.	Displacement in lbs.	Displacement-length coefficient.	Beam.	Draft.	α.	β.	γ.	m.	n.
I	500	26·6	1·365	0·467	1·6	81	0·14	2·7	584
II	1000	53·2	1·930	0·660	2·0	160	0·18	3·0	540
III	1500	79·8	2·364	0·809	2·0	240	0·18	3·2	540
IV	2000	119·7	2·895	0·991	2·5	360	0·18	3·5	540

Cylindrical coefficient = 0·68; midship-section coefficient = 0·90; water-line length = 20·51 feet; beam/draft = 2·923.

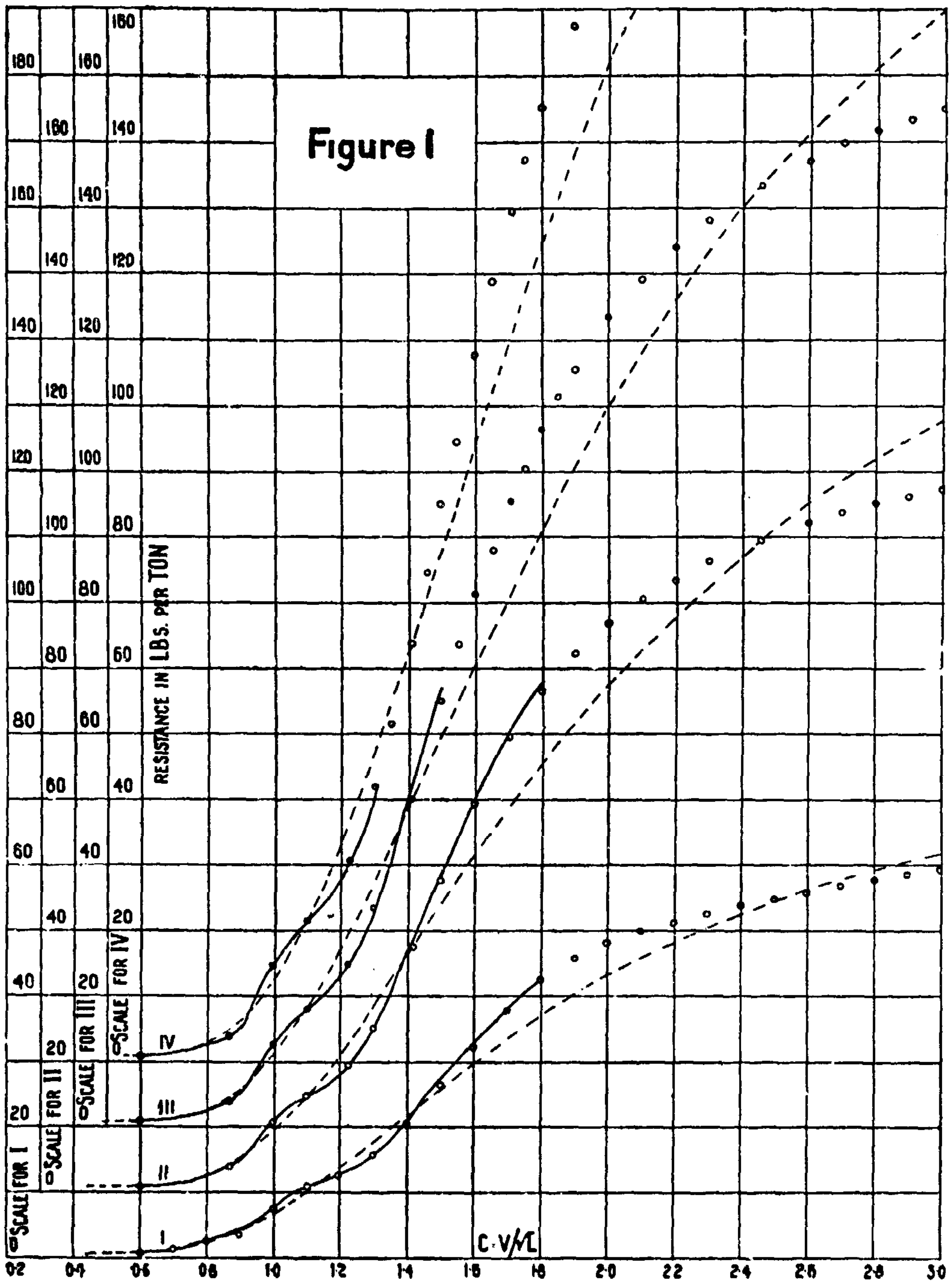
The curves in fig. 1 indicate the results of the analysis; in each case the continuous curve is the experimental curve of residuary resistance; the points marked by circles have been calculated from the formula (2), while the broken curve is a mean curve graphed from the expression $\alpha e^{-m/9c^2} + \beta e^{-m/c^2}$. The calculated curves have been extended as far as $c = 3$, in order to include the highest theoretical point of intersection of the mean and oscillating curves.

The third column in Table I refers to a coefficient of fineness used by Taylor in the paper referred to; it is defined as $D/(L/100)^3$, where D is the displacement of the model in salt water in tons, and L is the water-line length in feet. It is a method of estimating the proportions of a model by the displacement of a ship of the same lines and of 100-feet length.

From the numbers in Table I, β appears practically proportional to the displacement. The resistance R has been calculated in pounds per ton displacement, so that dimensionally β is a pure number. In this series certain quantities are constant, namely, with the ordinary notation, L, B/H, $(B \times H \times L)/D$, and (area of midship-section)/$(B \times H)$. As far as this series is concerned we might regard β as proportional, for instance, to $(B \times H)/L^2$ or to the displacement-length coefficient.

*D. W. Taylor, 'Trans. Amer. Soc. Nav. Arch.,' vol. 16, p. 13 (1908).

The index m of the exponential increases slightly with the displacement, that is, with increasing beam and draft; this variation is in the direction



one might anticipate, as it indicates a greater diffusion of the pressure changes. In regard to the coefficients specifying the interference between

the bow and stern systems, γ is larger at the higher displacements, while n is less; both variations are consistent with a diminution of the distance between the axes of the simple equivalent pressure distributions.

To illustrate the smaller changes which are possible at the same displacement, three models are taken from the same paper, having different midship-section coefficients with the same area of midship-section; thus smaller coefficients are associated with greater beam and draft. One of these three is No. 1 of Table I: midship-section coefficient $= 0{\cdot}9$, $\beta = 81$, $m = 2{\cdot}7$. Another of the set I had already used in my previous paper with the results: midship-section coefficient $= 0{\cdot}7$, $\beta = 82{\cdot}5$, $m = 2{\cdot}53$. For the third of the series the same coefficient is $1{\cdot}1$, and there is a good agreement by taking $\beta = 79{\cdot}5$, $m = 2{\cdot}87$.

4. The next sets of experimental results are taken from a paper by D. W. Taylor* on the influence of length of parallel middle body. One must notice that the problem investigated is not quite the same as in the paper by W. Froude referred to above. In the latter case the bow and stern of the model were unaltered, but varying lengths of parallel middle body were inserted between them, so that the special effect was isolated as far as possible. In Taylor's experiments the models have constant length and displacement, but varying proportions of the length are occupied by a parallel middle body, and, of course, the bow and stern vary in form so as to keep the displacement constant; the effect is thus more complex theoretically. We may anticipate the direction of variation of some of the coefficients with increasing percentage of parallel middle body under these conditions. Since the ratio of the length of entrance and run to the length of the ship becomes less, the index m should decrease; also the effective distance apart of the bow and stern systems becomes greater, so at the same time γ should decrease and n increase.

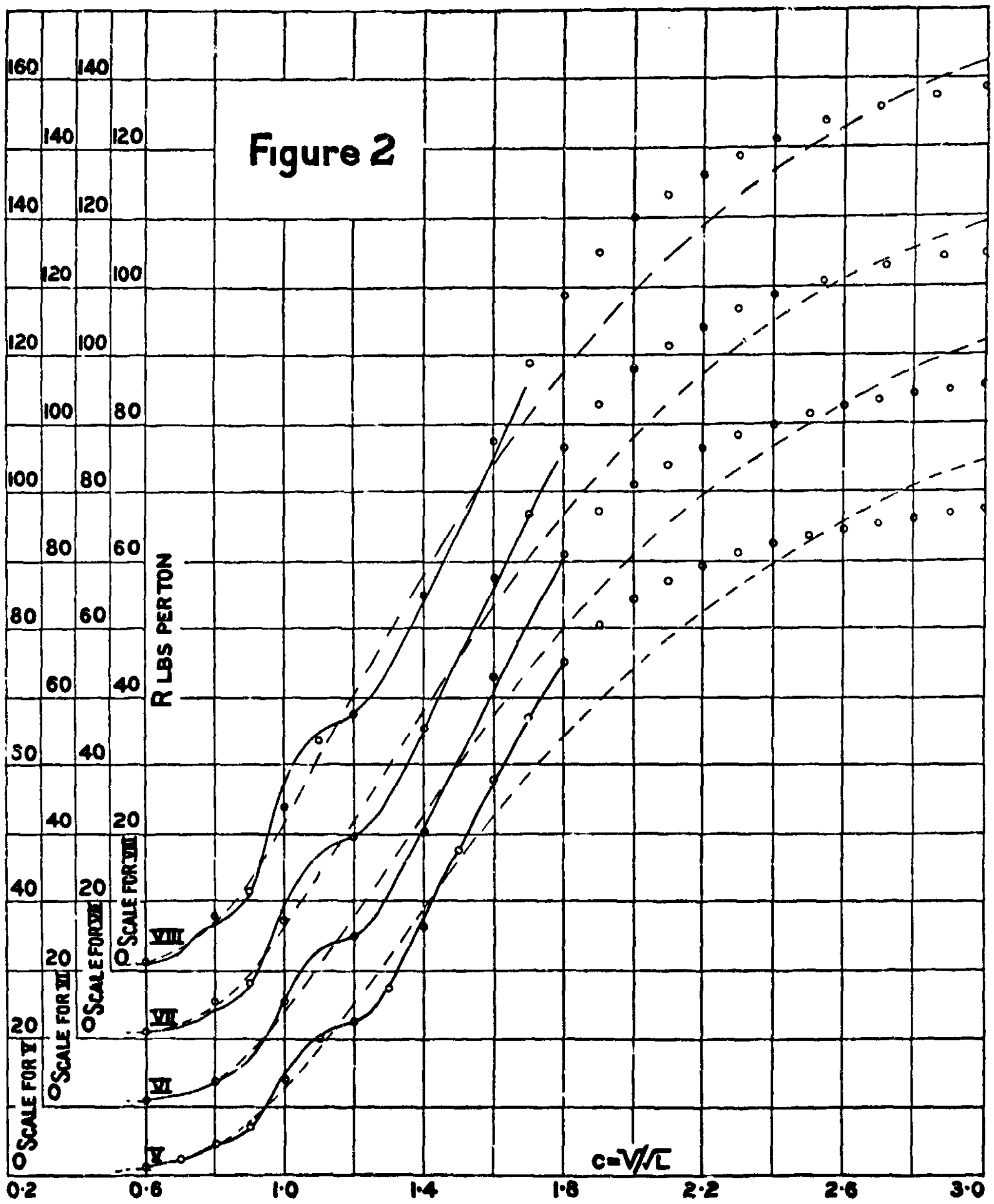
Table II.—Models V to VIII.

No.	Percentage of parallel middle body.	Cylindrical coefficient of ends.	α.	β.	γ.	m.	n.
V	0	0·740	2·5	135	0·17	2·53	584
VI	24	0·658	2·0	145	0·16	2·45	584
VII	36	0·594	2·0	155	0·14	2·35	605
VIII	48	0·500	2·0	165	0·12	2·10	650

Displacement = 1000 lbs.; length = 20·51 feet; beam = 1·682; draft = 0·673; displacement-length coefficient = 53·2; midship-section coefficient = 0·96; beam/draft = 2·5.

*D. W. Taylor, 'Trans. Amer. Soc. Nav. Arch.,' vol. 17, p. 171 (1909).

Table II contains the results for a set of four models of 1000 lbs. displacement, together with other data; the corresponding curves are shown in fig. 2 in the same manner as before.



From the curves in fig. 2, it will be seen that the calculated curves express the general variations in the manner anticipated above. The numerical agreement is best throughout the range for Model V, while for the other curves the agreement at lower values of c is not so good; this appears to be

associated with the change in shape of the ends of the model. Although for the whole length of each model the cylindrical coefficient is 0·74, on account of increasing proportion of parallel middle body the ends become finer; this is indicated in the third column of Table II. The formula (2), in its present form, gives best numerical agreement for models with fuller ends, that is, with fairly high cylindrical coefficients; this point is examined further below.

The same remarks apply to a second set of four models, taken from the same paper, having a displacement of 1500 lbs. The results are given in Table III and the curves in fig. 3. For Curve XII, a point in connection with the interference-coefficients γ and n may be noticed. Whatever value of n is used, if the simple theory is to be adequate, there must be certain relations between the values of c at which the humps and hollows occur; beginning with the highest values and working down to lower speeds, the successive values of c at which hollows, humps, and mean values occur must be proportional to the reciprocals of the sequence

$$\sqrt{0},\quad \sqrt{0\cdot5},\quad \sqrt{1},\quad \sqrt{1\cdot5},\quad \sqrt{2},\quad \sqrt{2\cdot5},\quad \ldots\ldots\ldots$$

In all the curves given here the graphs have been extended to $c = 3$, so as to include, in most cases, the highest mean value, corresponding to the second term in the above series. In most cases it was possible to choose n so that this relation was approximately satisfied, but the difficulty increases apparently at higher displacements, such as in Model XII. The mean curve shown for this case in fig. 3 represents the curve $R = 255c^{-2\cdot3/c^2}$; determining the value of n from the intersections of this with the actual curve, the numbers obtained from the higher positions are larger than those from the lower speeds. In consequence, the circles showing the theoretical continuation of the curve have been calculated with $\gamma = 0\cdot15$ and $n = 610$, without attempting a fit over the whole curve.

Table III.—Models IX to XII.

No.	Percentage of parallel middle body.	Cylindrical coefficient of ends.	α.	β.	γ.	m.	n.
IX	0	0·740	2·5	200	0·19	2·7	584
X	24	0·658	2·2	210	0·17	2·64	584
XI	36	0·594	2·0	235	0·14	2·6	610
XII	48	0·500	—	255	—	2·3	—

Displacement = 1500 lbs.; length = 20·51 feet; beam = 2·060; draft = 0·824; displacement-length coefficient = 79·8; midship-section coefficient = 0·96; beam/draft = 2·5.

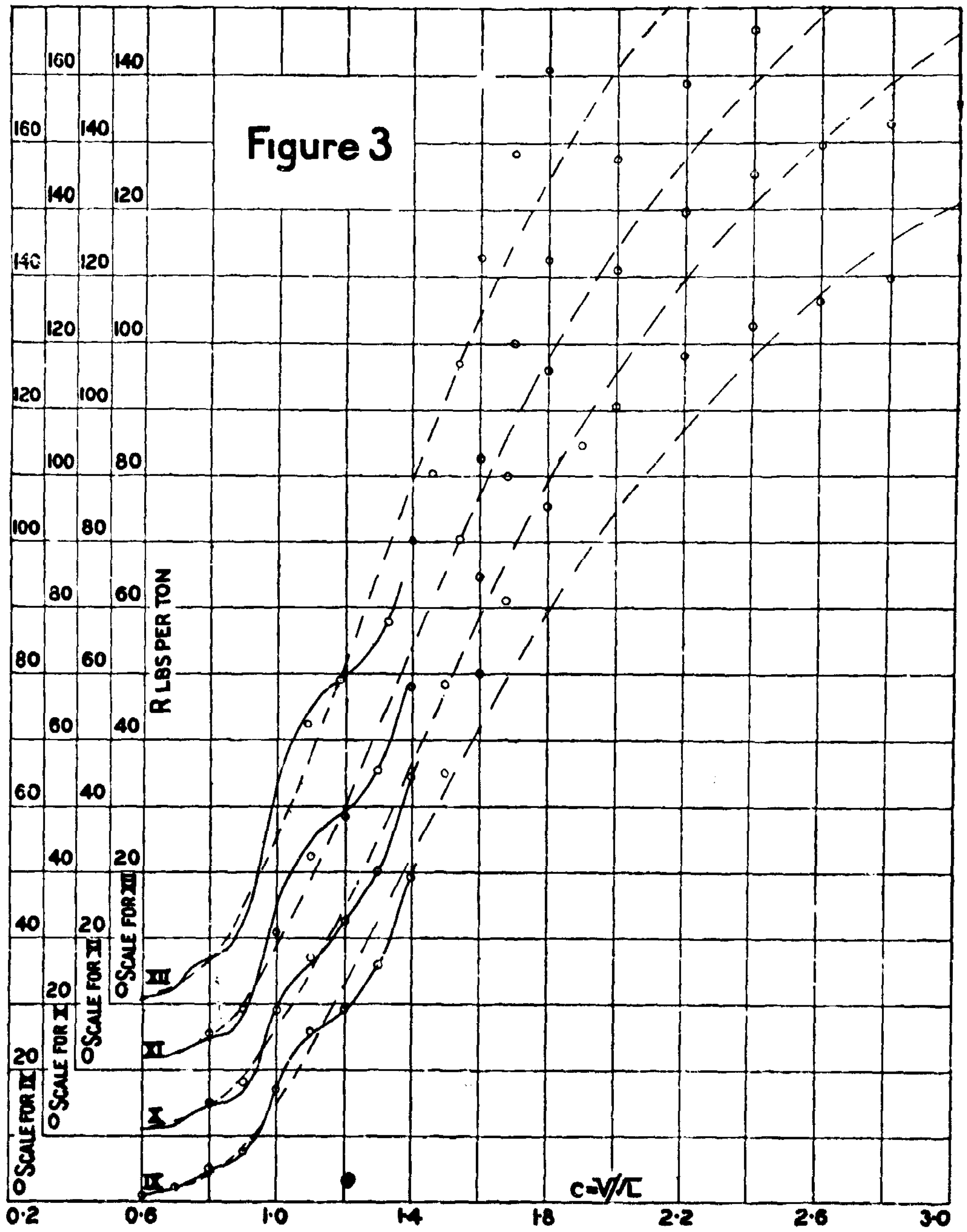


Figure 3

5. The curves in the previous sections have been examined chiefly from a theoretical point of view, that is, with the object of testing in these cases the general adequacy of a certain type of simple equivalent pressure distribution. One might try also to classify the coefficients of the formula, so as to obtain empirical expressions for them in relation to the form of

the model. The latter is frequently specified by various coefficients of fineness, which of course give only an approximate estimate of form, and in any case do not make a set of independent variables; no attempt is made here beyond giving all the available data for each model. With the results given above and in the previous paper, one can find an approximate estimate of the leading coefficient β, at least for forms similar to those already examined. It was noticed that, other things being equal, β was proportional to the displacement-length coefficient; also for given values of the latter β appears to be approximately proportional to the ratio of beam to draft. This seems reasonable, since wave-making is largely a surface effect; that is, for a disturbance travelling below the surface the wave-making falls off rapidly with its depth. In several of the cases already examined, it happens that β is numerically only slightly larger than the product of the two ratios mentioned, that is, β is a little greater than $(B/H) \times D/(L/100)^3$, with all the quantities in the units specified above. This result is used now to make an approximate estimate of total effective horse-power for a certain ship, as it affords opportunity for introducing other points of interest. The data for the ship are those of the "Turbinia," as far as they are available from the published record of trials.*

Turbinia.—Displacement $= 44\frac{1}{2}$ tons; length $= 100$ feet; beam $= 9$ feet; draft $= 3$ feet; cylindrical coefficient $= 0{\cdot}66$; speed $= 31$ knots.

The displacement-length coefficient is 44·5, while the ratio of beam to draft is 3; since the cylindrical coefficient is less than those already examined, we take β as about 5 per cent. greater than the product of these two ratios, that is, $\beta = 140$. Following out the indications of the previous cases, m should be nearly 3; as we shall calculate quantities for $c = 3{\cdot}1$ the exponential e^{-m/c^2} only varies slowly with m, so that $m = 3$, with sufficient approximation. Under the same conditions we take $n/c^2 = 60^\circ$ and $\gamma = 0{\cdot}15$, also $\alpha = 2$. Calculating from formula (2) with these values, we obtain an estimate of 410 for the effective horse-power of the ship at 31 knots due to wave-making, with the possibility of this being slightly in defect; any of the usual approximate formulæ, with simple powers of the speed, when extended to this high value of c give possibly twice this estimate, a result which is much too high. If we take the area of wetted surface (S) as 970 square feet and the frictional coefficient (f) as 0·0095, we may calculate the frictional effective horse-power from the expression $0{\cdot}00307 f S V^{2{\cdot}83}$; it is 470 at 31 knots. We obtain thus an approximate estimate of 880 for the total effective horse-power of the ship at 31 knots. It is stated in the record referred to above that the total effective horse-power at 31 knots is 946,

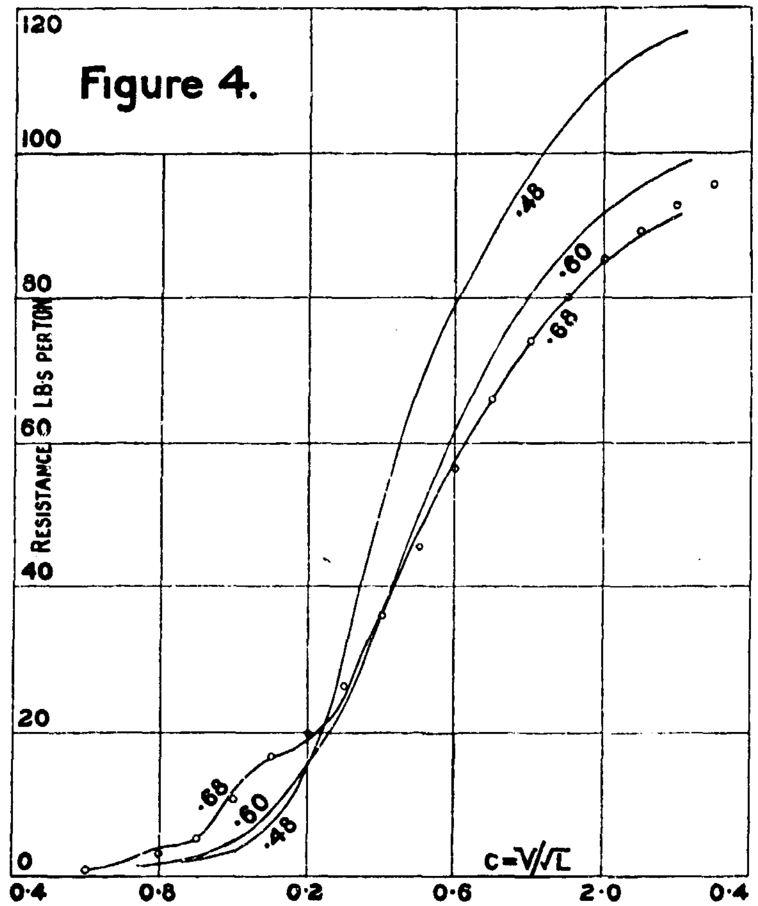
* C. A. Parsons, 'Trans. Inst. Nav. Arch.,' 1897, vol. 38, p. 232.

obtained by Froude's method from tank experiments on a model of the ship; no details of the calculation are given. Although the estimate above is only approximate, another possible factor should be noted; this is the influence of the finite depth of the tank. It has been stated that, from recorded experimental data, this effect becomes appreciable when the length of the waves exceeds twice the depth; this means approximately when $c > 1 \cdot 9h/L$, h being the depth of water and L the length of the model. This appears to agree with the curves of fig. 11 of my previous paper, which were obtained from theoretical considerations. The effect of shallow water is an excessive increase in the resistance for a considerable range, but if the speed is made high enough the resistance may become even less than in deep water at the same speed. It seems possible that the tank experiments quoted above come within the range of excess of transverse wave-making resistance. It is stated that, assuming a propulsive coefficient of 60 per cent., the value of 946 means a corresponding indicated horse-power of 1576; it may be noted that the estimate of 880 corresponds to the same indicated horse-power with an efficiency of about 56 per cent. In this connection the following remark may be quoted from a recent discussion: "Is it possible that this is one contributing cause to the large propulsive coefficients obtained by torpedo craft compared with those obtained in full-sized vessels, viz., that the tank effective horse-power of torpedo craft models is over-rated, because of excessive transverse wave-making resistance in the 'shoal water' of the tank"?*

6. It must be noted that all the preceding calculations refer to rather full-ended models, that is, with a cylindrical coefficient of about 0·68 and upwards. It was upon such a type that the original experiments of Froude were performed, and it seems that the characteristic interference effects occur specially in such vessels; the latter are associated with the idea of two fairly distinct systems of pressure disturbance at bow and stern respectively. Now if the ends are made finer it is reasonable to imagine the two systems coalescing into what could be more accurately interpreted as one pressure system. This would be more diffused over the length of the ship, so the equivalent index n should be larger; further, since for constant displacement finer ends mean larger beam and draft, the limiting coefficient β should be larger. Consequently, for decreasing cylindrical coefficient, at constant displacement, the curves of residuary resistance should be intersecting curves, lower at low speeds and then ultimately higher. This is illustrated in the curves in fig. 4, which have been superposed to show the point in question. The curves are taken from a series of 1000-lb. models by D. W. Taylor, of

* E. Wilding, 'Trans. Inst. Nav. Arch.,' 1909, vol. 51, p. 160.

constant midship-section coefficient 0·926, and with the ratio of beam to draft 2·923; the cylindrical coefficients are 0·68, 0·60, and 0·48.



The curve with a coefficient of 0·68 is represented quite well by formula (2) with $\alpha = 2$, $\beta = 155$, $\gamma = 0{\cdot}14$, $m = 2{\cdot}9$, $n = 584$; the circles represent calculated points. The typical oscillations are clearly visible in this curve but they appear to be absent altogether from the other two curves; the general character of the latter curves is in accordance with the remarks made above. One may even obtain some numerical agreement with γ zero and larger values of β and m, but it is unsatisfactory without a further examination of intermediate st ges and their equivalent pressure distributions; another point is that with a larger value of m the first term in the formula, which represents the diverging waves, becomes more important, and adds another reason for deferring the study of finer-ended models.

HARRISON AND SONS, Printers in Ordinary to His late Majesty, St. Martin's Lane.

(Excerpt from the Proceedings of the
University of Durham Philosophical Society,
Vol. IV., Part 2.)

THE DISPLACEMENT OF THE PARTICLES IN A CASE OF FLUID MOTION.

By T. H. HAVELOCK, M.A., D.Sc.

[Read March 3rd, 1911.]

The leading features of the motion induced by the passage of a cylinder through a perfect fluid are well known, but certain aspects of the permanent displacement of the fluid particles are less familiar. The following notes on these were suggested by an unexplained paradox which is mentioned in recent treatises, such as Lanchester's Aero-dynamics and Taylor's Speed and Power of Ships; it was found later that the same difficulty is mentioned by Maxwell in a paper on the paths of the particles. The present remarks are arranged as follows:

1. From the ordinary theory of the fluid motion is deduced a simple proof of Rankine's formula for the radius of curvature of the path of a particle, and the solution is then completed in terms of elliptic functions.
2. After drawing paths of particles, curves are obtained for the subsequent positions of lines of particles which were abreast of the cylinder at certain times.
3. A graphical study of the deformation of a group of particles as it passes near the cylinder suggests a difference between the behaviour of an ideally continuous fluid and one which is molecular.

4. A discussion of the paradox that the fluid appears to have a permanent forward displacement ultimately. The difficulty is shown to arise from the introduction of infinities without precise definition of conditions. Analytically the ambiguity occurs in the form of a double integral whose value depends upon the order of performing the integrations.

1.—A circular cylinder of radius a and of infinite length moves through an infinite fluid with uniform velocity u at right angles to its axis. The fluid is assumed to be perfectly continuous, frictionless, and incompressible; and under these conditions a certain continuous motion is determined in the fluid. Let the diagram in Fig. 1 represent a section at right angles to the axis of the cylinder, the circle with centre O being a section of the cylinder at any instant.

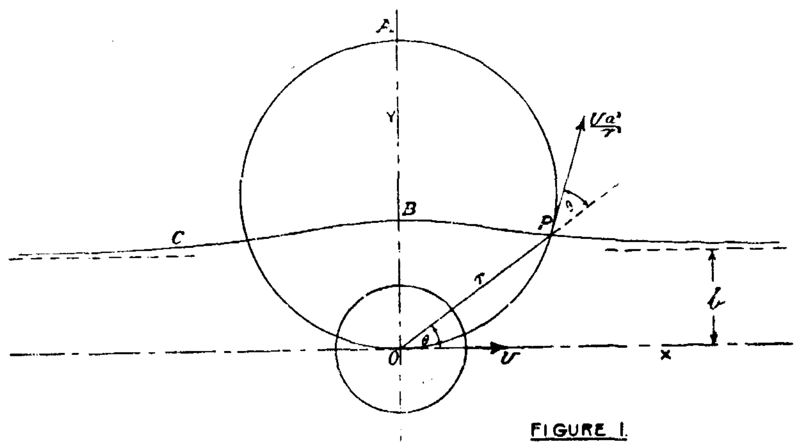


FIGURE I.

The fluid at any point $P(r,\theta)$ is moving with velocity ua^2/r^2 in a direction making an angle 2θ with Ox, that is tangentially to a circle through P touching the axis of x at O. Thus the fluid at points on a circle such as OPA is moving tangentially to the circle at each point at a given instant. This solution gives the actual velocity of the fluid at any point at any time; it does not follow the motion of a given particle of the fluid. If we fix attention upon a fluid

particle at P in Fig. 1, we could trace its path relatively to the cylinder by superposing on its actual motion a backward velocity u parallel to Ox. It can be shown that this relative path is a curve PBC whose equation in terms of y and r is

$$y\left(1-\frac{a^2}{r^2}\right)=b, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where b is the distance of the particle from the axis Ox when at an infinite distance before or behind the cylinder. These curves, given by different values of b, would be the actual paths of the fluid particles if the cylinder were at rest and the fluid were streaming past it. In the case under consideration the cylinder is moving and the fluid is at rest at infinity; hence the actual path of a particle may be imagined as the path in (1) referred to axes moving with uniform velocity u. The equation of the path was first obtained by Rankine[1] in the form of a relation between the ordinate y and the radius of curvature ρ. It can be deduced from Fig. 1.

We have $\rho d(2\theta)/dt =$ velocity of particle at P $= ua^2/r^2$. By writing down the velocity of P relative to O in a direction at right angles to OP we have

$$r\frac{d\theta}{dt}=u\sin\theta+\frac{ua^2}{r^2}\sin\theta.$$

From these two equations, with y for $r\sin\theta$, we obtain $2\rho y(1+a^2/r^2)=a^2$. But relatively to the cylinder the particle lies on the curve given by equation (1) above; hence, substituting for a^2/r^2 we find the result

$$\frac{1}{\rho}=\frac{4}{a^2}(y-\tfrac{1}{2}b) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As Rankine pointed out, this represents in general a case of the 'elastic curve'; and, in fact, the path of a particle is one loop of a coiled elastica. We can complete now the solution of (2). For any given particle, fixed by the value of b, if θ

[1] W. J. M. Rankine, *Phil. Trans. A.*, vol. 154, p. 369, 1864. The result is erroneously attributed to Maxwell in the article on hydromechanics in the *Encyclopædia Britannica*, 11th ed.

is the angle between the tangent to the path and the axis Ox, we have on integrating (2)

$$y = \tfrac{1}{2}b + (\tfrac{1}{4}b^2 + \tfrac{1}{2}a^2 + \tfrac{1}{2}a^2 \cos\theta)^{\frac{1}{2}} \tag{3}$$

We choose the axis Oy so that x is zero when y is a maximum, and measure the arc s from this point. Then

$$x = \left(1 - \frac{2}{k^2}\right)s + \frac{a}{k}\mathrm{E}am\left(\frac{2s}{ka}\right);\ y = \tfrac{1}{2}b + \frac{a}{k}dn\left(\frac{2s}{ka}\right); \tag{4}$$

where the modulus of the elliptic functions is $k=(1+b^2/4a^2)^{-\frac{1}{2}}$.

In terms of elliptic integrals which are usually tabulated, namely $\mathrm{F}(k,\phi)$ and $\mathrm{E}(k,\phi)$ and the corresponding complete integrals K and E, we find the following results: the letters refer to the symmetrical curve in fig. 2.

(i) At the point B. $y = \frac{1}{2}b + a/k$; $\rho = \frac{1}{4}ka$.

(ii) At C, the widest part of the loop. $y = \frac{1}{2}b + \frac{a}{k}\sqrt{(1-k^2)}$;

$$x = \tfrac{1}{2}ka\left\{\left(1 - \frac{2}{k^2}\right)\mathrm{F}(k, 45°) + \frac{2}{k^2}\mathrm{E}(k, 45°)\right\};$$

$$\rho = \tfrac{1}{4}ka/\sqrt{(1 - \tfrac{1}{2}k^2)}.$$

(iii) At an end point A. $y = b$; $x = \frac{1}{2}ka\left\{\left(1 - \frac{2}{k^2}\right)\mathrm{K} + \frac{2}{k^2}\mathrm{E}\right\}$;

$$\rho = a^2/2b.$$

These data are generally sufficient for drawing the curve with considerable accuracy. From the periodicity of the elliptic functions we can also write down the total length of the path ABCD; it is equal to $k\mathrm{K}a$. As a numerical example, one finds that the total distance covered by a particle initially at a distance a from the axis, as the cylinder moves from an infinite distance behind to an infinite distance in front of the particle, is approximately $2a$; this is the curve denoted by 1·0 in Fig. 2. It need hardly be pointed out that although the limiting length of path is finite, the time involved becomes infinite.

In Fig. 2, some curves are drawn for various values of the ratio of b to a; except in one case, only half of each complete path is shown. For b zero, the path is infinite in length and is given by

$$x = a\tanh(2s/a) - s;\ y = a\,\mathrm{sech}(2s/a).$$

The cross-marks on the curves indicate the spaces covered in successive equal intervals of time by particles which were simultaneously at similar points (B) of their paths.

2.—With the help of these curves we can trace the changes in any line containing always the same particles. For this purpose we draw the relative stream lines given by (1), for the same values of b/a as are shown in Fig. 2. We superpose this diagram on Fig. 2, with the axes of x coinciding, and draw a curve through the intersections of corresponding actual and relative paths; displacing one diagram parallel to the direction of motion, we mark again the intersections and obtain the displaced position of the same set of particles. For instance, with the actual paths as in Fig. 2, we obtain by this method the successive positions of a line of particles which at some instant formed a straight line abreast of the cylinder; these curves are shown for one quadrant in Fig. 3.

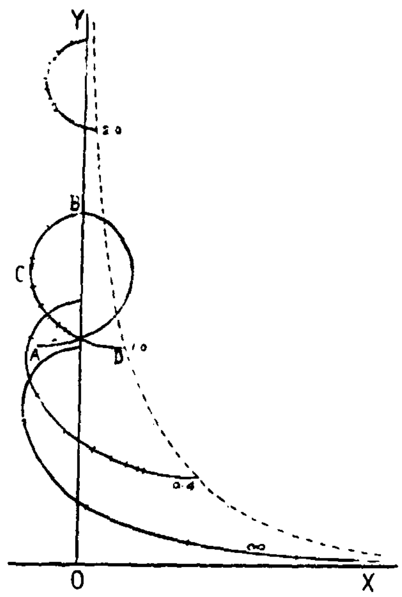


FIGURE 2

The diagram can also be described in another manner. The cylinder moves from left to right. At the instant represented in Fig. 3, AB is a line of particles abreast of the cylinder; the successive curves to the left are the present positions of particles which were abreast of the cylinder at certain equal intervals of time previously. The unit of time T is that taken by the cylinder to move through one-quarter of its diameter. Thus the curve $C^1D^1E^1$ represents the present position of particles which were abreast of the cylinder at CDE at a time 5T previously. It may be noticed that the circumference of the cylinder forms part,

in the limit, of one of the relative stream lines; so that the same particles are always in contact with the cylinder, as the ordinary ideal theory requires.

3.—To trace out the deformation of other lines of particles, it is necessary to adjust first the curves in Fig. 2. For instance, to obtain curves which have been drawn by Maxwell, we arrange the paths in Fig. 3 so that the initial points (A) lie in a straight line perpendicular to Ox; then by the same process as before, we obtain the successive forms of a line of particles which lay in a straight line initially

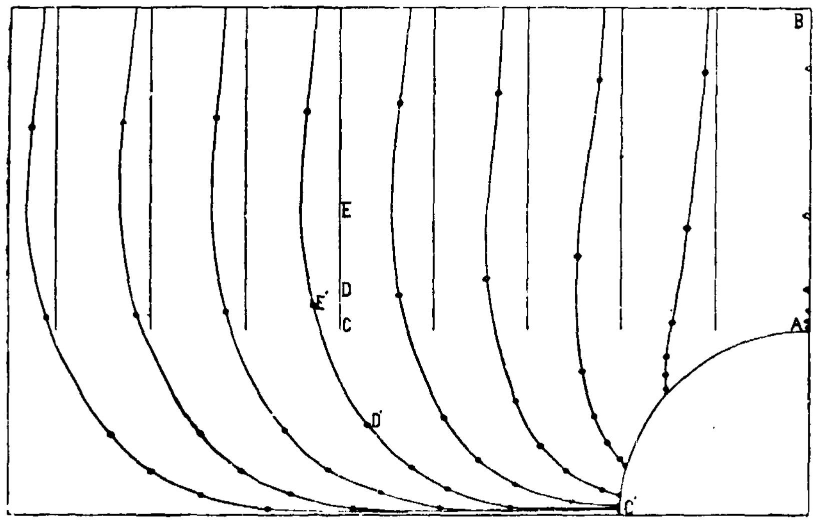


FIGURE 3

when at a great distance in front of the cylinder. We could trace similarly the deformation of groups of particles.

Fig. 4 was obtained by this method; it illustrates the extreme deformation which occurs near the cylinder. Consider for a moment that the cylinder is at rest and the fluid streams past it from left to right. The three enclosed areas, equal in magnitude, are successive positions of the same group of particles.

It has been mentioned already that the ordinary solution of this problem assumes that the fluid is infinitely divisible

into parts retaining the characteristic properties of a fluid. We introduce other limitations when we regard the fluid as made up of a large, but finite, number of particles or molecules which retain their identity during the motion. For such a molecular fluid, it is known that solutions obtained by continuous analysis imply that the molecules move in such a way that their order of arrangement does not alter. Also if we consider a group of molecules forming an element of volume round some point at any time, the same molecules will form an element of volume in the neighbourhood of some other point at any subsequent time; that is, the deformation of an element of volume must be infinitesimal. An inspection of Fig. 4 shows that this condition is not fulfilled in the

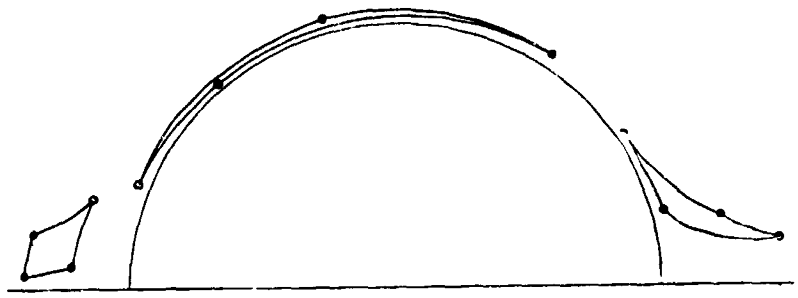


FIGURE 4.

vicinity of the cylinder. One can imagine a curve drawn round the cylinder, not symmetrical fore and aft, within which the conditions are certainly not satisfied. These considerations may help to remove an apparent absurdity. If we examine curves, as Maxwell's, showing the successive forms of lines of particles originally straight in advance of the cylinder, we notice that the cylinder never penetrates through any such line, all of them being looped always round the cylinder. Quite apart from other considerations which enter in the case of an actual fluid, we are relieved from this conclusion by remembering that, on account of molecular constitution alone, there is a region round the cylinder within which the solution obtained by continuous analysis does not represent the true state of motion.

4.—We consider now the paradox to which reference has been made above, returning to the solution of the first two sections. If we imagine the fluid to be contained within a fixed vessel it is clear that, as the cylinder is moved forward, an equal volume of fluid must be displaced backwards. The same argument should hold continuously as we suppose the containing vessel increased indefinitely, and hence in the limit, when we consider motion in an infinite fluid subject to its being at rest at infinity. Thus there should be a permanent displacement of the fluid backwards on the whole. But, according to the paths drawn in Fig. 2, we find that every particle comes to rest ultimately at some point D in advance of its initial position A; so that there appears to be a displacement of the liquid forwards. The interest of this paradox lies partly in its recurrence in various writings. Lanchester[1] states the difficulty and leaves it with the remark: "it is evident that some subtle error must exist in Rankine's argument, the exact nature of which it is difficult to ascertain." Taylor[2] points out how with a finite displacement of the cylinder it can be verified that the fluid is displaced backwards, but with an infinite displacement one has the curious result of a permanent forward displacement. Maxwell[3] raised the same problem many years ago; he admits it as a real difficulty and disposes of it thus: "It appears that the final displacement of every particle is in the forward direction. It follows from this that the condition fulfilled by the fluid at an infinite distance is not that of being contained in a fixed vessel; for in that case there would have been, on the whole, a displacement backwards equal to that of the cylinder forwards. The problem actually solved differs from this only by the application of an infinitely small forward velocity to the infinite mass of fluid such as to generate a finite momentum."

The difficulty arises chiefly from a loose use of the idea of

[1] F.W. Lanchester, *Aerodynamics*, vol. 1, *Aerodonetics*, p. 20, 1909.
[2] D.W. Taylor, *Speed and Power of Ships*, p. 10, 1910.
[3] J.C. Maxwell, *Scientific Papers*, vol. ii., p. 210, 1870.

infinity as if it implied a definite state or time, rather than the limiting value of a process which must be defined precisely in each case; unless this is done the problem is really indeterminate. From this point of view, Maxwell's statement seems inadequate, in that it accepts the forward displacement as definitely proved; on the other hand it points to a root of the matter, namely, the conditions at infinity. Leaving this till later, we discuss the previous solution as it stands, first stating the possibilities in general terms and then treating them analytically.

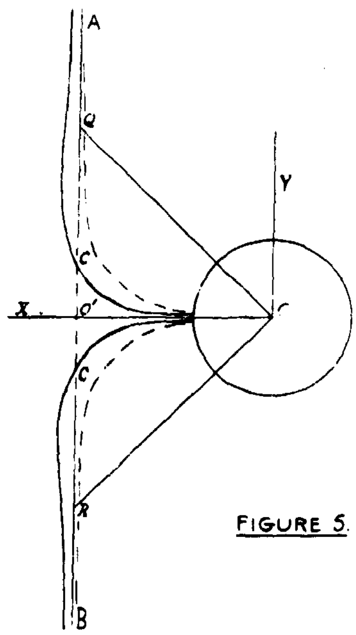


FIGURE 5.

In Fig. 5, O is the centre of the cylinder; the curved line represents the particles which were abreast of the cylinder when the centre was at O^1. The flow of fluid backwards is given by the difference between the areas behind and in front of the line AO^1B. As O^1O increases, the points C move outwards along the line AO^1B: the dotted curve, which is entirely in front of AO^1B, shows the ultimate position of the same particles, according to the paths in Fig. 4, when O^1O becomes infinite.

(A) Let x, y be co-ordinates of any point P on the line AO^1B referred to the centre O. If we fix any value of y, however large, we can make P be within the range O^1C by making x large enough. This is the argument which leads to a permanent forward displacement. It clearly lays more stress on the infinity of extent of liquid fore and aft of the cylinder.

(B) On the other hand, if we fix x, no matter how large, we can make the point P be beyond C on the line O^1A by making y large enough. By giving more weight to the infinity of liquid abreast of the cylinder, this argument

denies that the limit of the dotted curve in Fig. 5 can ever be attained. These two arguments can, of course, be stated simply in terms of the flow of liquid at any instant across a line behind the cylinder. If we draw lines OQ, OR at 45° to the line OO¹, then at any time the flow across AO¹B is forwards in the range QO¹R and backwards beyond Q and R. According to (A), the range QO¹R can be made infinite by taking x large enough; while the argument (B) points to the region within lines at 45° and 90° to the axis OO¹.

Analytically, the matter reduces to the evaluation of a double integral which gives different values according to the order in which the integrations are performed. We can see this by writing down an expression for the total momentum of the whole liquid in the direction of motion of the cylinder.

Referring to Fig. 1, we have $ua^2r^{-2}\cos 2\theta$ for the component fluid velocity at any point, or $ua^2(x^2 - y^2)/(x^2 + y^2)^2$ in terms of rectangular coordinates. Thus with s for the density of the fluid, the total momentum forward is given by

$$\mathrm{M} = ua^2s\iint\frac{x^2 - y^2}{(x^2 + y^2)^2}dxdy,$$

where the integration extends throughout the fluid.

We divide the integration into two parts, writing f for $(x^2 - y^2)/(x^2 + y^2)^2$. First, the region abreast of the cylinder, extending to infinity in both directions, gives without ambiguity

$$\mathrm{M}_1 = 4ua^2s\int_0^a dx\int_{\sqrt{a^2-x^2}}^{\infty} fdy = -\pi sa^2u.$$

For the rest of the fluid, fore and aft of the cylinder, we have

$$\mathrm{M}_2 = 4ua^2s\iint fdxdy,$$

where x ranges from a to ∞, and y from 0 to ∞.

The integral M_2 has different values according to the order in which the integrations are performed. We have

$$\mathrm{M}_2 = 4ua^2s\int_0^{\infty} dy\int_a^{\infty} fdx = 2\pi sa^2u.$$

This evidently corresponds with argument (A) above. Adding M_1 and M_2, we have a total momentum forwards of $\pi s a^2 U$ and this agrees with the permanent forward displacement.

On the other hand, we have

$$M_2 = 4ua^2 s \int_a^\infty dx \int_0^\infty f dy = 0.$$

This is the argument (B), and it gives a total momentum of $\pi s a^2 u$ backwards.

We may write the integral M_2 as a limit in the form

$$M_2 = \operatorname*{Lim}_{b,\, c \to \infty} 4ua^2 s \int_a^b dx \int_0^c f dy.$$

In this integral with b, c finite and a not zero, the order of integration may be inverted without changing the value: we have in either case

$$M_2 = 4ua^2 s \operatorname*{Lim}_{b,\, c \to \infty} \left(\tan^{-1}\frac{b}{c} - \tan^{-1}\frac{a}{c}\right).$$

This form brings out the indeterminateness of the problem, for the limit can have any value we please between $\pi/2$ and O according to the limiting value of the ratio b/c. The argument (A) above supposes that b and c are both infinite in such a way that b is infinitely greater than c; while we obtain the result of argument (B) by supposing c/b infinite in the limit. Another special case would be to suppose b and c to become indefinite in a ratio of equality. Then M_2 is $\pi s a^2 u$ and the total momentum of the fluid is zero. In this case we picture the fluid as of equally infinite extent in and at right angles to the line of motion. Up to the present we have taken the solution of the fluid motion without considering the conditions under which it was obtained. These included the condition that the fluid should be at rest at infinity, that is, the velocity should become infinitesimal as the distance from the cylinder increased indefinitely. If we could imagine the fluid to be contained in a fixed boundary at infinity, the condition to be satisfied there would be the vanishing of the normal component of velocity. At first sight, there would not seem to be much

difference between the two cases, the latter being included in the former. But we have seen that it is necessary to define conditions more precisely in order to avoid ambiguity. We may illustrate this by a definite problem in initial motion for which the solution is known.

Let the fluid be contained within a fixed concentric cylinder of radius c, and let the inner cylinder be suddenly started with velocity u. If ϕ is the velocity potential of the initial fluid motion, the boundary condition at the outer cylinder is that $d\phi/dr$ should be zero. The value of ϕ is

$$\phi = \frac{ua^2}{c^2 - a^2}\left(r + \frac{c^2}{r}\right)\cos\theta.$$

The second part of ϕ represents the fluid motion already studied, with an additional factor $c^2/(c^2-a^2)$. Superposed on this there is a uniform flow backwards of amount $ua^2/(c^2-a^2)$. The total momentum can be found by integrating throughout the liquid as before. In this case there is no ambiguity and it is easily shown that the second term in ϕ contributes nothing to the momentum. Adding the part due to the uniform flow, we find the total fluid momentum to be $\pi s a^2 u$ backwards; this result is independent of the radius of the outer cylinder, and, of course, agrees with elementary considerations.

Now suppose the radius c to become infinite. The fluid motion then differs from that studied in the previous sections only by a superposed uniform flow backwards of infinitesimal magnitude; but when integrated through the infinite extent of liquid it gives use to a finite momentum $\pi s a^2 u$ backwards. Further, in any finite time the additional term makes no more than an infinitesimal difference to the paths of the particles; but if we attempted to extend the solution to "infinite" time we should be faced with various ambiguities in making any allowance for the extra term.

The velocity potential ϕ for a finite extent of fluid is determinate when the values of ϕ or $\delta\phi/\delta n$ are given over all the boundaries. If the outer boundary becomes infinite

and ϕ is said to vanish at infinity, the solution is indeterminate by an infinitesimal amount. Consequently the total momentum or flow may be indeterminate to a finite amount. On the other hand, the total kinetic energy of the fluid motion, involving a summation of the square of the velocity, is only indeterminate to an infinitesimal extent.

In conclusion, it appears that the problem is indeterminate unless the infinite boundary of fluid can be defined as the limit of some particular form, and further in that case the conditions satisfied at the boundary must also be considered. At the best the question of what happens in an infinite fluid after an infinite time leads to unreal difficulties; the above discussion may serve to show in what way these arise when we attempt to force to this extent ordinary solutions which give consistent results when treated in a legitimate manner.

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Ship Resistance: The Wave-making Properties of Certain Travelling Pressure Disturbances.

By T. H. HAVELOCK, M.A., D.Sc., Armstrong College, Newcastle-on-Tyne.

(Communicated by Sir J. Larmor, F.R.S. Received October 7,—Read November 27, 1913.)

1. In previous papers* I have investigated the wave-making resistance of a ship by comparing it with a certain simple type of pressure disturbance travelling over the surface of the water. In a recent paper† on the effect of form and size on the resistance of ships, by Messrs. Baker and Kent of the National Physical Laboratory, reference is made to this point of view. The main work of these authors consists in the examination of model results and the deduction of empirical formulæ of practical value. In addition, they connect the wave-making properties with the pressure distribution and have obtained graphs of the latter for various ship forms under certain conditions; these curves show a range of negative pressure, or defect of pressure, between the positive humps of excess pressure corresponding to the bow and stern. The authors remark that this will have an effect upon the wave-making, but conclude that it is sufficient for their purpose to be able to state that such pressure disturbances, as they have shown to exist when a ship is in motion, will produce waves which will vary more or less in accordance with the theory referred to above.

Under the circumstances it seems advisable to extend the mathematical

* 'Roy. Soc. Proc.,' 1909, A, vol. 82, p. 276; *also* 1910, A, vol. 84, p. 197; *also* 'Proc. Univ. Durh. Phil. Soc.,' 1910, vol. 3, p. 215.

† G.S. Baker and J. L. Kent, Trans. Inst. Nav. Arch., vol. 55(ii), p. 37 (1913).

theory by working out the wave-making properties of other distributions of pressure. Although no attempt has been made to connect the distribution directly with ship form, the following examples have been chosen with a view to general inferences which can be drawn in this respect. In particular, the distributions graphed by Baker and Kent can be represented, in type at least, by a mathematical expression for which the corresponding Fourier integral can be evaluated, so that one can compare the result with that obtained from simpler forms. Although the expression for the wave-making resistance becomes more complicated, it is not essentially different from that obtained previously; it appears in general to be built up of terms involving the same type of exponential e^{-a/v^2}, together with oscillating factors representing interference effects between prominent features of the pressure distribution.

2. We confine our attention to two-dimensional fluid motion. We may imagine it to be produced in a deep canal of unit breadth, with vertical sides, by the horizontal motion of a floating pontoon with plane sides fitting closely to the walls of the canal but without friction. We assume that, as regards transverse wave-making, this is effectively equivalent to some travelling distribution of pressure impressed upon the surface of the water.

Let Ox be in the direction of motion of the disturbance, and let y be the surface elevation of the water. Suppose the distribution of pressure to be given by

$$p = f(x). \tag{1}$$

For a line distribution we may suppose the disturbance to be inappreciable except near the origin and to be concentrated there in such a manner that the integral pressure P is finite, where

$$\mathrm{P} = \int_{-\infty}^{\infty} f(x)\, dx. \tag{2}$$

When this disturbance moves along the surface of water, of density ρ, with velocity v, the main part of the surface disturbance consists of a regular train of waves in the rear given by

$$g\rho y = -2\kappa \mathrm{P} \sin \kappa x, \tag{3}$$

where the length λ of the waves is

$$\lambda = \frac{2\pi}{\kappa} = \frac{2\pi v^2}{g}.$$

We can generalise this result for any form of pressure distribution $f(x)$, which is likely to occur, by the Fourier method. We have in general

$$g\rho y = -2\kappa \int_{-\infty}^{x} f(\xi) \sin \kappa (x-\xi)\, d\xi \tag{4}$$

$$= -2\kappa (\phi \sin \kappa x - \psi \cos \kappa x), \tag{5}$$

where $$\phi = \int_{-\infty}^{\infty} f(\xi)\cos\kappa\xi\,d\xi, \qquad \psi = \int_{-\infty}^{\infty} f(\xi)\sin\kappa\xi\,d\xi. \tag{6}$$

The mean energy per unit area of the wave motion given by (5) is $2\kappa^2(\phi^2+\psi^2)/g\rho$. Now the head of the disturbance advances with velocity v, while the rate of flow of energy in the train of waves is the group velocity $\frac{1}{2}v$; hence the net rate of gain of energy per unit area is $\frac{1}{2}v$ times the above expression for the energy. If we equate this product to Rv, then R may be called the wave-making resistance per unit breadth; and we have

$$R = \kappa^2(\phi^2+\psi^2)/g\rho. \tag{7}$$

We have in each case to evaluate the complex integral

$$\chi = \phi + i\psi = \int_{-\infty}^{\infty} f(\xi)\,e^{i\kappa\xi}\,d\xi. \tag{8}$$

In the examples which follow, the integral has a finite, definite value which can be obtained in Cauchy's manner by integrating round a closed simple contour in the plane of the complex variable ξ. The function $f(\xi)$ is such that (i) it has no critical points other than simple poles in the semi-infinite plane situated above the real axis for ξ; (ii) it has no critical points on the real axis; and (iii) its value tends to zero as ξ becomes infinite. Further, the quantity κ is restricted to real, positive values. Under these conditions it can be shown* that

$$\int_{-\infty}^{\infty} f(\xi)\,e^{i\kappa\xi}\,d\xi = 2\pi i\Sigma A,$$

where ΣA is the sum of the residues of the integrand at the poles of $f(\xi)$ situated above the real axis. If a is a pole, A is given by the value of $(\xi-a)f(\xi)e^{i\kappa\xi}$ when $\xi = a$. Alternatively, in the following examples $f(\xi)$ is of the form $F(\xi)/G(\xi)$, none of the zeros of $G(\xi)$ coinciding with those of $F(\xi)$, and A is given by $F(a)e^{i\kappa a}/G'(a)$.

3. For the sake of comparison the results which have been obtained previously may be repeated briefly. If

$$p = f(\xi) = \frac{A}{\xi^2+\alpha^2}, \tag{9}$$

the poles are at $\xi = \pm i\alpha$, of which the positive one alone concerns us. Hence we have

$$\chi = \int_{-\infty}^{\infty} \frac{Ae^{i\kappa\xi}}{\xi^2+\alpha^2}\,d\xi = 2\pi i \left| \frac{Ae^{i\kappa\xi}}{\xi+i\alpha} \right|_{\xi=i\alpha} = \frac{\pi}{\alpha} Ae^{-\alpha\kappa}. \tag{10}$$

Hence from (7) $$R = \frac{\pi^2A^2}{g\rho\alpha^2}\kappa^2 e^{-2\alpha\kappa} = \frac{\pi^2 gA^2}{\rho\alpha^2 v^4}e^{-2g\alpha/v^2}.\dagger \tag{11}$$

* Jordan, 'Cours d'Analyse,' vol. 2, § 270.

† *Cf.* Lamb, 'Hydrodynamics,' 1932 edn. p. 415.

If A is a constant and independent of the speed v, the graph of R as a function of v rises to a maximum and then falls slowly but continually to zero as v increases indefinitely. Thus, for an assigned pressure disturbance of this type whose magnitude is independent of the speed, there is a certain speed beyond which the resistance R continually decreases.

On the other hand, if the pressure disturbance is that produced by the motion of a floating, or submerged, body, it is clear that it will depend upon the speed. Since we may suppose the pressures in question to be the excess or defect of pressure due to the speed, it seems a plausible first approximation to assume that the distribution is not altered appreciably in type and that the magnitude is proportional to v^2. Thus if in (11) we make A proportional to v^2 we obtain

$$R = \text{const.} \times e^{-2ga/v^2}. \tag{12}$$

The value of R now tends to a finite limiting value as v increases indefinitely.

If the quantity A, specifying the magnitude of the pressure disturbance, varies as v^n, then the graph of R rises to a maximum for some finite value of v, provided n is positive and less than 2; the nearer n is to 2 the higher is the speed at which the maximum occurs. For the present we assume that n is equal to 2; in any case it does not affect the results of a qualitative comparison of different types of distribution.

The scope of the assumption may be illustrated by a certain case. Prof. Lamb* has worked out directly the wave-making resistance R due to a circular cylinder of small radius a, submerged with its centre at a constant depth f, and moving with uniform velocity v; he finds that R varies with the speed according to the law $v^{-4}e^{-2gf/v^2}$. If we attempt to represent the disturbance approximately by some equivalent surface pressure distribution, the type which suggests itself naturally is

$$p = A(f^2-x^2)/(f^2+x^2)^2.$$

It can be shown† that this distribution, together with the assumption that A is proportional to v^2, leads to the same law of variation of resistance with speed.

4. In a certain sense the generalisation from a line disturbance to any diffused distribution of pressure may be regarded analytically as a case of interference; the final result is due to the mutual interference of the line elements into which we may analyse the given distribution. However, the idea of interference in ship waves has usually been associated, after the work

* H. Lamb, 'Ann. di Matematica,' vol. 21, Ser. 3, p. 237.
† 'Roy. Soc. Proc.,' A, vol. 82, p. 300.

of W. Froude, with the superposition of bow and stern wave-systems, that is, when the whole system may be separated into two fairly distinct parts. I have represented this previously by a positive pressure system of type (9) associated with the bow, followed by a similar negative system associated with the middle of the run. Thus if l is the distance between the centres of the two systems we have in the present notation

$$p = \frac{A_1}{(\xi - \frac{1}{2}l)^2 + \alpha^2} - \frac{A_2}{(\xi + \frac{1}{2}l)^2 + \alpha^2}. \tag{13}$$

Substituting in (8) and evaluating the integrals we find

$$\phi = (A_1 - A_2)\, e^{-\alpha\kappa} \cos \tfrac{1}{2}\kappa l, \qquad \psi = (A_1 + A_2)\, e^{-\alpha\kappa} \sin \tfrac{1}{2}\kappa l.$$

Hence from (7)

$$g\rho R = \kappa^2 (A_1^2 + A_2^2 - 2 A_1 A_2 \cos \kappa l)\, e^{-2\alpha\kappa}. \tag{14}$$

The graph of R is a mean curve similar to (12) with oscillations superposed upon it, humps and hollows corresponding to minima and maxima of $\cos \kappa l$ or $\cos (gl/v^2)$.

It is of interest to note that if A_1 and A_2 are equal, we have

$$R = \text{const.} \times e^{-2\alpha\kappa} \sin^2 \tfrac{1}{2}\kappa l. \tag{15}$$

Thus, in a hollow, R would be actually zero if the two pressure systems were equal in magnitude. This, of course, follows at once from general principles; if we have a pressure system followed at a fixed distance by an equal and similar system, then there are certain wave-lengths and corresponding speeds for which the main regular waves due to the two systems cancel each other out exactly. A moving body which would produce such a state of affairs would be, in Lord Kelvin's phrase, a waveless pontoon. Of course, this does not occur in ship forms, and there are several reasons why it could not be expected to do so. In fact we have in general to suppose A_2 much less than A_1 in (13). However, it is conceivable that some change of form might give more effective interference effects of this kind and so deepen the hollows in the resistance curve, though possibly as a practical suggestion it may be subject to the same limitation as in other cases, namely, even if the wave-making resistance were lessened in this way probably the alterations would so increase frictional and other resistances that there might be no gain on the whole.

5. Baker and Kent have pointed out that in certain cases the pressure distribution at the entrance of a ship form is not simply a hump of excess pressure, but is a hump followed by a hollow of negative pressure. They assign to the interference of these two parts a certain subsidiary interference effect in the resistance which may become important when it coincides with

one due to the bow and stern systems. This follows on general grounds, and might be represented analytically as in § 4, but it is worth while examining other distributions with this character.

In the first place consider one which does not give the desired interference effect, namely,

$$p = \frac{A\xi}{\xi^2 + \alpha^2}. \tag{16}$$

The graph has been drawn for certain numerical values of the constants and is curve A in fig. 1.

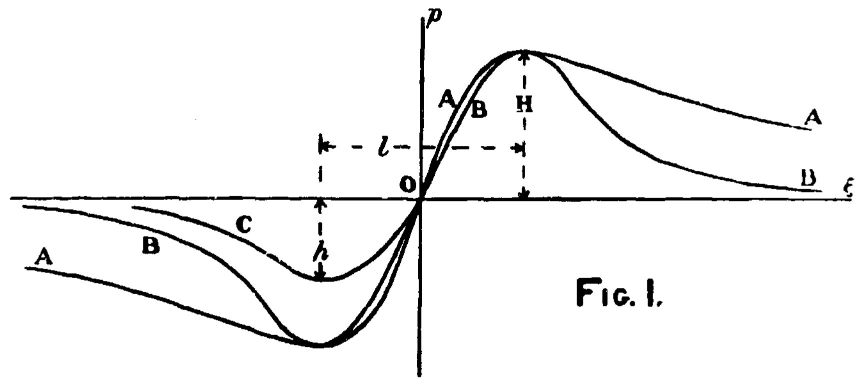

Fig. 1.

We have $$\chi = \int_{-\infty}^{\infty} \frac{A\xi e^{i\kappa\xi}}{\xi^2 + \alpha^2} d\xi = i\pi A e^{-\alpha\kappa}.$$

Hence, from (7) and (8),

$$\phi = 0; \qquad \psi = \pi A e^{-\alpha\kappa};$$

$$g\rho R = \pi^2 \kappa^2 A^2 e^{-2\alpha\kappa}. \tag{17}$$

We have here the same form for R as a function of v as in (11) for the single hump of positive pressure; we do not get the interference effect which might have been expected. This may be explained by remarking that the pressure falls away from the maximum only slowly; in other words, the hump and hollow are not sufficiently pronounced for their individuality to show directly in the final formula. In the previous section, where the distribution is $1/(\xi^2 + \alpha^2)$ instead of $\xi/(\xi^2 + \alpha^2)$, the maximum and minimum are more pronounced and we get a typical oscillating term in the final result. This view may be confirmed by another example.

6. Consider

$$p = \frac{A\xi}{\xi^4 + 4\alpha^4}. \tag{18}$$

This distribution is graphed in curve B of fig. 1, arranged so as to have the same minimum and maximum as for (16); the curves A and B illustrate

clearly the difference in question. Numerically, if ξ_0 is the position of the maximum, at $3\xi_0$ the value of p from (16) has fallen to $3/5$ of the maximum, while from (18) it has fallen to $1/7$ of the same value.

The poles of the function in (18) are $\pm(1\pm i)\alpha$; thus from (8) we have

$$\chi = 2\pi i \left| \frac{\xi e^{i\kappa\xi}}{\{\xi+\alpha(1-i)\}\{\xi+\alpha(1+i)\}\{\xi-\alpha(1-i)\}} \right|_{\alpha(1+i)}$$
$$+2\pi i \left| \frac{\xi e^{i\kappa\xi}}{\{\xi-\alpha(1+i)\}\{\xi+\alpha(1+i)\}\{\xi-\alpha(1-i)\}} \right|_{-\alpha(1-i)}$$
$$= \frac{i\pi}{2\alpha^2} e^{-\alpha\kappa} \sin \alpha\kappa.$$

The wave-making resistance, R, is given by

$$4g\rho\alpha^4 \mathrm{R} = \kappa^2\pi^2\mathrm{A}^2 e^{-2\alpha\kappa} \sin^2 \alpha\kappa. \tag{19}$$

We have now the oscillating factor $\sin^2\alpha\kappa$. There will be, for instance, a hump on the resistance curve when $2\alpha\kappa = \pi$, that is, when the half wave-length is equal to 2α. It may be noticed that this is nearly, but not exactly, the distance between the maximum and minimum of p; from (18) it follows that the latter distance is $2\alpha\sqrt[4]{(4/3)}$, or approximately $2{\cdot}15\alpha$.

We also have R exactly zero in the hollows in the resistance curve, a result which follows from the numerical equality of the positive and negative pressures at equal distances from the origin. We can make the negative pressures less by considering an unsymmetrical distribution.

7. Let the pressure be

$$p = \frac{\xi}{\xi^4 - \beta\xi + 4\alpha^4}. \tag{20}$$

In this case the graph would be as in fig. 1, with the curve B for positive ξ and the curve C for negative values.

If the poles of (20) are $a_1 \pm ib_1$ and $a_2 \pm ib_2$, we have

$$\left.\begin{aligned} a_1 + a_2 &= 0, \\ a_1^2 + b_1^2 + a_2^2 + b_2^2 + 4a_1a_2 &= 0, \\ 2\{a_1(a_2^2 + b_2^2) + a_2(a_1^2+b_1^2)\} &= \beta, \\ (a_1^2+b_1^2)(a_2^2+b_2^2) &= 4\alpha^4. \end{aligned}\right\} \tag{21}$$

In forming the function χ by the previous method we have two parts. The part for the pole a_1+ib_1 is

$$2\pi i \left| \frac{\xi e^{i\kappa\xi}}{\{\xi-(a_1-ib_1)\}\{\xi-(a_2+ib_2)\}\{\xi-(a_2-ib_2)\}} \right|_{a_1+ib_1}$$
$$= \frac{\pi}{b_1} \cdot \frac{(a_1+ib_1)e^{i\kappa a_1}e^{-\kappa b_1}}{(a_1-a_2)^2-(b_1^2-b_2^2)+2ib_2(a_1-a_2)}. \tag{22}$$

There is also a similar expression corresponding to the pole a_2+ib_2: from (21) we see that the result can be written in the form

$$\chi = (A_1+iB_1)e^{i\kappa a_1-\kappa b_1}-(A_2+iB_2)e^{i\kappa a_2-\kappa b_2}.$$

Hence for the resistance we have, from (7),

$$g\rho R/\kappa^2 = (A_1^2+B_1^2)e^{-2b_1\kappa}+(A_2^2+B_2^2)e^{-2b_2\kappa}$$
$$-2\{(A_1A_2+B_1B_2)\cos 2a_1\kappa-(A_2B_1-A_1B_2)\sin 2a_1\kappa\}e^{-(b_1+b_2)\kappa}. \quad (23)$$

We notice how the presence of the smaller negative pressure complicates the mathematical expressions. On the other hand, all the terms are of the same type as in simpler cases; we have three terms involving the same exponential function, the third having an oscillating factor $\cos(2\kappa a_1+\epsilon)$, where

$$\tan\epsilon = (A_2B_1-A_1B_2)/(A_1A_2+B_1B_2). \quad (24)$$

The humps and hollows on the curve for R will not coincide exactly with those obtained by graphing

$$e^{-2\kappa(b_1+b_2)}\cos(2a_1\kappa+\epsilon), \quad \text{with} \quad \kappa = g/v^2,$$

but the agreement will be sufficiently close for present purposes.

Accordingly, the maxima for R will be near speeds for which

$$2a_1\kappa+\epsilon = n\pi; \qquad n = 1, 3, 5, \ldots.$$

The corresponding speeds and wave-lengths are given by

$$v^2 = \frac{2ga_1}{n\pi-\epsilon}; \qquad \lambda = \frac{4\pi a_1}{n\pi-\epsilon}. \quad (25)$$

In the previous case of symmetry, with the result in (19), the humps occur at wave-lengths $4a/n$, that is when the wave-length is equal to or an odd sub-multiple of a certain length; a similar statement in terms of velocity brings in the series 1, $1/\sqrt{3}$, $1/\sqrt{5}$, etc. In the present case we see from (24) that this arrangement is somewhat disturbed by the presence of the phase ϵ, a quantity which may possibly be small compared with π. A complete algebraical study might be made, but possibly a simpler way would be to start from a graph of the pressure curve and carry out the integrations involved in (8) by graphical methods. We can also obtain information by working out some numerical examples; one may suffice at present namely,

$$p = \frac{\xi}{\xi^4-180\xi+2419}. \quad (26)$$

The pressure curve is of the form BC, shown in fig. 1, with

$$h/H = 0{\cdot}541; \qquad l = 10{\cdot}66.$$

Further with the previous notation,

$$a_1 = -a_2 = 5, \qquad b_1 = 4, \qquad b_2 = \sqrt{34}.$$

Working out the numerical values from (22) we obtain for the resistance, omitting a constant factor,

$$R = 516e^{-8\kappa} + 353e^{-11\cdot 66\kappa} - 857\cos(10\kappa - \epsilon)e^{-9\cdot 83\kappa}, \tag{27}$$

with $\kappa = g/v^2. \qquad \tan\epsilon = 0\cdot 017.$

We verify that in this case ϵ is, in fact, very small, consequently the simple relation between speeds at which there are humps is not appreciably altered. The absolute position of these humps on the R,v curve may be slightly displaced. For instance, the final hump occurs when 10κ is equal to π, that is when the half wave-length is equal to 10; on the other hand, the distance l between the maximum and minimum on the pressure curve is $10\cdot 66$ units.

8. We turn now to more complicated distributions of pressure similar to those obtained by Baker and Kent, to which reference has already been made. We can build up a rational algebraic fraction which has at least the salient features of these curves; for instance, the graph of fig. 2 is represented by

$$p = \frac{2\xi^2 - \lambda^2 - \mu^2}{\xi^4 - (\lambda^2 + \mu^2)\xi^2 + \frac{1}{2}(\lambda^4 + \mu^4)}, \tag{28}$$

where λ and μ are constants. We have, on the curve,

$$OA = \lambda, \quad OB = \sqrt{[\tfrac{1}{2}(\lambda^2 + \mu^2)]}, \quad OC = \mu, \quad AE = CD = 2/(\mu^2 - \lambda^2),$$
$$OF = 2(\lambda^2 + \mu^2)/(\lambda^4 + \mu^4).$$

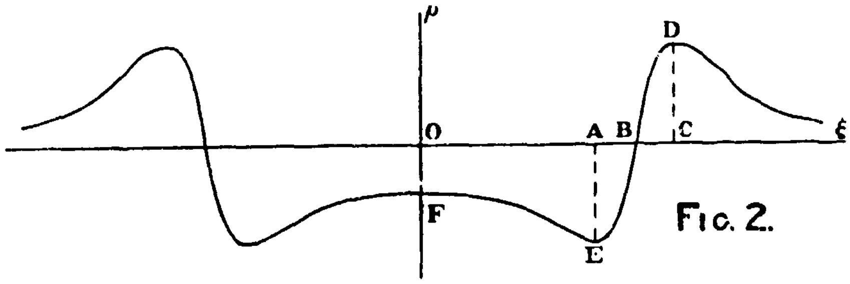

FIG. 2.

With different values of λ and μ, one could obtain variations in the relative prominence of OF compared with CD, and in other features.

If the roots of the denominator in (28) are $\pm(a \pm ib)$, we have

$$\left.\begin{aligned} 2(a^2 - b^2) &= \lambda^2 + \mu^2, \\ 2(a^2 + b^2)^2 &= \lambda^4 + \mu^4. \end{aligned}\right\} \tag{29}$$

Using these relations in evaluating the integral χ, we obtain

$$\chi = 2\pi i \left| \frac{2(\xi^2 - a^2 + b^2)e^{i\kappa\xi}}{\{\xi - (a - ib)\}\{\xi + (a + ib)\}\{\xi + (a - ib)\}} \right|_{a+ib}$$
$$+ 2\pi i \left| \frac{2(\xi^2 - a^2 + b^2)}{\{\xi - (a + ib)\}\{\xi - (a - ib)\}\{\xi + (a + ib)\}} \right|_{-a+ib}$$

On simplification this leads to

$$\chi = 2\pi e^{-b\kappa}(b\cos\kappa a - a\sin\kappa a)/(a^2+b^2).$$

Hence from (7) the corresponding resistance is given by

$$(a^2+b^2)g\rho R = 4\pi^2\kappa^2 e^{-2b\kappa}\sin^2(\kappa a-\epsilon), \tag{30}$$

with $$\tan\epsilon = b/a.$$

We have in (30) a form very similar to those we have already studied. The phase ϵ means a bodily displacement of the series of humps and hollows; but, again, ϵ is small under the usual circumstances, when the difference between μ and λ is small compared with either.

Further, because of the symmetry of the distribution fore and aft, there are values of κ, with corresponding speeds, for which R is zero; we have seen that to avoid this result we must suppose the magnitude of the pressure p to be less in the vicinity of the run than at the entrance. We could introduce this want of symmetry by considering

$$p = \frac{\xi^2 - c^2}{\{(\xi-a_1)^2+b_1^2\}\{(\xi-a_2)^2+b_2^2\}} \tag{31}$$

In the expression for the integral χ we should have a part corresponding to each of the poles a_1+ib_1 and a_2+ib_2; in consequence, the resistance R would be similar in form to the expression in (23).

From (30) we notice that the wave-lengths corresponding to humps on the resistance curve are submultiples of $2a$; also when λ and μ are nearly equal, $2a$ is of the order 2μ, the distance between the two positive pressure humps. The typical interference effects in this example are due to the interference of the bow and stern systems; in order to get a secondary interference effect between the positive and negative parts at the bow these must have separate individuality to a greater degree, as we saw in § 5. For instance, we could consider two distributions like (20), one associated with the entrance, the other reversed and associated with the run; we should then have a very general type of distribution represented by

$$p = \frac{A_1(\xi-\frac{1}{2}l)}{(\xi-\frac{1}{2}l)^4-\beta_1(\xi-\frac{1}{2}l)+4\alpha_1^4} - \frac{A_2(\xi+\frac{1}{2}l)}{(\xi+\frac{1}{2}l)^4+\beta_2(\xi+\frac{1}{2}l)+4\alpha_2^4}. \tag{32}$$

It is unnecessary to graph this or to put down expressions for χ and R. We should obtain a sum of expressions like (23) involving sines and cosines of $2\kappa a_1$ and of $2\kappa a_2$, and, in addition, of κl. There would be in general various possibilities of subsidiary interference effects; the main one would be the bow and stern interference represented by κl, and the next in importance that between the positive and negative parts at the bow represented by $2\kappa a_1$. There would also be the possibility of these two effects adding together at certain speeds.

9. One could obtain more maxima, or increased waviness, in the pressure curve by introducing higher powers of ξ into the fractions we have used. With the same general method for evaluating the integral χ it follows that we should obtain expressions of the same type, only more complicated in form.

The various examples which have been studied cover a wide range of distributions of the type which one would expect to be associated with the motion of a ship, in respect to the formation of transverse waves. It may be said that the corresponding resistance curves do not differ essentially from those obtained from a simple distribution, only with the introduction of additional coefficients there is possible a wider range of variation.

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The Initial Wave Resistance of a Moving Surface Pressure.

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1. The study of the water waves produced by the motion of an assigned pressure distribution over the surface has hitherto been limited to the steady state attained when the system has been moving with uniform velocity for a very long time. In his latest series of papers on water waves, Lord Kelvin* made an elaborate graphical and numerical study of cognate problems, and expressed the hope of applying his methods to calculate the initiation and continued growth of canal ship-waves due to the sudden commencement and continued application of a moving, steady surface pressure.

In the following paper, I have not attempted any analysis of the surface elevation itself, but I have proceeded directly to the calculation of the corresponding wave resistance. At present the wave resistance is known only for the steady state for certain localised pressure systems in uniform motion, and it seems desirable to attempt some estimate of the time taken to attain this state when we take into account the beginnings of the motion. One might examine the effect of initial acceleration, but I have limited the problem by considering only the case of a system which is suddenly established, and is at the same instant set in motion with uniform velocity.

* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 456 (1906).

The work is arranged in the following order: a general expression for the wave resistance as a function of the time, an exact solution for a certain waveless system, a comparison of this solution and the group approximation, and an approximate solution for certain systems which leave regular waves in their rear.

2. Consider, first, the effect of a single impulse applied to the surface of deep water, with no initial displacement of the surface. Take the axis of y vertically upwards, the axis of x horizontal, and the origin in the undisturbed surface. If the impulse is given by $\mathrm{F}(x)$, and if the Fourier method is applicable, the elevation at any time t is given by

$$-\pi g \rho y = \int_0^\infty \kappa \mathrm{V} \sin(\kappa \mathrm{V} t)\, d\kappa \int_{-\infty}^{\infty} \mathrm{F}(\alpha)\, e^{i\kappa(x-\alpha)}\, d\alpha, \tag{1}$$

where $\mathrm{V} = (g/\kappa)^{\frac{1}{2}}$, and the real part of the integral is to be taken. The effect of a pressure system, whether stationary or moving, can be obtained by integrating (1) suitably with respect to the time. For the pressure system may be considered as a succession of impulses; to each impulse there corresponds a fluid motion with definite velocity potential, and the velocity potential of the fluid motion at any instant is the sum of the velocity potentials due to all the previous impulses. Similarly, the corresponding surface elevations are simply superposed, and we obtain the required solution by an integration.

For a pressure system moving with uniform velocity c, we have to substitute $x+ct$ for x in (1) and then integrate with respect to t between the limits 0 and t. But the solution so obtained is indeterminate to a certain extent, for we can superpose on it any infinite train of waves of wave velocity c. The so-called practical solution is found by choosing the amplitude of this train so as to annul the main regular waves in front of the travelling system. The integrals are, in fact, indeterminate, and are evaluated by taking their principal value, in Cauchy's sense of the term. Another way of avoiding this difficulty is to introduce small frictional terms proportional to the velocity. The integrals are then determinate, though more complicated in form; however, the final results, after the analysis is completed, can be simplified by taking the frictional coefficient as small as we please. We shall use this method, and it is sufficient for our purpose to write, instead of (1),

$$-\pi g \rho y = \int_0^\infty e^{-\mu c t} \kappa \mathrm{V} \sin(\kappa \mathrm{V} t)\, d\kappa \int_{-\infty}^{\infty} \mathrm{F}(\alpha)\, e^{i\kappa(x-\alpha)}\, d\alpha, \tag{2}$$

where, ultimately, μ is to be considered small.*

* Compare Lamb, 'Hydrodynamics,' 1932 edn. p. 348.

Consider, then, a pressure system

$$p = \mathrm{F}(x), \tag{3}$$

which is suddenly established, and is at the same instant set in motion with uniform velocity c along the axis of x.

Putting $x = \varpi + ct$, the surface elevation at any time t after the start is given by

$$-\pi g \rho y = \int_0^t e^{-\mu c u} du \int_0^\infty \kappa \mathrm{V} e^{i\kappa(\varpi + cu)} \sin(\kappa \mathrm{V} u)\, d\kappa \int_{-\infty}^\infty \mathrm{F}(\alpha) e^{-i\kappa\alpha} d\alpha. \tag{4}$$

For simplicity, we shall confine ourselves to pressure systems which are symmetrical with respect to the origin; so that

$$\phi(\kappa) = \int_{-\infty}^\infty \mathrm{F}(\alpha) e^{-i\kappa\alpha} d\alpha = 2\int_0^\infty \mathrm{F}(\alpha) \cos \kappa\alpha\, d\alpha. \tag{5}$$

Also we shall use only localised distributions for which the integrals are finite and determinate; the systems will be finite and continuous and such that the integral pressure is finite, that is, the integral $\int_{-\infty}^\infty \mathrm{F}(\alpha)\, d\alpha$ convergent. Carrying out the integration with respect to u, we obtain

$$-2\pi g \rho y = \int_0^\infty \kappa \mathrm{V} \phi(\kappa) e^{i\kappa\varpi} \left\{ \frac{1}{\kappa(\mathrm{V}+c) + i\mu c} + \frac{1}{\kappa(\mathrm{V}-c) - i\mu c} \right\} d\kappa$$
$$- e^{-\mu c t} \int_0^\infty \kappa \mathrm{V} \phi(\kappa) e^{i\kappa\varpi} \left\{ \frac{e^{i\kappa(\mathrm{V}+c)t}}{\kappa(\mathrm{V}+c) + i\mu c} + \frac{e^{-i\kappa(\mathrm{V}-c)t}}{\kappa(\mathrm{V}-c) - i\mu c} \right\} d\kappa. \tag{6}$$

The first integral represents the steady state, while the second gives the deviation from it when we take into account the beginning of the motion.

3. From the first integral in (6) we have, with $\kappa_0 = g/c^2$,

$$-\pi g \rho y = \kappa_0 \lim_{\mu \to 0} \int_0^\infty \frac{\kappa \phi(\kappa) e^{i\kappa\varpi} d\kappa}{\kappa(\kappa_0 - \kappa) + \mu^2 - 2\mu\kappa i}. \tag{7}$$

The integral is to be evaluated first, before we make μ zero, otherwise it is indeterminate. The interpretation for certain types of localised pressure system is well known; in such cases the solution takes the form

$$y = f(\varpi), \qquad \varpi > 0,$$
$$y = -\frac{2\kappa_0}{g\rho} \phi(\kappa_0) \sin \kappa_0 \varpi + f(-\varpi), \qquad \varpi < 0. \tag{8}$$

This solution represents an infinite train of regular waves in the rear of the moving system, together with a disturbance symmetrical fore and aft which becomes negligible at a distance depending upon the concentration and the velocity. For our present purpose, all the examples we use are included under the case

$$\phi(\kappa) = \kappa^n e^{-a\kappa}, \qquad n > 0, \qquad a > 0. \tag{9}$$

To verify the solution (8) in this case, regard κ in (7) as a complex variable $re^{i\theta}$.

For ϖ positive, integrate round a sector of radius R bounded by the lines $\theta = 0$ and $\theta = \beta$ $(0<\beta<\frac{1}{2}\pi)$. Under the specified conditions, it can be shown that the integral along the arc $r = R$ tends to zero as R is made infinite. In this way the integral (7) is transformed into an integral, along the line $\theta = \beta$, in which we can make μ zero.

For ϖ negative, integrate round a sector of radius R bounded by the lines $\theta = 0$ and $\theta = \beta$, with $-\tan^{-1}2\mu/\kappa_0>\beta>-\frac{1}{2}\pi$. We get a similar result, except that the integrand has now a simple pole within the sector at the point $\kappa_0-2\mu i$ approximately. The residue at this pole gives the term in (8) which represents the regular train of waves in the rear of the system. It can also be verified that in this case y and $\partial y/\partial\varpi$ are finite and continuous throughout.

Returning to the general expression (6), the second integral represents the deviation from the steady state. It contains $\exp\{i\kappa(\varpi+ct)\}$ as a factor, and we see from its form that it represents the effect at time t of a certain initial distribution of velocity and displacement. To illustrate this point, consider a stationary pressure system which is suddenly established at a given instant and maintained constant. The effect is the same as if there had been in existence up to the given instant two equal and opposite systems with their ultimate static effect upon the water surface fully established, the negative system being then suddenly annulled. Thus the subsequent effect is the steady state of the positive system combined with the effect of an initial displacement equal to the steady state of an equal negative system. In the same way, for a pressure system which is suddenly established and started in uniform motion, the effect is the superposition of the steady state of this system and the disturbance due to initial conditions given by the steady state of an equal negative system in uniform motion. We shall find this principle of use in a later section.

4. The wave resistance R_1 in the steady state is usually obtained from energy principles applied to the regular waves. The front of the train advances with velocity c, while the rate of flow of energy across any fixed vertical plane in the rear is the corresponding group velocity $\frac{1}{2}c$; from the amplitude of the regular waves in (8), by equating the net rate of gain of fluid energy to R_1c, it follows that

$$R_1 = \kappa_0^2\{\phi(\kappa_0)\}^2/g\rho. \qquad (10)$$

Some consideration is necessary before we can apply this method to the motion before the steady state has been attained.

Begin with a case in which there is no ambiguity, namely, when the waves are produced by a rigid body moving horizontally through the liquid. We can apply the general hydrodynamical principle that the rate of increase of total energy of the fluid is equal to the activity of the pressure taken over all the bounding surfaces. If we equate the rate of increase of energy to the product of a force R and the velocity of the rigid body, it follows that R is simply the total fluid pressure on the moving body resolved horizontally. This result can easily be verified by direct calculation for the steady state, whether the waves are produced by the motion of a rigid body or by the motion of an assigned surface pressure; in fact, the two cases are identical in the steady state, for we can imagine the surface pressure to be applied by a rigid cover which fits the water surface everywhere.

Consider now the problem before the steady state has been established. If the waves are caused by a moving rigid body, we can use either definition for the wave resistance, we can calculate it from the rate of increase of fluid energy or from the total horizontal pressure on the body. We are not discussing this case, simply because so far the analysis has proved too complicated to allow of suitable reduction. We replace this problem by that of the motion of an assigned surface pressure. Now we can calculate the rate of increase of the total energy of the fluid when the pressure system is in motion. But it would not be satisfactory to divide this quantity by the velocity of the pressure system and define the quotient as the wave resistance, for part of the increase of fluid energy is independent of the motion of the pressure system. For instance, if a stationary pressure system is suddenly established and maintained steady, the activity of the surface pressure is not zero immediately after the initial instant; there is a subsequent flow of energy, whose rate ultimately subsides to zero. From these considerations it seems that we should get results more comparable with the wave resistance of a rigid body by adopting the alternative method of calculation. In what follows we shall therefore calculate for any instant the total horizontal component of the surface pressure regarded as applied normally to the surface of the water; and we shall define this to be the wave resistance.

With the usual limitation that the slope of the surface is everywhere small, we have from this definition

$$R = -\int_{-\infty}^{\infty} F(\varpi)\frac{\partial y}{\partial \varpi}\,d\varpi. \tag{11}$$

We can verify that this gives the same result (10) for the steady state. For instance, taking the expressions in (8), the part which is symmetrical with respect to the origin gives no contribution to R, and we obtain

$$g\rho \mathrm{R}_1 = 2\int_{-\infty}^{0} \mathrm{F}(\varpi) \,.\, \kappa_0^2 \phi(\kappa_0) \cos \kappa_0 \varpi \, d\varpi = \kappa_0^2 \{\phi(\kappa_0)\}^2,$$

or if we work directly from the integral (7), we have

$$\pi g\rho \mathrm{R}_1 = \kappa_0 \lim_{\mu \to 0} \int_0^{\infty} \frac{i\kappa^2 \{\phi(\kappa)\}^2 \, d\kappa}{\kappa(\kappa_0 - \kappa) + \mu^2 - 2\mu\kappa i}, \tag{12}$$

where the real part is to be taken. Under the general conditions specified for $\phi(\kappa)$, or, in particular, for the case given in (9), it can be shown that this leads to the same expression (10). The wave resistance in general is the sum of two parts, the steady value R_1, as given by (10), and the deviation R_2.

Using the definition (11) with the second integral in (6), we find

$$-2\pi g\rho \mathrm{R}_2 = \lim_{\mu \to 0} e^{-\mu ct} \int_0^{\infty} i\kappa^2 \mathrm{V} \{\phi(\kappa)\}^2 \left[\frac{e^{-i\kappa(\mathrm{V}-c)t}}{\kappa(\mathrm{V}-c) - \mu ci} + \frac{e^{i\kappa(\mathrm{V}+c)t}}{\kappa(\mathrm{V}+c) + \mu ci} \right] d\kappa. \tag{13}$$

5. Consider first a special case in which the pressure system is such that there are no regular waves left in the rear, a type which Kelvin called a waveless system. It follows from (10), (12), and (13), that this is the case when the system is such that $\phi(\kappa)$ is of the form $(\kappa - \kappa_0)\psi(\kappa)$, where $\psi(\kappa)$ remains finite. We have then

$$\int_{-\infty}^{\infty} \mathrm{F}(\varpi) \cos \kappa\varpi \, d\varpi = \phi(\kappa) = (\kappa - \kappa_0)\psi(\kappa). \tag{14}$$

If this system is made to travel with the velocity c, for which $2\pi/\kappa_0$ is the free wave-length, there will be no regular train of waves in the rear. The integrals (12) and (13) now remain finite and determinate with μ zero; we can thus simplify the expressions by making μ zero. The integral (12) vanishes, as does also the equivalent expression (10). Then, taking the real part of (13), we find for the total wave resistance of this system at any time t

$$-\pi g\rho \mathrm{R} = \kappa_0^{\frac{1}{2}} \int_0^{\infty} \kappa(\kappa - \kappa_0) \{\psi(\kappa)\}^2$$
$$\times \{\kappa^{\frac{1}{2}} \sin \kappa \mathrm{V}t \cos \kappa ct - \kappa_0^{\frac{1}{2}} \cos \kappa \mathrm{V}t \sin \kappa ct\} \, d\kappa. \tag{15}$$

It is of interest to examine this solution when the integral can be evaluated exactly in finite terms. Burnside* suggested some years ago a method of building up exact solutions of certain wave problems, and similar forms have been analysed in detail by Kelvin, after obtaining the solutions by a different method. The cases in which we can carry out the integrations in (15) lead to similar functions; we obtain them by taking

$$\psi(\kappa) = \kappa^{\frac{1}{2}} e^{-r\kappa}, \qquad r > 0. \tag{16}$$

* W. Burnside, 'Proc. Lond. Math. Soc.,' vol. 20, p. 31 (1888).

This case is the simplest of the type which allows of exact evaluation of (15), and for which the integral pressure is finite. To derive the corresponding pressure system, we make use of Euler's integrals of the form

$$\int_0^\infty \kappa^{n-1} e^{-\lambda\kappa\cos\alpha} \cos(\lambda\kappa\sin\alpha)\, d\kappa = \lambda^{-n}\Gamma(n)\cos n\alpha, \tag{17}$$

$$\lambda > 0, \qquad n > 0, \qquad -\tfrac{1}{2}\pi < \alpha < \tfrac{1}{2}\pi.$$

Using the Fourier integral theorem, combined with (16) and (17), we find

$$\begin{aligned} \pi F(x) &= \int_0^\infty (\kappa-\kappa_0)\kappa^{\frac{1}{4}} e^{-r\kappa} \cos\kappa x\, d\kappa \\ &= \Gamma(\tfrac{9}{4})(r^2+x^2)^{-9/8}\cos(\tfrac{9}{4}\tan^{-1}x/r) - \kappa_0\Gamma(\tfrac{5}{4})(r^2+x^2)^{-5/8}\cos(\tfrac{5}{4}\tan^{-1}x/r). \end{aligned} \tag{18}$$

The two terms of this expression are easily graphed when expressed in terms of the angle $\tan^{-1}(x/r)$; two numerical cases are shown later.

We can now find the resistance R for the system (18), travelling with velocity $c = \surd(g/\kappa_0)$. Substituting (17) in (15), and writing $\kappa = u^2$, $g^{\frac{1}{2}}t = q$, $ct = p$, we have

$$\begin{aligned} -\tfrac{1}{2}\pi g\rho R = &-\kappa_0 \int_0^\infty (u^6-\kappa_0 u^4) e^{-2ru^2} \sin pu^2 \cos qu\, du \\ &+ \kappa_0^{\frac{1}{2}} \int_0^\infty (u^7-\kappa_0 u^5) e^{-2ru^2} \cos pu^2 \sin qu\, du. \end{aligned} \tag{19}$$

The integrals involved can all be derived, by differentiation with respect to the parameters, from

$$\int_0^\infty e^{-(\rho-ip)u^2} \cos qu\, du = \tfrac{1}{2}[\pi/(\rho-ip)]^{\frac{1}{2}} e^{-q^2/4(\rho-ip)}. \tag{20}$$

Carrying out these operations, we obtain finally

$$\begin{aligned} -\pi^{1/2} g\rho R = {} & \kappa_0^{1/2}(4r^2+c^2t^2)^{-5/4} e^{-gt^2r/2(4r^2+c^2t^2)} \\ & \times [\kappa_0^{\frac{1}{2}}\{-\tfrac{15}{8}A\sin(\tfrac{7}{2}\theta-\phi) + \tfrac{45}{16}q^2A^2\sin(\tfrac{9}{2}\theta-\phi) - \tfrac{15}{32}q^4A^3\sin(\tfrac{11}{2}\theta-\phi) \\ & + \tfrac{1}{64}q^6A^4\sin(\tfrac{13}{2}\theta-\phi)\} + \kappa_0^{3/2}\{\tfrac{3}{4}\sin(\tfrac{5}{2}\theta-\phi) - \tfrac{3}{4}q^2A\sin(\tfrac{7}{2}\theta-\phi) \\ & + \tfrac{1}{16}q^4A^2\sin(\tfrac{9}{2}\theta-\phi)\} + \tfrac{105}{16}qA^2\cos(\tfrac{9}{2}\theta-\phi) - \tfrac{105}{32}q^3A^3\cos(\tfrac{11}{2}\theta-\phi) \\ & + \tfrac{21}{64}q^5A^4\cos(\tfrac{13}{2}\theta+\phi) - \tfrac{1}{128}q^7A^5\cos(\tfrac{15}{2}\theta-\phi) \\ & + \kappa_0\{-\tfrac{15}{8}qA\cos(\tfrac{7}{2}\theta-\phi) + \tfrac{5}{8}q^3A^2\cos(\tfrac{9}{2}\theta-\phi) - \tfrac{1}{32}q^5A^3\cos(\tfrac{11}{2}\theta-\phi)\}], \end{aligned} \tag{21}$$

where

$$q = g^{\frac{1}{2}}t; \quad A = (4r^2+c^2t^2)^{-\frac{1}{2}}; \quad \theta = \tan^{-1}(ct/2r); \quad \phi = gct^3/4(4r^2+c^2t^2).$$

6. Before working out numerical examples, it is convenient to record the asymptotic expansion suitable for large values of $ct/2r$. From (21), by

writing $\theta = \frac{1}{2}\pi - \tan^{-1}(2r/ct)$, and expanding the various terms, we get, up to and including terms in $(2r/ct)^{3/2}$,

$$-\pi^{1/2} g\rho R = \tfrac{9}{128} g^4 c^{-17/2} t^{-1/2} e^{-\beta} \cos(gt/4c + \tfrac{1}{4}\pi)$$
$$+ \tfrac{1}{64} g^3 c^{-15/2} t^{-3/2} \{37 + \beta(18\beta - 75)\} e^{-\beta} \cos(gt/4c - \tfrac{1}{4}\pi), \quad (22)$$

where $\lambda_0 = 2\pi c^2/g$ and $\beta = \pi r/\lambda_0$. If the pressure system has moved through n wave-lengths, we have $ct = n\lambda_0$, and the ratio of the amplitudes of the two terms in (22) is

$$\frac{1}{9\pi n}\left\{37 + \frac{\pi r}{\lambda_0}\left(\frac{18\pi r}{\lambda_0} - 75\right)\right\}, \quad (23)$$

an expression which gives some estimate of the approximation obtained by using only the first term of (22). It depends not only upon the distance travelled, but also upon the ratio of the effective breadth of the system to the free wave-length for the assigned velocity.

Compare this approximation with that obtained by applying Kelvin's group method directly to the integral expression for the wave resistance.

Under certain conditions,* an approximate value of an integral of the form

$$\int_a^b F(\kappa) e^{if(\kappa)} d\kappa$$

is given by

$$\frac{\sqrt{\pi}\, F(\alpha)}{\sqrt{|\frac{1}{2} f''(\alpha)|}} e^{i\{f(\alpha) \pm \frac{1}{4}\pi\}}, \quad (24)$$

the upper or lower sign being taken in the exponential according as $f''(\alpha)$ is positive or negative, and α being a root of $f'(\alpha) = 0$. It is assumed that the circular function in the integral goes through a large number of periods within the range of integration, while $F(\kappa)$ changes comparatively slowly; in addition, the quotient $f'''(\alpha)/\{f''(\alpha)\}^{3/2}$ must be small.

Apply this to the form for F given in (13). The second term within the square brackets contributes nothing to the approximation; from the first term we have, with $ct = n\lambda_0$,

$$f(\kappa) = -\kappa(V - c)t = -g^{\frac{1}{2}} t \kappa^{\frac{1}{2}} + ct\kappa.$$

Hence

$$\alpha = g/4c^2; \quad f''(\alpha) = 2c^3 t/g; \quad f'''(\alpha)/\{f''(\alpha)\}^{3/2} = 3/\sqrt{(\pi n)}.$$

From (13) and (24) the group value of R is

$$R = -\frac{1}{2\pi g\rho} \lim_{\mu \to 0} e^{-\mu ct} \sqrt{\left(\frac{\pi g}{c^3 t}\right)} \frac{i g^{1/2} \alpha^{3/2} \{\phi(\alpha)\}^2}{g^{1/2}\alpha^{1/2} - c\alpha - i\mu c} e^{i(gt/4c - \pi/4)}. \quad (25)$$

Taking the real part of this expression and putting μ zero, we obtain

$$R = -\frac{g^{1/2}\{\phi(g/4c^2)\}^2}{4\pi^{1/2}\rho c^{7/2} t^{1/2}} \cos(gt/4c + \tfrac{1}{4}\pi). \quad (26)$$

* Lamb, 'Hydrodynamics,' 1932 edn. p. 395.

It should be noted that for a pressure system which leaves regular waves in its rear, we cannot take (26) as an approximation for the limiting value of (13) when $\mu \to 0$, except under certain further limitations. For the present this difficulty does not arise, as we are considering a waveless system with $\phi(\kappa)$ of the form $(\kappa - \kappa_0)\,\psi(\kappa)$; we have seen that in this case the integrals (12) and (13) remain finite and determinate with μ zero.

In particular, for the forms (16) and (18), the group formula (26) reduces to

$$-\pi^{1/2} g\rho\,\mathrm{R} = \tfrac{9}{128}\, g^4 c^{-17/2} t^{-1/2} e^{-\pi r/\lambda_0} \cos\left(gt/4c + \tfrac{1}{4}\pi\right). \qquad (27)$$

which, from (22), agrees with the first term in the asymptotic expansion of the exact solution for this case.

Instead of expressing R as a function of the time t, we can use the distance travelled, or again the number n of free wave-lengths λ_0 through which the system has moved; in the last case the circular function in (27) becomes $\cos \frac{1}{4}(2n+1)\,\pi$. The form of (27) agrees with the definition of the wave resistance as the resolved total pressure. For after a sufficient time, the surface in the neighbourhood of the moving origin consists chiefly of the simple waves whose group velocity is the velocity c of the pressure system; thus the wave-length there is $4\lambda_0$.

7. Consider now two numerical examples of the exact solution (21) with different values of the ratio r/λ_0.

In the first place, we shall adopt units used by Kelvin, for comparison and for simplicity of calculation.

Case i: $g = 4$; $r = 1$; $\lambda_0 = 2$; $\kappa_0 = \pi$; $c = 2/\sqrt{\pi}$.

From (18) the pressure system F (x) can be obtained by graphing

$$4\pi \cos^{5/4}\theta \cos\tfrac{5}{4}\theta - 5\cos^{9/4}\theta \cos\tfrac{9}{4}\theta,$$

where $\theta = \tan^{-1}(x/r)$. The graph is shown in curve (1) of fig. 1; the curve has maxima near $x = \pm\, 0{\cdot}2$, though they are almost inappreciable on the diagram.

It is convenient to graph the resistance curve upon a base $\xi = ct/2r$; in this particular case ξ is also the number of wave-lengths λ_0 through which the system has moved. The angles of the formula (21) are now

$$\theta = \tan^{-1}\xi; \qquad \phi = \pi\xi^3/2(1+\xi^2).$$

It is unnecessary to repeat the expression (21) with these values; each of the 14 terms can be easily calculated for any given value of ξ. The results are shown in curve (1) of fig. 2; to obtain the curve 15 points were calculated by the formula (21).

The wave resistance decreases ultimately to zero, as it should for a waveless

system, but it approaches the steady state very slowly. This is explained when we examine the graph of the pressure system in this case. The

FIG 1

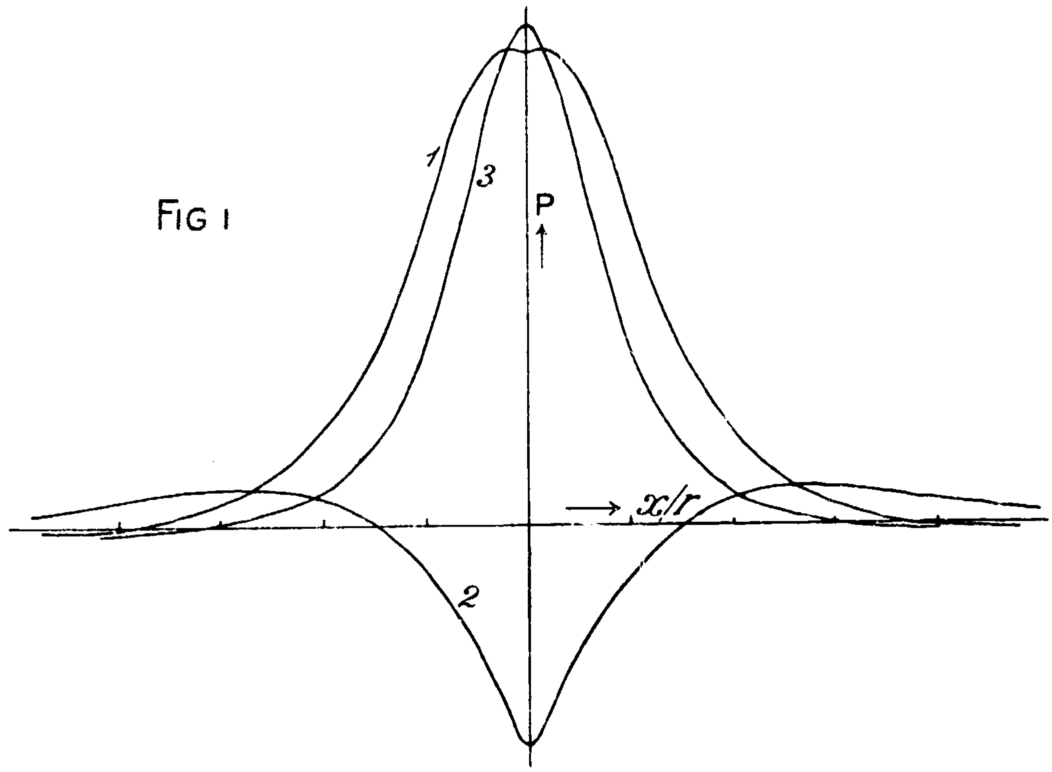
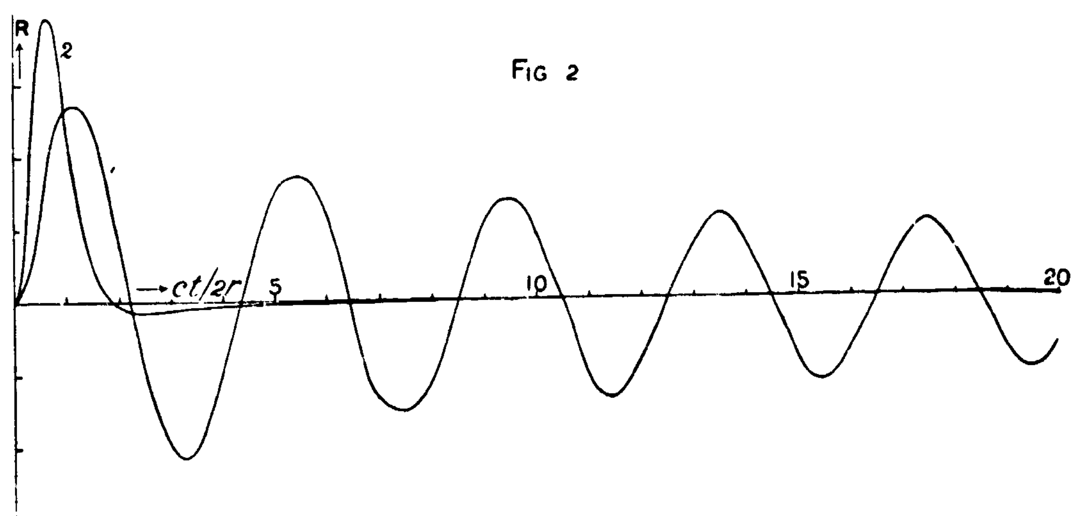


FIG 2



waveless character is due to the mutual interference effects produced by the peaks of the pressure graph, and fig. 1 shows how inappreciable the peaks

are in this case. Hence the slowness with which the steady state is attained and the probable lack of stability of the steady state.

To compare the group approximation with the exact solution, we have from (27)

$$-\pi^{1/2} g\rho \mathrm{R} = 9 \times 2^{-15/2}\pi^4 n^{-1/3} e^{-\pi^2} \cos \tfrac{1}{4}(2n+1)\pi. \tag{28}$$

The following is a comparison of the values of $10^2\pi^{\frac{1}{2}} g\rho \mathrm{R}$, as given by (28) and the exact formula (21):

n.	Group.	Exact.
9	$-23 \cdot 73$	$-26 \cdot 45$
16	$+17 \cdot 77$	$+16 \cdot 04$
25	$-14 \cdot 21$	$-14 \cdot 93$
100	$+\ 7 \cdot 11$	$+\ 6 \cdot 94$

Case ii.—As a second numerical example, we take one which might correspond more to practical conditions, in that the pressure system is similar to that associated with the motion of a ship model in an experimental tank. Using foot-second units, we take

$$g = 32; \quad r = 2; \quad c = 20; \quad \kappa_0 = 0 \cdot 08; \quad \lambda_0 = 25\pi.$$

The pressure system is graphed in curve (2) of fig. 1, from the expression $8 \cos^{5/4}\theta \cos \frac{5}{4}\theta - 125 \cos^{9/4}\theta \cos \frac{9}{4}\theta$. We notice the contrast between this and the previous case. We should now expect the steady state to be attained quickly and to be much more stable. This is brought out very clearly by the resistance curve, which has been graphed from (21), and is shown in curve (2) of fig. 2; after the initial peak, the subsequent oscillations can scarcely be shown on the scale of the diagram.

A comparison of the exact formula and the group approximation gives similar results to the previous case, for in both the numerical value of the ratio (23) is of the order $1/n$, in spite of the difference in the values of r/λ_0 for the two cases.

It should be remarked that the two cases cannot be compared as regards absolute values from the curves shown, because the scale for the ordinates has been chosen arbitrarily in each case. The maximum value of R, that is, the value at the prominent peak on curve (2), is given by $g\rho \mathrm{R} = 7 \times 10^{-3}$. We can obtain some idea of the magnitude by the following comparison: We have chosen the pressure system so that it is waveless at a particular velocity, namely, 20 feet per second. Now, imagine the same system to be

driven at any other steady velocity ; it will have a steady resistance, which we can calculate from the formula (10); in this case it is

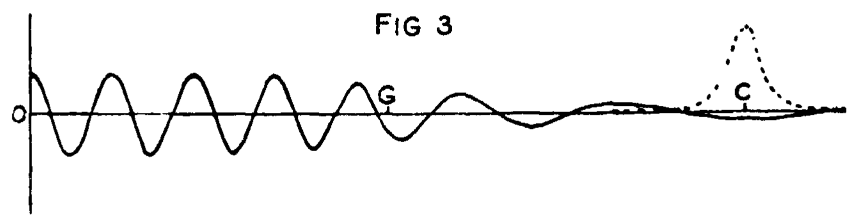
$$g\rho R = \kappa^{5/2}(\kappa - 0{\cdot}08)^2 e^{-4\kappa}, \qquad \kappa = g/V^2. \tag{29}$$

This steady wave resistance has a maximum at a velocity of about 5·25 ft./sec., and the value of $g\rho R$ is then $16{\cdot}4 \times 10^{-3}$. Hence the maximum resistance due to the sudden starting of the system at its waveless speed is about one-half the maximum steady resistance at any uniform speed.

8. We have been able to obtain an exact solution for a special type of waveless system ; we leave this now to consider more generally a symmetrical localised pressure system, which is suddenly established and set in motion.

We have seen that the surface elevation at any time is found by superposing the steady state of the system and the effect due to initial conditions given by the steady state of an equal negative system in uniform motion. Apply this to a case in which the steady state consists of an infinite train of regular waves in rear of the system, together with a localised displacement symmetrical with respect to the moving origin. Let O be the fixed origin and starting point, and C the position at time t. The deviation from the ultimate steady state consists of the effect due to a certain initial distribution of displacement and velocity localised round O, together with the subsequent state of a semi-infinite train of regular waves, which at the initial instant had a definite front at the point O. We may describe the latter part in general terms as a regular train with a front, more or less definite according to the time, at a point G corresponding to the group velocity, and in advance of G a disturbance which may be called the forerunner. If OC is sufficiently large, and if we require the surface elevation only at points sufficiently far in advance of G, the forerunner is given with considerable accuracy by Kelvin's group method of approximation. The argument is represented diagrammatically in fig. 3, the continuous line showing the elevation and the dotted line the travelling pressure system.

FIG 3



The wave resistance being defined as the total horizontal component of the pressure system, we divide it into two parts. The first part is the final steady value $\kappa_0^2\{\phi(\kappa_0)\}^2/g\rho$ as given in (10), and the second is the deviation given by the integral in (13). The latter represents the resolved pressure

system as if the surface elevation were that due to the stoppage of a negative system, as represented in fig. 3.

For a concentrated pressure system, the value of the integral (13) will be given approximately by the Kelvin group method, if the time is sufficiently large; that is, if C is sufficiently far in advance of G for us to neglect the contribution of the applied pressure acting on the surface to the rear of G.

Without attempting to specify these conditions more precisely, we shall apply the method to the type of system used in the previous sections; from the previous exact solution we have been able to estimate somewhat the degree of accuracy of the group approximation.

The group value of (13) is given in (26). Hence the wave resistance, for sufficiently large values of t, is given by

$$R = \frac{g}{\rho c^4}\left\{\phi\left(\frac{g}{c^2}\right)\right\}^2 - \frac{g^{1/2}\{\phi(g/4c^2)\}^2}{4\pi^{1/2}\rho c^{7/2} t^{1/2}}\cos(gt/4c+\tfrac{1}{4}\pi). \qquad (30)$$

9. Apply this to the pressure distribution

$$\pi F(x) = \Gamma(\tfrac{9}{4})(r^2+x^2)^{-9/8}\cos\{\tfrac{9}{4}\tan^{-1}(x/r)\}, \qquad (31)$$

for which $\phi(\kappa) = \kappa^{5/4}e^{-r\kappa}$, with $\kappa = g/c^2$. The graph of this distribution is shown in curve (3) of fig. 1.

We have

$$\rho R = \frac{g^{7/2}}{c^9}e^{-2gr/c^2} - \frac{g^3}{128\pi^{1/2}c^{17/2}t^{1/2}}e^{-gr/2c^2}\cos(gt/4c+\tfrac{1}{4}\pi). \qquad (32)$$

The value of R oscillates about the final steady value. The relative deviation is given by the ratio of the two terms, namely,

$$2^{-15/2}\pi^{-1}n^{-1/2}e^{3\pi r/\lambda_0}\cos\tfrac{1}{4}(2n+1)\pi,$$

where λ_0 is the wave-length of the regular train and $ct = \lambda_0 n$. We may obtain numerical values by using the two cases of the previous sections.

For Case i we have $r = 1$, $\lambda_0 = 2$, and we find the following comparison between R_1, the final steady resistance, and R_2, the deviation given by the second term of (32):—

n.	R_2/R_1.
9	+0·046
16	−0·035
25	+0·027
100	−0·014

Hence, after the system has moved through nine wave-lengths, the deviation is less than 5 per cent.

In Case ii, $r = 2$ and $\lambda_0 = 25\pi$. We find that when $n = 9$, the deviation is already less than 0·06 per cent.

10. Consider now a simpler type of localised pressure distribution, namely,

$$\pi F(x) = r/(r^2+x^2). \tag{33}$$

This type leads to a steady wave resistance whose variation with the velocity is more like that of a ship model. We have $\phi(\kappa) = e^{-r\kappa}$, and (30) gives

$$\rho R = \frac{g}{c^4} e^{-2gr/c^2} - \frac{g^{1/2}}{4\pi^{1/2} c^{7/2} t^{1/2}} e^{-gr/2c^2} \cos(gt/4c + \tfrac{1}{4}\pi). \tag{34}$$

The relative deviation is now 32 times as large as in the previous case, since

$$\frac{R_2}{R_1} = \frac{e^{3\pi r/\lambda_0}}{2^{5/2}\pi n^{1/2}} \cos \tfrac{1}{4}(2n+1)\pi.$$

With $r = 1$, $\lambda_0 = 2$, the value of R_2/R_1 is about 0·5 for $n = 100$. We should have to take n of the order 10,000 before bringing the deviation from the steady value below 5 per cent.

On the other hand, with $r = 2$, $\lambda_0 = 25\pi$, the deviation is under 2 per cent. when $n = 9$, or at about 35 seconds after the beginning of the motion; it is less than 2 per cent. when $n = 4$, or after a travel of rather more than 300 feet.

11. The waves produced by the horizontal motion of a circular cylinder of small radius travelling at a considerable depth h below the surface may be compared with those produced by the surface pressure

$$\pi F(x) = Ac^2(h^2-x^2)/(h^2+x^2)^2. \tag{35}$$

We assume that the intensity of the system is proportional to the square of the velocity. It appears that the steady wave resistance is then the same function of the velocity as in the motion of the cylinder;* for we have

$$\phi(\kappa) = Ac^2\kappa e^{-\kappa h},$$

and hence

$$\rho R = \frac{A^2 g^3}{c^4} e^{-2gh/c^2} - \frac{A^2 g^{5/2}}{64\,\pi^{1/2} c^{7/2} t^{1/2}} e^{-gh/2c^2} \cos(gt/4c + \tfrac{1}{4}\pi). \tag{36}$$

As a numerical example, take the case when the velocity is such that the steady resistance R_1 has its maximum value; that is, when $c^2 = gh$. Then we have

$$\frac{R_2}{R_1} = \frac{e^{3/2}}{2^{13/2}\pi n^{1/2}} \cos \tfrac{1}{4}(2n+1)\pi. \tag{37}$$

The value of the ratio means a deviation from the steady value of about 0·8 per cent. when $n = 3\tfrac{1}{2}$, that is, when the system has travelled through a distance $7\pi h$.

* Lamb, 'Hydrodynamics,' 1932 edn. p. 410.

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Some Cases of Wave Motion due to a Submerged Obstacle.

By T. H. HAVELOCK, F.R.S.

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1. As far as I am aware, only one case of wave motion caused by a submerged obstacle has been worked out in any detail, namely the two-dimensional motion due to a circular cylinder; for this case, Prof. Lamb has given a solution applicable when the cylinder is of small radius and is at a considerable depth.* The method can be extended to bodies of different shape, and my object in this paper is to work out the simplest three-dimensional case, the motion of a submerged sphere.

The problem I have considered specially is the wave resistance of the submerged body. In the two-dimensional case, this is calculated by considerations of energy and work applied to the train of regular waves. But for a moving sphere the wave system is more complicated, like the well-known wave pattern for a moving point disturbance, and similar methods are not so easily applied; I have therefore calculated directly the horizontal resultant of the fluid pressure on the sphere. Before working out this case, the analysis for the circular cylinder is repeated, because it is necessary to carry the approximation a stage further than in Prof. Lamb's solution in order to verify that the resultant horizontal pressure on the cylinder is the same as the wave resistance obtained by the method of energy.

The stages in approximating to the velocity potential may be described in terms of successive images; the first stage ϕ_1 is the image of a uniform stream in the submerged body, the second stage ϕ_2 is the image of ϕ_1 in the free surface, the third ϕ_3 is the image of ϕ_2 in the submerged body, and so on. In order to keep the integrals convergent, a small frictional coefficient is introduced in the usual manner; after the calculations have been carried out, the coefficient is made zero. Further, the solution for uniform motion is built up so that expressions can be found for the velocity potential at any time after the starting of the motion, although only the final steady state has been studied in detail. The wave resistance of a sphere is found to have the form const. $\times\, \alpha^{3/2} e^{-\alpha/2}\, W_{1,1}(\alpha)$, in which α is $2gf/c^2$, with f the depth of the sphere and c its velocity; $W_{1,1}(\alpha)$ is a confluent hypergeometric function. In order to graph the wave resistance as a function of the velocity, expansions have been found for this particular variety of the function

* H. Lamb, 'Ann. di Matematica,' vol. 21, p. 237; also 'Hydrodynamics,' 6th edn. (1932) p. 410.

$W_{k,m}(\alpha)$; it belongs to the logarithmic case for which a general expansion is not available.

In general form the graph of the resistance is very similar to that of the circular cylinder.

Circular Cylinder.

2. The steady state for uniform motion of the cylinder may be attacked directly, as in Prof. Lamb's solution, but we shall adopt his suggestion of building it up from simple oscillations. Take the axis of x in the free surface of the water, and the axis of y vertically upwards. A circular cylinder, of radius a, is making small oscillations parallel to Ox with velocity $c\cos\sigma t$, the axis of the cylinder being horizontal and perpendicular to Ox, and the mean position of the centre being the point $(0, -f)$. A first appproximation when the depth f is sufficiently large is found by ignoring the surface effect altogether and putting

$$\phi = ca^2(x/r^2)e^{i\sigma t}; \qquad r^2 = x^2+(y+f)^2. \tag{1}$$

This satisfies the boundary condition at the surface of the cylinder. For the next step, add a term X_1 to the velocity potential so as to satisfy the conditions at the free surface, but ignoring meantime the disturbance produced thereby at the surface of the cylinder. The term X_1 must be a potential function and it must satisfy the condition for deep water, namely, $\partial X_1/\partial y = 0$ for $y = -\infty$; these conditions are fulfilled by

$$X_1 = e^{i\sigma t}\int_0^\infty \alpha(\kappa)e^{\kappa y}\sin\kappa x\,d\kappa, \tag{2}$$

where α is a function of κ to be determined. This form is chosen because we can satisfy the conditions at the free surface by using an equivalent form for (1), since

$$x/r^2 = \int_0^\infty e^{-\kappa(y+f)}\sin\kappa x\,d\kappa; \qquad y+f > 0. \tag{3}$$

The surface elevation is expressed similarly by

$$\eta = e^{i\sigma t}\int_0^\infty \beta(\kappa)\sin\kappa x\,d\kappa. \tag{4}$$

In order to keep the various integrals convergent, we assume that the liquid has a slight amount of friction proportional to velocity; in the sequel the results are simplified by making the frictional coefficient μ tend to zero. In these circumstances the pressure equation is

$$p/\rho = \text{const.} + \frac{\partial p}{\partial t} - gy + \mu\phi - \tfrac{1}{2}q^2. \tag{5}$$

Hence the conditions at the free surface are, neglecting the square of the velocity,

$$\partial\phi/\partial t - gy + \mu\phi = \text{const.}; \quad -\partial\phi/\partial y = \partial\eta/\partial t.$$

Here ϕ is the velocity potential after (2) has been added to (1); thus the equations for α and β are

$$\left.\begin{aligned} ca^2\kappa e^{-\kappa f}-\kappa\alpha &= i\sigma\beta \\ i\sigma ca^2e^{-\kappa f}+i\sigma\alpha-g\beta+\mu ca^2e^{-\kappa f}+\mu\alpha &= 0 \end{aligned}\right\} \tag{6}$$

From these we obtain the expressions for X_1 and η, namely

$$X_1 = ca^2e^{i\sigma t}\int_0^\infty \frac{g\kappa+\sigma^2-i\mu\sigma}{g\kappa-\sigma^2+i\mu\sigma}\,e^{-\kappa(f-y)}\sin\kappa x\,d\kappa, \tag{7}$$

$$\eta = ca^2e^{i\sigma t}\int_0^\infty \frac{2\kappa(\mu+i\sigma)}{g\kappa-\sigma^2+i\mu\sigma}\,e^{-\kappa f}\sin\kappa x\,d\kappa. \tag{8}$$

The expression for X_1 can be divided into two parts

$$X_1 = -ca^2e^{i\sigma t}\int_0^\infty e^{-\kappa(f-y)}\sin\kappa x\,d\kappa - 2ca^2e^{i\sigma t}\int_0^\infty \frac{g\kappa e^{-\kappa(f-y)}\sin\kappa x\,d\kappa}{\sigma^2-i\mu\sigma-g\kappa}. \tag{9}$$

If we regard X_1 as the image of the oscillating cylinder in the free surface, we see from the form of the first integral in (9) that part of the image is a negative doublet at the image point $(0, f)$. We obtain next the velocity potential of the motion produced by a sudden small displacement of the cylinder, and we take this to be equivalent to a momentary doublet of constant strength. Suppose then that at a time τ a doublet is suddenly created, maintained constant for a time $\delta\tau$, and then annihilated. The velocity potential at any subsequent time t is given by a Fourier synthesis of the preceding results for an oscillating cylinder, and we have

$$\phi = \frac{\delta\tau}{\pi}\int_0^\infty e^{i\sigma(t-\tau)}[\phi]\,d\sigma, \tag{10}$$

where $[\phi]$ is the sum of (1) and (9), omitting the factor $e^{i\sigma t}$.

Carrying out this integration for the value of ϕ in (1) and for the first part of (9) gives simply the momentary doublet at the centre of the cylinder and the negative doublet at the image point. These doublets last for a short time $\delta\tau$; the subsequent fluid motion is contributed by the second part of (9). For this we have to evaluate the real part of

$$\int_0^\infty \frac{e^{i\sigma(t-\tau)}}{\sigma^2-i\mu\sigma-g\kappa}\,d\sigma; \qquad t-\tau>0. \tag{11}$$

We obtain the value by contour integration; further we simplify the result by neglecting μ^2. We shall make μ zero ultimately, but we must retain it sufficiently to keep the integrals convergent; however, at one or two stages, superfluous terms may be omitted when it is clear that the final limiting values will not be affected. We find for (11) the value

$$-\pi\, e^{-\frac{1}{2}\mu(t-\tau)}\sin\{\kappa V(t-\tau)\}\,\kappa V$$

writing V for $\sqrt{(g/\kappa)}$ whenever it serves to simplify the notation. Hence the velocity potential of the subsequent fluid motion after the cylinder has been given a small displacement at time τ is

$$\phi = 2ca^2\delta\tau e^{-\frac{1}{2}\mu(t-\tau)}\int_0^\infty \kappa \mathrm{V} e^{-\kappa(f-y)} \sin \kappa x \sin \kappa \mathrm{V}(t-\tau)\, d\kappa. \tag{12}$$

Finally we obtain the velocity potential for a cylinder in uniform motion by substituting $x+c(t-\tau)$ for x, noting that hereafter x will refer to a moving origin immediately over the centre of the cylinder; we then integrate with respect to τ from the start of the motion up to the instant in question. We could in this way obtain results for any stage of the motion, but we limit the discussion to the final steady state; for this we take $-\infty$ as the lower limit in integrating with respect to τ. Before writing down the result, we must remember to introduce the integrated effect of the original momentary doublet in (1) and its negative image, which were not included in (11); these clearly add up to steady doublets. Hence we find for the steady state

$$\phi = \mathrm{D}-\mathrm{D}_1+2ca^2\int_0^\infty e^{-\kappa(f-y)}(\mathrm{A}\sin \kappa x+\mathrm{B}\cos \kappa x)\, d\kappa, \tag{13}$$

where D represents the doublet ca^2x/r^2 at the point $(0, -f)$, D_1 an equal doublet at the point $(0, f)$, and

$$2\mathrm{A} = \frac{\kappa^2\mathrm{V}(\mathrm{V}+c)}{\kappa^2(\mathrm{V}+c)^2+\frac{1}{4}\mu^2}+\frac{\kappa^2\mathrm{V}(\mathrm{V}-c)}{\kappa^2(\mathrm{V}-c)^2+\frac{1}{4}\mu^2},$$

$$4\mathrm{B} = \frac{\mu\kappa\mathrm{V}}{\kappa^2(\mathrm{V}-c)^2+\frac{1}{4}\mu^2}-\frac{\mu\kappa\mathrm{V}}{\kappa^2(\mathrm{V}+c)^2+\frac{1}{4}\mu^2}. \tag{14}$$

3. Before proceeding further we may obtain the surface elevation from (13) for comparison. The surface condition is now

$$-\partial\phi/\partial y = \partial\eta/\partial t = -c\,\partial\eta/\partial x.$$

Hence we have

$$\eta = 2a^2f/(x^2+f^2)-2a^2\int_0^\infty(\mathrm{A}\cos \kappa x-\mathrm{B}\sin \kappa x)\, e^{-\kappa f}\, d\kappa, \tag{15}$$

in which $\kappa_0 = g/c^2$. Further, since μ is to be small, we may omit irrelevant terms and put

$$\mathrm{A} = -\kappa_0(\kappa-\kappa_0)/\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\},$$

$$\mathrm{B} = \kappa_0(\mu/c)/\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\}. \tag{16}$$

The integral in (15) can then be written as

$$\int_0^\infty\left\{\frac{e^{-i\kappa x}}{\kappa-\kappa_0-i\mu/c}+\frac{e^{i\kappa x}}{\kappa-\kappa_0+i\mu/c}\right\}e^{-\kappa f}\, d\kappa. \tag{17}$$

We transform these integrals by contour integration in the plane of a complex variable κ, treating separately the cases of x positive and x negative; after making μ zero in the final results we obtain

$$\eta = \frac{2a^2f}{x^2+f^2} + 4\pi a^2\kappa_0 e^{-\kappa_0 f}\sin\kappa_0 x + 2a^2\kappa_0\int_0^\infty \frac{m\cos mf - \kappa_0\sin mf}{m^2+\kappa_0^2}e^{mx}\,dm\,; \quad x<0,$$

$$\eta = \frac{2a^2f}{x^2+f^2} + 2a^2\kappa_0\int_0^\infty \frac{m\cos mf - \kappa_0\sin mf}{m^2+\kappa^2}e^{-mx}\,dm\,; \qquad x>0. \tag{18}$$

These agree with Lamb's results for the circular cylinder in a uniform stream.

The wave resistance R is derived from the regular waves in the rear, by considering the rate of increase of energy and taking into account the propagation of energy in a regular train; we have

$$\mathrm{R} = \tfrac{1}{4}g\rho\,(\text{amplitude})^2 = 4\pi^2 g\rho a^4\kappa_0^2 e^{-2\kappa_0 f}. \tag{19}$$

4. We have now to obtain the resistance R by direct summation of the horizontal component of fluid pressure on the cylinder. It is clearly necessary to proceed to a further stage with the velocity potential, since we have assumed so far that the surface effect is negligible in the neighbourhood of the cylinder. If we write (13) as

$$\phi = \mathrm{D}+\mathrm{X}_1, \tag{20}$$

the doublet D is the first approximation, satisfying the boundary conditions on the cylinder; X_1 is the image of the doublet in the free surface, found by satisfying the conditions there. The next step is to find X_2, the image of X_1 in the cylinder, ignoring then the effect of X_2 at the free surface. It follows that X_2 is the image of X_1 in the cylinder, found as if the cylinder were at rest in a field defined by X_1. Taking polar co-ordinates with the origin at the centre of the circular section of the cylinder, we have

$$x = r\cos\theta\,; \qquad y+f = r\sin\theta\,; \tag{21}$$

also the conditions for X_2 are that it should be a potential function, the components of velocity must vanish as r becomes infinite, and

$$\partial(\mathrm{X}_1+\mathrm{X}_2)/\partial r = 0, \qquad \text{for } r=a. \tag{22}$$

But from (13), X_1 consists of a summation of terms of the form

$$e^{\kappa y}\begin{matrix}\cos\\ \sin\end{matrix}\,\kappa x.$$

We obtain X_2 by replacing each term by the expressions

$$e^{-\kappa f}e^{\kappa a^2\sin\theta/r}\begin{matrix}\cos\\ \sin\end{matrix}(\kappa a^2\cos\theta/r),$$

and the above conditions for X_2 are then satisfied. This process amounts simply to inversion; we may think of X_1 as due to a line distribution of

sources and X_2 is then a circle of sources on the inverse of this line with respect to the cylinder. We have now for the velocity potential to this stage

$$\phi = D + 2ca^2\int_0^\infty e^{-\kappa(f-y)}\{(A-\tfrac{1}{2})\sin\kappa x + B\cos\kappa x\}\,d\kappa$$
$$+2ca^2\int_0^\infty e^{-\kappa f+\kappa a^2 y/r^2}\{(A-\tfrac{1}{2})\sin(\kappa a^2 x/r^2) + B\cos(\kappa a^2 x/r^2)\}\,d\kappa. \quad (23)$$

We have put $A-\frac{1}{2}$ for A so as to include under the integral sign the doublet previously denoted by D_1.

The method could theoretically be carried on step by step; however, we stop at this stage because it is sufficient for obtaining the wave resistance R from the pressure equation to the same approximation as by the energy method.

We have
$$R = \int_0^{2\pi} ap\cos\theta\,d\theta; \quad (24)$$
$$p/\rho = -c\partial\phi/\partial x - gy + \mu\phi - \tfrac{1}{2}q^2. \quad (25)$$

If we write (23) as $\phi = D + X_1 + X_2$, and omit terms which obviously contribute nothing to the value of R, we have, when $r = a$,

$$\frac{p}{\rho} = -c\frac{\partial}{\partial x}(X_1+X_2) + \mu(X_1+X_2) - \frac{1}{a^2}\frac{\partial D}{\partial\theta}\frac{\partial(X_1+X_2)}{\partial\theta}$$
$$= (2c/a)\sin\theta\,\partial(X_1+X_2)/\partial\theta + \mu(X_1+X_2), \quad (26)$$

where we have used (22) and the value of D. From (23), omitting the doublets D and D_1, which will from symmetry give no contribution to R when μ is zero, we have

$$p = 4ca^2\int_0^\infty e^{-2\kappa f+\kappa a\sin\theta}\{2\kappa cA\sin\theta\sin(\phi-\theta) + \mu A\sin\phi$$
$$+ 2\kappa cB\sin\theta\cos(\phi-\theta) + \mu B\cos\phi\}\,d\kappa, \quad (27)$$

where $\phi = \kappa a\cos\theta$. Substituting in (24) we have an expression for R. We may now change the order of integration and take first that with respect to θ; we can carry this out, after some transformation, by means of the integrals

$$\int_0^\pi e^{h\cos\theta}\cos(h\sin\theta - n\theta)\,d\theta = \pi h^n/\Gamma(n+1),$$
$$\int_0^\pi e^{h\cos\theta}\cos(h\sin\theta + n\theta)\,d\theta = 0, \quad (28)$$

where n is a positive, odd integer. In fact the integration with respect to θ gives simply $\pi\kappa a(\kappa cB + \mu A)$; hence we have

$$R = 4\pi\rho ca^4\int_0^\infty \kappa(\kappa cB + \mu A)e^{-2\kappa f}\,d\kappa, \quad (29)$$

where A and B are given by (14), or by (16) since we suppose μ small. Thus we have

$$\mathrm{R} = 4\pi\rho c a^4 \lim_{\mu\to 0} \int_0^\infty \frac{\mu\kappa_0^2\kappa e^{-2\kappa f}\,d\kappa}{\{\kappa-(\kappa_0+i\mu/c)\}\{\kappa-(\kappa_0-i\mu/c)\}}$$

$$= 4\pi\rho c a^4 \operatorname{Lim} \mu\{2\pi i\kappa_0^3 e^{-2\kappa_0 f}/2i(\mu/c)+\text{finite quantity}\}$$

$$= 4\pi^2 g^3 \rho a^4 c^{-4} e^{-2gf/c^2}, \tag{30}$$

which is the same as the previous expression (19).

Sphere.

5. A sphere of radius a is at depth f below the surface and is moving with uniform velocity c parallel to the axis of x. The origin is in the free surface, the axis of z being drawn vertically upwards. As before, the first approximation is a doublet D given by

$$\phi = ca^3x/2r^3; \qquad r^2 = x^2+y^2+(z+f)^2. \tag{31}$$

For the purpose of satisfying the conditions at the free surface we have

$$\phi = \mathrm{D} = -\tfrac{1}{2}ca^3\frac{\partial}{\partial x}\int_0^\infty e^{-\kappa(z+f)}\,\mathrm{J}_0\{\kappa\sqrt{(x^2+y^2)}\}\,d\kappa; \qquad z+f>0. \tag{32}$$

This suggests at once suitable forms for the next approximation and for the free surface; the equations are similar to (6) of the previous case, and we obtain in the same way

$$\phi = \mathrm{D}-\mathrm{D}_1+\mathrm{X}_1, \tag{33}$$

where D_1 is a doublet at the image point $(0, 0, f)$ and

$$\mathrm{X}_1 = -ca^3\frac{\partial}{\partial x}\int_0^\infty \sqrt{(g\kappa)}\,e^{-\kappa(f-z)}\,d\kappa\int_0^\infty e^{-\frac{1}{2}\mu u}\,\mathrm{J}_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}]\sin(\kappa\mathrm{V}u)\,du. \tag{34}$$

The corresponding surface elevation is

$$\eta = a^3\int_0^\infty e^{-\kappa f}\,\mathrm{J}_0\{\kappa\sqrt{(x^2+y^2)}\}\,\kappa\,d\kappa$$

$$-a^3\int_0^\infty \sqrt{(g\kappa)}\,e^{-\kappa f}\,\kappa\,d\kappa\int_0^\infty e^{-\frac{1}{2}\mu u}\,\mathrm{J}_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}]\sin(\kappa\mathrm{V}u)\,du. \tag{35}$$

The first term represents the effect of the doublets D and D_1. It can be verified by approximate methods that the second term includes a main part like the well-known wave pattern for ship waves. Since the expression in (35) gives finite and continuous values for the surface elevation, it might be of interest to examine some points in detail; for instance, the elevation near the lines corresponding to the lines of cusps for a moving point disturbance. However, we pass now to the calculation of the resultant horizontal pressure

on the sphere. We have to find X_2 the image of X_1 in the sphere; for this we first put X_1 into a different form by using

$$\pi J_0[\kappa\{(x+cu)^2+y^2\}^{\frac{1}{2}}] = \int_0^\pi \cos\{\kappa(x+cu)\cos\phi\}\cos(\kappa y\sin\phi)\,d\phi. \quad (36)$$

From (36) and (34), after carrying out the integration with respect to u, we obtain

$$\pi X_1 = ca^3\int_0^\infty e^{-\kappa(f-z)}\,\kappa d\kappa \int_0^\pi \{A\sin(\kappa x\cos\phi)+B\cos(\kappa x\cos\phi)\}$$
$$\times\cos(\kappa y\sin\phi)\cos\phi\,d\phi, \quad (37)$$

where A and B are given by (14) after writing $c\cos\phi$ for c.

For convenience in the following analysis, we transfer the origin to the centre of the sphere, noting that in (37) we shall have $\exp.(-2\kappa f+\kappa z)$ in place of $\exp.(-\kappa f+\kappa z)$. Also we use polar co-ordinates

$$x = r\cos\alpha\,; \qquad y = r\sin\alpha\cos\beta\,; \qquad z = r\sin\alpha\sin\beta.$$

The conditions for X_2 are that it must be a potential function, the disturbance due to it must ultimately vanish as we recede from the sphere, and on the sphere

$$\partial(X_1+X_2)/\partial r = 0. \quad (38)$$

To avoid repetition of expressions like (37), we take out of it a typical term and write

$$X_1 = e^{\kappa z}\sin(\kappa x\cos\phi)\cos(\kappa y\sin\phi). \quad (39)$$

We know that the function

$$r^{-1}e^{\kappa a^2 z/r^2}\sin(\kappa a^2 x\cos\phi/r^2)\cos(\kappa a^2 y\sin\phi/r^2) \quad (40)$$

satisfies the first two conditions for X_2, but we find it does not fulfil (38). An additional term is required, and it can be found in the following way. Suppose that on the sphere we have

$$e^{\kappa z}\sin(\kappa x\cos\phi)\cos(\kappa y\sin\phi) = \Sigma A_m Y_m(\alpha,\beta), \quad (41)$$

where the right-hand side is an expansion in surface spherical harmonics. Then for the term (39), all the conditions for X_2 are satisfied by

$$ar^{-1}e^{\kappa a^2 z/r^2}\sin(\kappa a^2 x\cos\phi/r^2)\cos(\kappa a^2 y\sin\phi/r^2)-\Sigma a^{m+1}A_m Y_m/(m+1)\,r^{m+1}. \quad (42)$$

Suppose, similarly, that on the sphere we have

$$e^{\kappa z}\cos(\kappa x\cos\phi)\cos(\kappa y\sin\phi) = \Sigma B_m Y_m(\alpha,\beta). \quad (43)$$

Then the complete expression for X_2 is

$$\pi X_2 = ca^3 \int_0^\infty e^{-2\kappa f} \kappa d\kappa \int_0^\pi ar^{-1} e^{\kappa a^2 z/r^2} \cos(\kappa a^2 y \sin\phi/r^2)\cos\phi$$
$$\times \{A \sin(\kappa a^2 x \cos\phi/r^2) + B\cos(\kappa a^2 x\cos\phi/r^2)\}\, d\phi$$
$$-ca^3 \int_0^\infty e^{-2\kappa f} \kappa d\kappa \int_0^\pi \sum_m (AA_m + BB_m)(m+1)^{-1}(a/r)^{m+1} Y_m \cos\phi\, d\phi. \tag{44}$$

We have now

$$\phi = D - D_1 + X_1 + X_2 = D + X, \tag{45}$$

and the pressure equation is

$$p/\rho = -c\,\partial\phi/\partial x - gz + \mu\phi - \tfrac{1}{2}q^2. \tag{46}$$

The wave resistance, or the resultant horizontal pressure on the sphere, is

$$R = \int_0^\pi d\alpha \int_0^{2\pi} a^2 p \sin\alpha\cos\alpha\, d\beta. \tag{47}$$

Omitting terms which, from symmetry, will give no contribution to R, we have

$$\frac{p}{\rho} = -c\frac{\partial X}{\partial x} + \mu X - \frac{\partial D}{\partial r}\frac{\partial X}{\partial r} - \frac{1}{r^2}\frac{\partial D}{\partial \alpha}\frac{\partial X}{\partial \alpha} - \frac{1}{r^2\sin^2\alpha}\frac{\partial D}{\partial \beta}\frac{\partial X}{\partial \beta}. \tag{48}$$

But when $r = a$, we have

$$\partial D/\partial\beta = 0; \qquad \partial D/\partial\alpha = -\tfrac{1}{2}ca\sin\alpha; \qquad \partial X/\partial r = 0,$$

hence
$$p/\rho = (3c/2a)\sin\alpha\,\partial X/\partial\alpha + \mu X. \tag{49}$$

We must now substitute (49) in (47) and use the value of X given by the sum of (37) and (44) on the sphere; it is clear that we may omit the doublet D_1 as it will not affect the limiting value of R when μ is zero.

6. Consider, in the first place, the contribution of the first term in the value of p given in (49). In the repeated integrals which are obtained, we may change the order of integration, and we shall carry out first the summation over the surface of the sphere. We notice that, when $r = a$, the first term in the value of X_2 in (44) is equal to the value of X_1; the additional part of X_2 is the term involving the expansions in spherical surface harmonics. Choose a typical term from the latter part, and we find we have to evaluate

$$\int \tfrac{3}{2}\sin\alpha\cos\alpha\,(\partial Y_m/\partial\alpha)\, dS, \tag{50}$$

taken over the surface of unit sphere.

But this integral is equal to

$$-3\int P_2(\cos\alpha)\, Y_m(\alpha,\beta)\, dS. \tag{51}$$

Hence, the only term which has a value different from zero is the term in Y_2, the surface harmonic of the second order. From the manner in which

the expansions were introduced, in (41) and (43), it follows that the contribution of the second term in (44) is one-third of that of the first term; hence, summing up the result so far as the first term of (49) is concerned, we have

$$\pi R' = -5c^2a^4\rho\int_0^\infty e^{-2\kappa f}\kappa d\kappa\int_0^\pi \cos\phi\, d\phi\int_0^{2\pi}d\beta\int_0^\pi \sin\alpha\, P_2(\cos\alpha)\, e^{\kappa a\sin\alpha\sin\beta}$$
$$\times\cos(\kappa a\sin\alpha\cos\beta\sin\phi)$$
$$\times\{A\sin(\kappa a\cos\alpha\cos\phi)+B\cos(\kappa a\cos\alpha\cos\phi)\}\,d\alpha. \quad (52)$$

Taking the integration with respect to β, we find it is equal to

$$2\int_0^\pi e^{\kappa a\sin\alpha\cos\beta}\cos(\kappa a\sin\alpha\sin\phi\sin\beta)\,d\beta = 2\pi I_0(\kappa a\sin\alpha\cos\phi), \quad (53)$$

where $I_0(x)$ is the Bessel function $J_0(ix)$, a result which may be obtained by direct expansion and integration term by term. For the integration with respect to α the term in A in (52) obviously gives zero, and we are left with

$$2\pi\int_0^\pi I_0(\kappa a\cos\phi\sin\alpha)\cos(\kappa a\cos\phi\cos\alpha)\, P_2(\cos\alpha)\sin\alpha\, d\alpha. \quad (54)$$

Here also we may expand in powers of κa and integrate term by term; it can be shown that the integral of the coefficient of $(\kappa a)^n$ vanishes except for the single term κ^2a^2; thus we find that (54) reduces to

$$-(2\pi/5)\kappa^2a^2\cos^2\phi.$$

7. We have now to consider the term μX in the value for p in (49). We might omit this term, on general grounds, as giving no contribution to R ultimately when μ vanishes; for X is the velocity potential for a sphere at rest in a given field X_1. However, it may be left in, and we have a similar calculation. Taking the second integral in (44), we find it is now only the term in Y_1 which counts; hence the contribution of this part is one-half of that of the first integral in (44). Further, it is the term involving A which gives a value different from zero when integrating with respect to α, and instead of (54) we have

$$2\pi\int_0^\pi I_0(\kappa a\cos\phi\sin\alpha)\sin(\kappa a\cos\phi\cos\alpha)\, P_1(\cos\alpha)\sin\alpha\, d\alpha,$$

which reduces to $(4\pi/3)\kappa a\cos\phi$.

8. Collecting the various results, we have now

$$R = 2ca^6\rho\int_0^\infty e^{-2\kappa f}\kappa^2 d\kappa\int_0^\pi(\kappa c\,B\cos\phi+\mu A)\cos^2\phi\, d\phi, \quad (55)$$

a form which may be compared with the corresponding expression for the cylinder in (29).

A and B are given by (14) when we replace c by $c\cos\phi$; putting these values in (55), we see that we may change the order of integration. Further, as we make μ vanish ultimately, we may use simplified forms of A and B corresponding to (16). These give

$$R = 4\kappa_0{}^2 ca^6\rho\mu \int_0^{\pi/2} \sec^2\phi\, d\phi \int_0^\infty \frac{\kappa^2 e^{-2\kappa f}\, d\kappa}{(\kappa-\kappa_0\sec^2\phi)^2+(\mu/c)^2\sec^2\phi}.$$

To obtain the limiting value for μ zero we may treat this like the similar expressions in (30); or, alternatively, we may put $(\mu/c)\sec\phi = 1/n$, and use the general result

$$\lim_{n\to\infty}\int_a^b \frac{nf(x)\,dx}{1+n^2(x-\alpha)^2} = \frac{\pi}{2}\ \{f(\alpha-0)+f(\alpha+0)\}.$$

The apparent difficulty with regard to values of ϕ near $\pi/2$ is overcome by noticing that with the particular functions involved in R no extra contribution arises from such terms near the upper limits of the variables. Carrying out the integration in κ in this way, and changing the remaining variable by putting $\tan\phi = t$, we obtain

$$R = 4\pi g^4\rho a^6 c^{-6} e^{-2gf/c^2}\int_0^\infty (1+t^2)^{3/2} e^{-2gft^2/c^2}\, dt. \qquad (56)$$

The remaining integral can be expressed in terms of known functions. Possibly the simplest method is to use the confluent hypergeometric function* defined, for real positive values of α and for real values of k and m for which $k-m-\frac{1}{2}\leqslant 0$, by

$$W_{k,m}(\alpha) = \frac{e^{-\alpha/2}\alpha^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty u^{-k-\frac{1}{2}+m}(1+u/\alpha)^{k-\frac{1}{2}+m}e^{-u}\,du. \qquad (57)$$

We have now the wave resistance of the sphere given by

$$R = \tfrac{1}{2}\pi^{3/2}g\rho a^6 f^{-3}\alpha^{3/2}e^{-\alpha/2}W_{1,1}(\alpha); \qquad \alpha = 2gf/c^2. \qquad (58)$$

8. For purposes of calculation, we require expansions of $W_{1,1}(\alpha)$. This function belongs to the logarithmic type of confluent hypergeometric function, and general expansions are not available in this case; however, they can be obtained without difficulty for $W_{1,1}$. In the first place, the differential equation satisfied by $W_{1,1}$ is

$$\frac{d^2y}{d\alpha^2}+\left(-\frac{1}{4}+\frac{1}{\alpha}-\frac{3}{4\alpha^2}\right)y = 0. \qquad (59)$$

We use the ordinary methods for solving by means of power series. The roots of the indicial equation are $\frac{3}{2}$ and $-\frac{1}{2}$; hence one of the fundamental

* E. T. Whittaker, 'Bull. Amer. Math. Soc.,' vol. 10, p. 125; also Whittaker and Watson, 'Modern Analysis,' Chap. XVI.

solutions will contain logarithms. Calculating the coefficients step by step, we obtain as a fundamental system

$$\left.\begin{aligned} y_1 &= \alpha^{3/2}\left(1-\tfrac{1}{3}\alpha+\tfrac{7}{96}\alpha^2-\tfrac{1}{96}\alpha^3+\tfrac{11}{9216}\alpha^4-\ldots\right) \\ y_2 &= y_1\log\alpha+\alpha^{-3/2}\left(-\tfrac{8}{3}-\tfrac{8}{3}\alpha+\tfrac{2}{9}\alpha^3-\tfrac{95}{1152}\alpha^4+\ldots\right) \end{aligned}\right\}. \tag{60}$$

We know that $W_{1,1}$ is a linear function of y_1 and y_2; however, it is simpler to obtain an expansion directly and use (60) to verify it. For this purpose we use the equivalent contour integral for the confluent hypergeometric function,

$$W_{k,m} = \frac{\alpha^k e^{-\alpha/2}}{2\pi i}\int_{-\infty i}^{\infty i}\frac{\Gamma(s)\,\Gamma(-s-k-m+\frac{1}{2})\,\Gamma(-s-k+m+\frac{1}{2})}{\Gamma(-k-m+\frac{1}{2})\,\Gamma(-k+m+\frac{1}{2})}\,\alpha^s ds, \tag{61}$$

where the contour has loops if necessary, so that the poles of $\Gamma(s)$ and those of $\Gamma(-s-k-m+\frac{1}{2})\,\Gamma(-s-k+m+\frac{1}{2})$ are on opposite sides of it. The integral can be evaluated by the method of residues. When $k = m = 1$, the poles at which the residues have to be found are simple poles at $s = -\frac{1}{2}, -\frac{3}{2}$, together with double poles at $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. The latter series gives rise to logarithmic residues. Carrying out the calculation, we obtain

$$W_{1,1} = \pi^{-1/2}\alpha e^{-\alpha/2}\left(\alpha^{-3/2}+\frac{3}{2}\alpha^{-1/2}\right)-\frac{3}{4\pi}\alpha^{3/2}e^{-\alpha/2}\left\{\log\alpha\sum_0^\infty\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\Gamma(p+3)}\alpha^p\right.$$
$$\left.+\tfrac{1}{2}\pi^{\frac{1}{2}}\left(\gamma-2\log 2-\frac{3}{2}\right)+\sum_1^\infty\alpha^p\frac{d}{dp}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\,\Gamma(p+3)}\right\}, \tag{62}$$

where γ is Euler's constant 0·5772.... The coefficients may be put into alternative forms more suited for calculation; for instance

$$\frac{d}{dp}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\,\Gamma(p+3)}$$
$$= \frac{1.3.5\ldots(2p-1)\,\pi^{\frac{1}{2}}}{2^p.\,p\,!\,(p+2)\,!}\left\{\gamma-2\log 2+\sum_1^p\frac{1}{n\,(2n-1)}-\sum_1^{p+2}\frac{1}{n}\right\}.$$

For numerical calculation we have

$$W_{1,1} = \frac{3}{8}\pi^{-1/2}\alpha^{3/2}e^{-\alpha/2}\ \left\{\frac{8}{3\alpha^2}+\frac{4}{\alpha}+\frac{3}{2}+\frac{5}{36}\alpha+\frac{11}{384}\alpha^2+\frac{7}{1280}\alpha^3+\ldots\right.$$
$$\left.-\left(\gamma+\log\frac{1}{4}\alpha\right)\left(1+\frac{1}{6}\alpha+\frac{1}{32}\alpha^2+\frac{1}{192}\alpha^3+\ldots\right)\right\}. \tag{63}$$

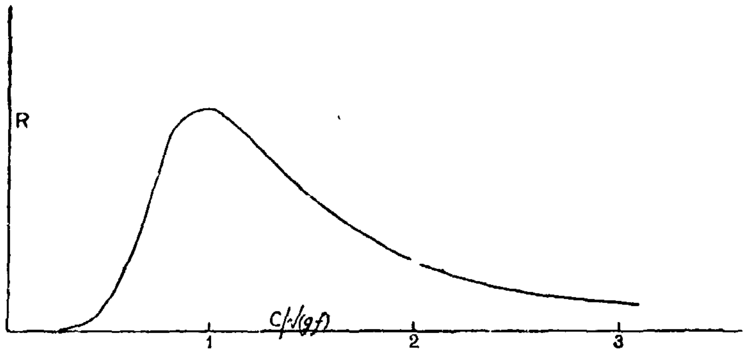
The expansion may be confirmed by comparison with the fundamental solutions of the differential equation given in (60); we find that

$$(8/3)\,\pi^{\frac{1}{2}}\,W_{1,1} = (2\log 2-\gamma-\tfrac{1}{8})\,y_1-y_2.$$

For large values of α the general asymptotic expansion of $W_{k,m}$ is available; and in this case we have

$$W_{1,1} \sim \alpha e^{-\alpha/2}\left(1+\frac{3}{4}\frac{1}{\alpha}+\frac{9}{32}\frac{1}{\alpha^2}-\frac{15}{128}\frac{1}{\alpha^3}+\ldots\right). \tag{64}$$

9. With (63) and (64) we can now calculate the resistance R from (58). For a given depth f, the variation of the resistance with the velocity is shown in the following curve, for which R has been calculated for various values of $c/\sqrt{(gf)}$.



The curve is very similar in form to the two-dimensional case of a circular cylinder. For small velocities, that is α large, if we take the first term of the asymptotic expansion (64), we have

$$R = \sqrt{(2\pi^3 g^7/f)} \cdot \rho a^6 c^{-5} e^{-2gf/c^2},$$

which may be compared with (30) for the cylinder. It is of interest to notice the similar law of variation of wave resistance with speed for the few cases of rigid bodies which have been worked out. The method adopted here can be applied to bodies of different forms, and it is hoped to illustrate later some interference effects.

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Periodic Irrotational Waves of Finite Height.

By T. H. HAVELOCK, F.R.S.

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1. The method of Stokes for waves of finite height on deep water consists in working upwards by successive steps from the infinitesimal wave towards the highest possible wave. Although lacking formal proof of convergence, it is generally accepted that the method is valid, but that it does not include the highest possible wave when the crests form wedges of 120°.

For the highest wave itself we have Michell's analysis by a distinct method, also involving an infinite series whose convergence has to be assumed.*

The theoretical position of Stokes' method has been stated concisely by Prof. Burnside in a recent criticism†:—

"The complete result would be to express the co-ordinates x and y in terms of ϕ and ψ in the form

$$x = -\phi + be^{\psi}\sin\phi + \sum_{2}^{\infty} b^{n} P_{n}(b)\, e^{n\psi}\sin n\phi,$$

$$y = -\psi + be^{\psi}\cos\phi + \sum_{2}^{\infty} b^{n} Q_{n}(b)\, e^{n\psi}\cos n\phi, \qquad (1)$$

where $P_n(b)$, $Q_n(b)$ are power series in b.

"These results have a meaning and can be used for actual approximate calculation only, if P_n, Q_n are convergent power series when b does not exceed some value, say b_0, while for suitable values of b and for real negative values of ψ, the series for x and y are convergent.

"Until the form of the power series P_n and Q_n have been determined, it is impossible to deal with their convergence. Assuming that they are convergent, it is clear from physical considerations that there must be an

* J. H. Michell, 'Phil. Mag.,' Ser. 5, vol. 36, p. 430 (1893).

† W. Burnside, 'Lond. Math. Soc. Proc.,' Ser. 2, vol. 15, p. 26 (1916).

upper limit b' for b in order that the series for x and y may be convergent for negative values of ψ, and there are no means of determining b'."

Prof. Burnside concludes that Stokes' method cannot be used for numerical calculation as it is not known whether the corresponding value of b is less than the above value b'.

In the following notes a general method is suggested, which includes waves of all possible heights, ranging from the highest wave down to the simple infinitesimal wave. The method consists of a simple and direct extension of Michell's analysis for the highest wave. The advantage is a theoretical one which may be expressed in this form: the parameter does not have, as in Stokes' series, an undetermined upper limit, but it enters in the form $e^{-2\alpha}$, where α may have any positive value, including zero.

It should be stated that here, again, we have infinite processes for which no formal proof of convergence is given: we have to rely meantime upon a numerical study of the series. However, in addition, we can compare the method with that of Stokes for waves short of the highest; in this case numerical results obtained by the two methods are the same, as might be expected.

Extending this comparison to the highest possible wave, we get a value for the quantity b' referred to previously, that is, the value of the parameter for which Stokes' series for the elevation become divergent. We obtain b' as $\frac{1}{3}-b_1$, where b_1 has the value 0·0414 approximately, or we have $b' = 0{\cdot}291...$, the value for b_1 being slightly less than the true value.

The discussion is arranged in the following order: Michell's form for the highest wave, its generalisation by means of the surface condition, method of approximation for the coefficients, calculation for the highest wave, the values when $e^{-2\alpha} = \frac{3}{4}$, comparison with Stokes' series, determination of b', further numerical examples and remarks upon the values of the coefficients.

2. It was shown by Stokes that the highest possible wave, under constant pressure at the free surface, has crests in the form of wedges of 120°. It follows directly from his argument, as a simple extension, that the crests will meet at the same angle for the highest possible wave under any assigned surface pressure provided the pressure is stationary in value over the crests.

Consider any assigned surface pressure of this character which is finite, continuous and periodic. To determine the form of the highest possible periodic wave, we may follow Michell's analysis for the case of constant pressure up to the stage at which the coefficients are determined from the given surface condition.

We might then begin with the form given in (5) below, but we may recall briefly Michell's argument. Take Ox horizontal, Oy vertical and

downwards, with O at a crest. Let the successive crests be given by $\phi = \pm n\pi$, n integral; and let the upper surface be $\psi = 0$. If θ be the inclination of the wave line to the horizontal at a point ϕ, assume

$$\frac{\partial\theta}{\partial\phi} = a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \ldots. \tag{2}$$

This is equivalent to assuming that under the given conditions for the surface pressure, the ratio of the curvature of the wave line to the velocity is finite and continuous throughout a wave-length; in that case $\partial\theta/\partial\phi$ can be expanded in a uniformly convergent Fourier series. In the numerical calculations which are needed later, the practical success of the method of approximation depends upon $a_1, a_2, \ldots$, being small compared with a_0, and, in fact, upon the series converging rapidly.

With the notation, $w = \phi + i\psi$, $z = x + iy$, $\mathrm{U} = \log(dz/dw)$, Michell showed that

$$\frac{d\mathrm{U}}{dw} - i(a_0 + a_1 e^{2iw} + a_2 e^{4iw} + \ldots), \tag{3}$$

is a function which is real over the surface $\psi = 0$, and possesses only simple poles, which are at the wave crests.

Suppose that near a crest, say $w = 0$, we have $dz/dw = \mathrm{A}w^n$, then $q^2 = \text{const.} \times r^{-2n/(n+1)}$, where q is fluid velocity and r is distance from the summit. But, since the pressure is constant in the neighbourhood of the crests, we have $q^2 = 2gy$, and hence $n = -\frac{1}{3}$. It follows that the function (3) differs by only a constant from the quantity $-\frac{1}{3}\Sigma(w - n\pi)^{-1}$. Hence, after adjusting the constants and integrating, we find for dz/dw the form

$$\frac{dz}{dw} = (-i \sin w)^{-1/3} e^{-iw/3} (1 + c_1 e^{2iw} + c_2 e^{4iw} + \ldots), \tag{4}$$

the real root of $(-i \sin w)^{-1/3}$ being taken along $\phi = 0$. The units are such that the wave velocity V, or the velocity at $\psi = \infty$, and the wave-length L are given by

$$\mathrm{V} = 2^{-1/3}; \qquad \mathrm{L} = \pi/\mathrm{V} = 2^{1/3}\pi.$$

It is convenient to invert (4) and write

$$\frac{dw}{dz} = (-i \sin w)^{1/3} e^{iw/3} (1 + b_1 e^{2iw} + b_2 e^{4iw} + b_3 e^{6iw} + \ldots). \tag{5}$$

3. The coefficients b_1, b_2, ..., are now to be determined by the pressure condition at the free surface. So far, we have stipulated only that the pressure at $\psi = 0$ shall be finite, continuous, periodic, and stationary at the points $\phi = n\pi$. For our present purpose we shall leave this pressure distribution undetermined, except for these conditions. We shall assume

that it is possible for some assigned stream-line below the surface, say the line $\psi = \alpha$, to be a line of constant pressure. Thus we shall determine $b_1, b_2, \ldots$, directly by applying the condition for constant pressure to (5) when $\psi = \alpha$. The surface $\psi = \alpha$ will then be a possible free wave surface, and free waves will be given by assigning any value to α in the range zero to infinity; thus, by working down from the highest possible wave, we include in one scheme free waves of any permissible height.

The condition that the pressure is constant for $\psi = \alpha$ is

$$q^2 = 2gy + \text{constant}. \tag{6}$$

It is convenient to use an equivalent form obtained by differentiating (6) with respect to ϕ, namely

$$\frac{\partial q^4}{\partial \phi} = 4g\frac{\partial \phi}{\partial y}. \tag{7}$$

This has to be used when

$$\frac{dw}{dz} = \{-i\sin(\phi + i\alpha)\}^{1/3} e^{i(\phi+i\alpha)/3}(1 + \beta_1 e^{2i\phi} + \beta_2 e^{4i\phi} + \ldots), \tag{8}$$

where $\beta_1 = b_1 e^{-2\alpha}, \qquad \beta_2 = b_2 e^{-4\alpha}, \ldots$.

Multiplying by the conjugate complex and squaring, we obtain

$$q^4 = e^{-4\alpha/3}(\sinh^2\alpha + \sin^2\phi)^{2/3}(D_0 + 2D_1\cos 2\phi + 2D_2\cos 4\phi + \ldots), \tag{9}$$

where

$$D_0 = 1 + 4\beta_1^2 + (2\beta_2 + \beta_1^2)^2 + (2\beta_3 + 2\beta_1\beta_2)^2 + \ldots,$$

$$D_1 = 2\beta_1 + 2\beta_1(2\beta_2 + \beta_1^2) + (2\beta_2 + \beta_1^2)(2\beta_3 + 2\beta_1\beta_2) + \ldots,$$

$$D_2 = 2\beta_2 + \beta_1^2 + 2\beta_1(2\beta_3 + 2\beta_1\beta_2) + (2\beta_2 + \beta_1^2)(2\beta_4 + \beta_2^2 + 2\beta_1\beta_3) + \ldots,$$

$$D_3 = 2\beta_3 + 2\beta_1\beta_2 + 2\beta_1(2\beta_4 + \beta_2^2 + 2\beta_1\beta_3) + \ldots,$$

....

....

Differentiating (9) with respect to ϕ, we can take out a factor $(\sinh^2\alpha + \sin^2\phi)^{-1/3}$, and can collect the other terms into a sine series in even multiples of ϕ. However, we take out also the common factor $\sin\phi$, because we then have $\partial q^4/\partial\phi$ in a form which reduces directly to the proper form for the highest wave ($\alpha = 0$), and, in addition, we find that the numerical calculations converge more rapidly. After some reduction, we obtain in this way

$$\frac{\partial q^4}{\partial \phi} = \tfrac{4}{3}e^{-4\alpha/3}\sin\phi(\sinh^2\alpha + \sin^2\phi)^{-1/3}(A_1\cos\phi + A_3\cos 3\phi + \ldots), \tag{10}$$

where

$$A_1 = D_0+\tfrac{5}{2}D_1+6D_2+3(3D_3+4D_4+\dots)-3(D_1+2D_2+3D_3+\dots)\cosh 2\alpha,$$
$$A_3 = \tfrac{5}{2}D_1+4D_2+3(3D_3+4D_4+\dots)-3(2D_2+3D_3+\dots)\cosh 2\alpha,$$
$$A_5 = 4D_2+\tfrac{11}{2}D_3+3(4D_4+5D_5+\dots)-3(3D_3+4D_4+\dots)\cosh 2\alpha,$$
$$A_7 = \tfrac{11}{2}D_3+7D_4+3(5D_5+\dots)-3(4D_4+\dots)\cosh 2\alpha,$$

....

....

For the other side of equation (7), using I to denote the imaginary part Q of a complex quantity $P+iQ$, we have

$$\frac{\partial\phi}{\partial y} = -\mathrm{I}\,e^{-\alpha/3}(\sinh^2\alpha+\sin^2\phi)^{1/6}e^{-i(\theta-\phi)/3}\,\Sigma\,\beta_r e^{2ri\phi}, \tag{11}$$

where $$Re^{i\theta} = \cos\phi\sinh\alpha - i\sin\phi\cosh\alpha.$$

To expand this in a form similar to (10), we notice that

$$e^{-i(\theta-\phi)/3} = (1-e^{-2\alpha}e^{2i\phi})^{1/6}(1-e^{-2\alpha}e^{-2i\phi})^{-1/6},$$
$$(\sinh^2\alpha+\sin^2\phi)^{1/2} = \tfrac{1}{2}e^{\alpha}(1-e^{-2\alpha}e^{2i\phi})^{1/2}(1-e^{-2\alpha}e^{-2i\phi})^{1/2}.$$

Hence we have

$$\frac{\partial\phi}{\partial y} = -\mathrm{I}\,\tfrac{1}{2}e^{2\alpha/3}(\sinh^2\alpha+\sin^2\phi)^{-1/3}\sum_{r=0}^{\infty}\beta_r(1-e^{-2\alpha}e^{2i\phi})^{2/3}(1-e^{-2\alpha}e^{-2i\phi})^{1/3}e^{2ri\phi}. \tag{12}$$

We now expand the two binomial factors after the sign of summation in series valid for the whole range of ϕ and for all positive values of α. We can then write down the coefficient of $e^{2ni\phi}$, and so obtain $\partial\phi/\partial y$, involving a series of sines of even multiples of ϕ; as before, we take out a common factor $\sin\phi$, and obtain the result

$$\frac{\partial\phi}{\partial y} = \tfrac{1}{3}e^{-4\alpha/3}\sin\phi\,(\sinh^2\alpha+\sin^2\phi)^{-1/3}(B_1\cos\phi+B_3\cos 3\phi+\dots), \tag{13}$$

where the B's are linear functions of the β's, with coefficients which are functions of $e^{-2\alpha}$. In practice, these can be obtained directly from (12) to any required degree of approximation; general expressions can be put in the following exact forms

$$B_{2n+1} = B_{0,2n+1}+\beta_1 B_{1,2n+1}+\dots+\beta_r B_{r,2n+1}+\dots,$$
$$B_{r,2n+1} = \sum_{s=n+r+1}^{\infty} C_{-2s} - \sum_{s=n-r+1}^{\infty} C_{2s},$$
$$C_{2s} = 3\,\frac{-\frac{2}{3}(-\frac{2}{3}+1)\dots(-\frac{2}{3}+s-1)}{s!}\,e^{-2(s-1)\alpha}F(-\tfrac{1}{3},-\tfrac{2}{3}+s,s+1,e^{-4\alpha}),$$
$$C_{-2s} = 3\,\frac{-\frac{1}{3}(-\frac{1}{3}+1)\dots(-\frac{1}{3}+s-1)}{s!}\,e^{-2(s-1)\alpha}F(-\tfrac{1}{3}+s,-\tfrac{2}{3},s+1,e^{-4\alpha}),$$
$$C_0 = 3e^{2\alpha}F(-\tfrac{1}{3},-\tfrac{2}{3},1,e^{-4\alpha}), \tag{14}$$

where F represents the hypergeometric series.

We can now apply the surface condition (7) by substituting the expansions (10) and (13) and equating coefficients of $\cos\phi$, $\cos 3\phi$, ...

We obtain, as in Stokes' method, an infinite series of equations of the form $A_{2n+1} = gB_{2n+1}$, from which the quantities g, β_1, β_2, β_3, ..., are to be obtained in practice by successive approximation from the equations taken in order.

Up to terms of the third order the equations are

$$
\begin{aligned}
&1+5\beta_1+12\beta_2+10\beta_1^2+28\beta_1\beta_2+5\beta_1^3+18\beta_3 \\
&\qquad\qquad -6\,(\beta_1+2\beta_2+\beta_1^2+5\beta_1\beta_2+\beta_1^3+3\beta_3)\cosh 2\alpha \\
&\qquad = g\,(B_{01}+\beta_1B_{11}+\beta_2B_{21}+\beta_3B_{31}), \\
&5\beta_1+8\beta_2+4\beta_1^2+28\beta_1\beta_2+5\beta_1^3+18\beta_3-6\,(2\beta_2+\beta_1^2+3\beta_1\beta_2+3\beta_3)\cosh 2\alpha \\
&\qquad = g\,(B_{03}+\beta_1B_{13}+\beta_2B_{23}+\beta_3B_{33}), \\
&8\beta_2+4\beta_1^2+11\beta_1\beta_2+11\beta_3-18\,(\beta_1\beta_2+\beta_3)\cosh 2\alpha \\
&\qquad = g\,(B_{05}+\beta_1B_{15}+\beta_2B_{25}+\beta_3B_{35}), \\
&11\beta_3+11\beta_1\beta_2 = g\,(B_{07}+\beta_1B_{17}+\beta_2B_{27}+\beta_3B_{37}).
\end{aligned} \tag{15}
$$

It might appear, from the quantity $\cosh 2\alpha$ on the left, and from the factor $e^{2\alpha}$ in the expressions for the B's in (14), that there are terms in these equations which become infinitely large as α increases indefinitely. But we have $\beta_1 = b_1e^{-2\alpha}$, $\beta_2 = b_2e^{-4\alpha}$, $\beta_3 = b_3e^{-6\alpha}$, ..., therefore, if we write the equations (15) as a set of equations for the coefficients b_1, b_2, ..., this difficulty disappears. In this connection we may recall the initial assumption that the series in (5), namely, $1+b_1e^{2iw}+b_2e^{4iw}+\ldots$, is absolutely convergent, otherwise the analysis has no meaning.

The infinite set of equations, given to the third order in (15), has to be treated by Stokes' method; that is, assuming the process to be convergent, the equations taken in succession yield approximations to g, β_1, β_2, ..., for any assigned numerical value of α. But there is a difference between these equations and the corresponding set in Stokes' analysis. In the latter, the first coefficient, say b, is arbitrary, and the successive equations have their lowest terms of order zero, one, two, and so on, respectively; thus g and the remaining coefficients are found as power series in b. But in (15), we have a term independent of the β's on the right-hand side of each equation; thus the solution, if practicable, leads to a set of numerical values of g, β_1, β_2,... for a definite numerical value of α. We may notice, in passing, that for a first rough approximation $gB_{01} = 1$; and as B_{01} does not differ much from unity for any value of α, the coefficients β_1, β_2, ..., are of the order of magnitude of B_{03}, B_{05}, ..., respectively.

4. The method of approximation used in the following calculations may be described by considering first the simplest form of the equations, namely,

when $\alpha = 0$. The hypergeometric series in (14) can be summed in this case and we find

$$\mathrm{B}_{r,2n+1} = \frac{18\sqrt{3}}{\pi} \frac{6r+1}{9(2n+1)^2-(6r+1)^2} \tag{16}$$

The equations (15) reduce to

$$\begin{aligned} 1-\beta_1+4\beta_1^2-2\beta_1\beta_2-\beta_1^3 &= k\left(\tfrac{1}{8}-\tfrac{7}{40}\beta_1-\tfrac{13}{160}\beta_2-\tfrac{19}{352}\beta_3\right), \\ 5\beta_1-4\beta_2-2\beta_1^2+10\beta_1\beta_2+5\beta_1^3 &= k\left(\tfrac{1}{80}+\tfrac{7}{32}\beta_1-\tfrac{13}{88}\beta_2-\tfrac{19}{280}\beta_3\right), \\ 8\beta_2+4\beta_1^2-7\beta_1\beta_2-7\beta_3 &= k\left(\tfrac{1}{224}+\tfrac{7}{196}\beta_1+\tfrac{13}{56}\beta_2-\tfrac{19}{136}\beta_3\right), \\ 11\beta_1\beta_2+11\beta_3 &= k\left(\tfrac{1}{440}+\tfrac{7}{392}\beta_1+\tfrac{13}{272}\beta_2+\tfrac{19}{86}\beta_3\right), \end{aligned} \tag{17}$$

with $k = 18g\sqrt{3}/\pi$.

These are Michell's equations for the highest wave. Without specifying any definite method of approximation, Michell states that sufficiently close values are given by

$$g = 0{\cdot}832, \qquad \beta_1 = 0{\cdot}0397, \qquad \beta_2 = 0{\cdot}0094, \qquad \beta_3 = 0{\cdot}002. \tag{18}$$

In order to compare results for different values of α, it is desirable to adopt some consistent scheme of approximation.

In general, in the equations (15), we substitute

$$\begin{aligned} g\mathrm{B}_{01} &= 1+k_1\beta_1+k_2\beta_1^2+k_3\beta_1^3+\ldots, \\ \beta_2 &= b_2\beta_1^2+b_3\beta_1^3+\ldots, \\ \beta_3 &= c_3\beta_1^3+c_4\beta_1^4+\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{19}$$

For a first approximation, write down the first two equations up to terms in β_1, and we get two equations from which to determine k_1 and β_1. The first of these equations is, in fact, independent of β_1 on account of the form of (19).

For a second approximation, retain the value of k_1 so determined, and write down the first three equations of (15) up to the terms in β_1^2, the first of the three being again independent of β_1; from these, we determine k_2, β_2, and a second approximation to β_1. For the third stage, using the values of k_1, k_2, and b_2 already found, and writing down the first four equations of (15) up to the terms in β_1^3, we determine k_3, b_3, c_3, and a third approximation to β_1. Using (19) we obtain the corresponding values of g, β_2, β_3, ..., at any stage. The nth approximation to β_1 is given by an equation of the nth degree in β_1; but there is no difficulty in practice as to the particular root since we follow it through from the first approximation.

The method is simple in plan, if somewhat tedious in practice; so it is not necessary to give the details of the following calculations.

Taking the particular case (17), we may write down one set of equations to illustrate the type. After the first two stages, we obtain $k_1 = 0{\cdot}4$, $k_2 = 7{\cdot}6$, $b_2 = 4{\cdot}67$: using these values we find for the next stage the equations

$$\left.\begin{aligned} k_3 - 0{\cdot}65 b_3 - 0{\cdot}432 c_3 &= 1{\cdot}18 \\ \beta_1^3(0{\cdot}1k_3 + 2{\cdot}82b_3 - 0{\cdot}543c_3 - 44{\cdot}25) + 17{\cdot}745\beta_1^2 - 3{\cdot}21\beta_1 + 0{\cdot}1 &= 0 \\ \beta_1^3(0{\cdot}036k_3 - 6{\cdot}14b_3 + 5{\cdot}88c_3 + 41{\cdot}68) - 34{\cdot}68\beta_1^2 + 0{\cdot}332\beta_1 - 0{\cdot}036 &= 0 \\ \beta_1^3(0{\cdot}018k_3 + 0{\cdot}38b_3 - 9{\cdot}1c_3 - 53{\cdot}77) + 2{\cdot}15\beta_1^2 + 0{\cdot}15\beta_1 + 0{\cdot}018 &= 0 \end{aligned}\right\} \tag{20}$$

Eliminating k_3, b_3, c_3, we get a cubic for β_1, of which the required root is 0·0407, the previous stages having given the values 0·0311, 0·039. Also from (20) we find $k_3 = 30$, $b_3 = 35$, $c_3 = 40$. Collecting the results to this stage, we find

$$g = 0{\cdot}833, \quad \beta_1 = 0{\cdot}0407, \quad \beta_2 = 0{\cdot}0106, \quad \beta_3 = 0{\cdot}0027. \tag{21}$$

These values are rather higher than those given by Michell (18). In order to determine β_1 more closely, the approximation has been carried to the fourth stage, with the result

$$g = 0{\cdot}833, \quad \beta_1 = 0{\cdot}0414, \quad \beta_2 = 0{\cdot}0114, \quad \beta_3 = 0{\cdot}0042, \quad \beta_4 = 0{\cdot}0014. \tag{22}$$

With these values, the ratio of h, the height of the wave, to L, the wave-length, is given by

$$\begin{aligned} h/\mathrm{L} &= (\text{velocity at trough})^2/2g\mathrm{L} \\ &= (1-\beta_1+\beta_2-\beta_3+\beta_4)^2/2^{4/3}g\pi = 0{\cdot}1418. \end{aligned} \tag{23}$$

An interesting point about the series $1+\beta_1 e^{2i\phi}+\beta_2 e^{4i\phi}+\ldots$ for the highest wave is the smallness of all the coefficients $\beta_1, \beta_2, \ldots$, compared with the first term, namely, unity; on the other hand, the numerical values obtained do not suggest a rapid convergence of the series after the first term. It appears, from the method of approximation, and from the fact that all the quantities $\mathrm{B}_{01}, \mathrm{B}_{03}, \ldots$, are positive, that successive approximations will increase the values of the coefficients. A test for the sum of the series, compared with the value of g, is obtained by considering the velocity near a crest.

Near $\phi = 0$, we have $dw/dz = \phi^{1/3}e^{-i\pi/6}(1+\beta_1+\beta_2+\ldots)$.

Therefore $q^2 = \phi^{2/3}(1+\beta_1+\beta_2+\ldots)^2$ and $z = \frac{3}{2}\phi^{2/3}e^{i\pi/6}/(1+\beta_1+\beta_2+\ldots)$; and since $q^2 = 2gy$, it follows that we should have $(1+\beta_1+\beta_2+\ldots)^3/g = 1{\cdot}5$. But with the values given in (22), this expression has the value 1·42. This is perhaps a severe test; a simpler criterion is to write down the successive convergents to any one coefficient; for example, these for the leading coefficient β_1 are 0·0311, 0·0390, 0·0407, and 0·0414.

5. Returning to the general equations (15), we consider a wave short of the highest and we select the case $e^{-2a} = \frac{3}{4}$. We shall find that this corresponds to a value of about $\frac{1}{4}$ for Stokes' parameter b.

The coefficients $B_{r, 2n+1}$ have to be calculated from the relations (14); the hypergeometric series are, of course, convergent, and the values can be obtained to any required degree of accuracy. Substituting the numerical values in (15) we obtain the equations

$$\begin{aligned}
1-1{\cdot}25\beta_1-0{\cdot}5\beta_2+3{\cdot}75\beta_1^2-3{\cdot}25\beta_1\beta_2-1{\cdot}25\beta_1^3-0{\cdot}75\beta_3 \\
= g\,(1{\cdot}081-2{\cdot}584\beta_1-1{\cdot}53\beta_2-1{\cdot}281\beta_3), \\
5\beta_1-4{\cdot}5\beta_2-2{\cdot}25\beta_1^2+9{\cdot}25\beta_1\beta_2+5\beta_1^3-0{\cdot}75\beta_3 \\
= g\,(0{\cdot}0166+2{\cdot}133\beta_1-2{\cdot}322\beta_2-1{\cdot}429\beta_3), \\
8\beta_2+4\beta_1^2-7{\cdot}75\beta_1\beta_2-7{\cdot}75\beta_3 = g\,(-0{\cdot}0157+0{\cdot}2767\beta_1+2{\cdot}237\beta_2-2{\cdot}274\beta_3) \\
11\beta_3+11\beta_1\beta_2 = g\,(\quad 0{\cdot}0125+0{\cdot}0865\beta_1+0{\cdot}3243\beta_2+2{\cdot}254\beta_3).
\end{aligned} \tag{24}$$

We carry out now the successive approximations described in the previous section. At the third stage, we find

$$g = 0{\cdot}9246, \quad \beta_1 = 0{\cdot}00273, \quad \beta_2 = -0{\cdot}0034, \quad \beta_3 = -0{\cdot}0013. \tag{25}$$

Comparing these values with those for the highest wave given in (21), we see that the β's are much smaller; on the other hand, there may be greater difficulty in obtaining their values accurately, because of the later stage at which the β's begin to diminish steadily in absolute value. We shall find this impression confirmed later when we try smaller values of e^{-2a}.

To find the ratio h/L for this wave, we have

$$(\text{velocity at crest})^2 = 2^{-2/3}(1-e^{-2a})^{2/3}(1+\beta_1+\beta_2+\beta_3+\ldots)^2,$$
$$(\text{velocity at trough})^2 = 2^{-2/3}(1+e^{-2a})^{2/3}(1-\beta_1+\beta_2-\beta_3+\ldots)^2.$$

Taking the difference, and dividing by $2g$, we find h; and since $L = 2^{1/3}\pi$, we have $h/L = 0{\cdot}0898$. Stokes' parameter b is, to a first approximation, $\pi h/L$; hence this wave corresponds to b equal to $\frac{1}{4}$ nearly.

6. We have now two methods for a wave of finite height, namely, that described above and Stokes' method. The two can be shown to be in agreement in any particular case.

From (8), we have, on the wave surface $\psi = \alpha$,

$$2^{-1/3}\frac{dz}{dw} = (1-e^{-2a}e^{2i\phi})^{-1/3}(1+\beta_1 e^{2i\phi}+\beta_2 e^{4i\phi}+\ldots)^{-1}. \tag{26}$$

For any wave below the highest possible, that is provided α is not zero, the first factor on the right of (26) can be expanded in a series valid for

all values of ϕ; hence, under these conditions, we have on the surface $\psi = \alpha$,

$$2^{-1/3}\frac{dz}{dw} = 1 + A_1 e^{2i\phi} + A_2 e^{4i\phi} + A_3 e^{6i\phi} + \ldots, \tag{27}$$

where

$$A_1 = \tfrac{1}{3} e^{-2\alpha} - \beta_1,$$
$$A_2 = \tfrac{2}{9} e^{-4\alpha} - \tfrac{1}{3}\beta_1 e^{-2\alpha} - (\beta_2 - \beta_1^2),$$
$$A_3 = \tfrac{14}{81} e^{-6\alpha} - \tfrac{2}{9}\beta_1 e^{-4\alpha} - \tfrac{1}{3}(\beta_2 - \beta_1^2) e^{-2\alpha} - (\beta_3 - 2\beta_1\beta_2 + \beta_1^3),$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now Stokes' method gives z, and dz/dw, in the form of a series like (27); write this as

$$cz = 1 + C_1 e^{2i\phi} + C_2 e^{4i\phi} + C_3 e^{6i\phi} + \ldots. \tag{28}$$

From Stokes' equations, C_2, C_3, ..., are obtained as power series in C_1; these have been carried up to the tenth order by Wilton,* whose results we quote now—in so far as they are needed here—

$$-C_1 = b,$$
$$C_2 = b^2 + 0{\cdot}5b^4 + 2{\cdot}417b^6 + 15{\cdot}597b^8 + 64{\cdot}08b^{10}$$
$$-C_3 = 1{\cdot}5b^3 + 1{\cdot}583b^5 + 8{\cdot}215b^7 + 55{\cdot}01b^9,$$
$$C_4 = 2{\cdot}667b^4 + 4{\cdot}347b^6 + 24{\cdot}01b^8 + 166{\cdot}2b^{10},$$
$$-C_5 = 5{\cdot}208b^5 + 11{\cdot}53b^7 + 67{\cdot}40b^9,$$
$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{2\pi c^2}{g\lambda} = 1 + b^2 + 3{\cdot}5b^4 + 19{\cdot}08b^6 + 154{\cdot}7b^8 + 1297b^{10}. \tag{29}$$

With the units adopted here, the last expression corresponds to $1/g$. Further, in Stokes' investigations the wave-length was taken as 2π, while in the above work we have used π; the result is that in comparing the two methods by means of (27) and (29), C_1, C_2, C_3, C_4, ..., correspond respectively to A_1, $\tfrac{1}{2}A_2$, $\tfrac{1}{3}A_3$, $\tfrac{1}{4}A_4$, ...

For the numerical calculations in the case $e^{-2\alpha} = \tfrac{3}{4}$, we use the values of β_1, β_2, and β_3 given in (25); then from (27) we obtain

$$A_1 = 0{\cdot}24727; \quad \tfrac{1}{2}A_2 = 0{\cdot}06385; \quad \tfrac{1}{3}A_3 = 0{\cdot}0249;$$
$$\tfrac{1}{4}A_4 = 0{\cdot}0115; \quad \tfrac{1}{5}A_5 = 0{\cdot}0058; \ \ldots\ldots \tag{30}$$

On the other hand, if we take b equal to A_1, we get from this series in (29)

$$-C_1 = 0{\cdot}24727; \quad C_2 = 0{\cdot}06382; \quad -C_3 = 0{\cdot}0248;$$
$$C_4 = 0{\cdot}0114; \quad -C_5 = 0{\cdot}0058; \ \ldots\ldots \tag{31}$$

* J. R. Wilton, 'Phil. Mag.,' Ser. 6, vol. 27, p. 385 (1914).

It is unnecessary to carry the calculation further to show the numerical agreement between the two methods for waves short of the highest. It may be noticed that in the above comparison we have gone up to the coefficient β_3 of the present method; to obtain the agreement shown above, we have had to use the Stokes' series as far as the tenth order in the parameter.

7. From the comparison between (27) and (28), we see that, for waves lower than the highest, we are in effect dealing with a Stokes' series whose parameter has the value $\frac{1}{3}e^{-2\alpha}-\beta_1$. If we applied Stokes' method directly to (27), we should obtain $A_2, A_3, \ldots$, in the usual way as power series in this parameter, and the quantity $e^{-2\alpha}$ would be a superfluous arbitrary parameter. On the other hand, the present method gives a definite value of β_1 for an assigned value of α, or theoretically gives a functional relation between β_1 and α. The method definitely connects a wave of any height with the highest possible wave, and any possible wave-form is given as one of a family whose limiting curve has crests consisting of wedges of 120°.

Consider the expansion from the form (26) to the corresponding Stokes' form (27) or (28). Assuming the convergence of the series with the β-coefficients, the expansion is valid over the whole range of ϕ for all positive values of α, excluding zero; it is also valid for α zero, with the exception of the points $\phi = n\pi$, n integral. In other words, the comparison confirms the view that Stokes' series for the elevation is valid throughout, with the exception of the actual crests of the highest possible wave.

We can now estimate the limiting value of the Stokes' parameter b for convergence at the crests. To do this, we compare the series (27) for the highest wave with a Stokes' series, for points other than the crests.

For the highest wave $\alpha = 0$, we found

$$\beta_1 = 0{\cdot}0414, \qquad \beta_2 = 0{\cdot}0114, \qquad \beta_3 = 0{\cdot}0042, \qquad \beta_4 = 0{\cdot}0014.$$

Hence the expansion should be a Stokes' series with the parameter $\frac{1}{3}-0{\cdot}0414$, or say $0{\cdot}2919$. Making the comparison between (27) and (29) with these values, we find

$$\begin{aligned} A_1 &= 0{\cdot}2919\,; & \tfrac{1}{2}A_2 &= 0{\cdot}0993\,; & \tfrac{1}{3}A_3 &= 0{\cdot}0523\,; \\ -C_1 &= 0{\cdot}2919\,; & C_2 &= 0{\cdot}0914\,; & -C_3 &= 0{\cdot}0429. \end{aligned} \qquad (32)$$

The agreement is sufficient to justify the comparison, when we remember that the β-coefficients have only been determined to the fourth stage, and further, that the Stokes' series (29) for the C-coefficients presumably converge slowly in this extreme case.

It should be remarked that we do not gain information from this comparison about the convergence of the Stokes' series for the separate coefficients

for higher values of the parameter; the result concerns the series for the elevation. We find that Stokes' series for the elevation becomes divergent at the crests when the parameter has the value 0·291..., so far as the numerical calculation has been carried.

In this connection reference may be made to Wilton,* who concluded that the Stokes' series certainly diverge for a parameter greater than $1/e$, and who estimated the limiting value to be in the neighbourhood of $\frac{1}{3}$.

Wilton works out in detail a numerical example with the parameter $b = 0{\cdot}316$, for comparison with the highest wave. According to the present analysis, this is beyond the limiting value for b; the series should be divergent at the crest. This may well be the case, notwithstanding that the coefficients C, as calculated by Wilton, diminish steadily as far as the order shown; since the series is supposed to be divergent only at the crests, one might expect the divergence to become evident numerically only after calculating a large number of terms. The example may serve as an illustration of Prof. Burnside's criticism, that it is necessary to know the limiting value of b before Stokes' series can be used with confidence for numerical calculation.

8. We may examine briefly the present method for waves of small height. It is of interest first to consider the exact expression

$$\frac{dw}{dz} = 2^{-1/3}(1-e^{2iw})^{1/3}. \tag{33}$$

We can integrate dz/dw and so obtain the equation of the stream-lines in finite form, and also exact expressions for the variations of pressure along any stream-line. To find how far (33) satisfies the condition for a free wave under constant pressure at a stream line $\psi = \alpha$, it is simpler to expand first before integrating; we can then express q^2 and y as cosine series. In this way we find at the wave surface $\psi = \alpha$, writing down the variable part only,

$$\begin{aligned}\text{Const.} \times (q^2-2gy) = &\left\{\tfrac{1}{3}ge^{-2\alpha}-\left(\tfrac{1}{3}e^{-2\alpha}-\tfrac{1}{27}e^{-6\alpha}-\tfrac{5}{729}e^{-8\alpha}-\ldots\right)\right\}\cos 2\phi \\ &+\left\{\tfrac{1}{9}ge^{-4\alpha}-\left(\tfrac{1}{9}e^{-4\alpha}-\tfrac{5}{243}e^{-8\alpha}-\ldots\right)\right\}\cos 4\phi \\ &+\left\{\tfrac{14}{243}ge^{-6\alpha}-\left(\tfrac{5}{81}e^{-6\alpha}-\tfrac{10}{729}e^{-10\alpha}-\ldots\right)\right\}\cos 6\phi \\ &+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{34}$$

Hence, if we take $g^{-1} = 1+\frac{1}{9}e^{-4\alpha}$, the pressure is constant up to, and including, terms in $e^{-4\alpha}$; and the next term is the small quantity $-\frac{1}{243}e^{-6\alpha}\cos 6\phi$. This value for $1/g$ is Stokes' expression $1+b^2$, the

* J. R. Wilton, *loc cit. ante.*

parameter b having the value $\frac{1}{3}e^{-2\alpha}$ to this order. It is, of course, impossible to make the right-hand side of (34) zero for all values of ϕ merely by choosing g a suitable function of α. However, the fact that (33) satisfies approximately, to the order shown above, the conditions for a free wave of any height, explains the smallness of the coefficients β_1, β_2,... even for the highest wave when α is zero.

Returning to the general equations (15), for a numerical example of a wave of moderate height we take $e^{-2\alpha} = 3/10$. In this case we shall only carry the approximation to the second stage, to illustrate the character of the coefficients. We have the following numerical values:

$$B_{01} = 1{\cdot}008\,; \qquad B_{11} = -8{\cdot}24\,; \qquad B_{21} = -7{\cdot}336\,;$$
$$B_{03} = -0{\cdot}002\,; \qquad B_{13} = 2{\cdot}056\,; \qquad B_{23} = -8{\cdot}3\,;$$
$$B_{05} = -0{\cdot}004\,; \qquad B_{15} = 0{\cdot}082\,; \qquad B_{25} = 2{\cdot}1.$$

From these we obtain $\beta_1 = -0{\cdot}0018$; $\beta_2 = -0{\cdot}00066$. Making the comparison with a Stokes' series, as in the previous sections, we find

$$A_1 = -C_1 = 0{\cdot}1018, \qquad \tfrac{1}{2}A_2 = C_2 = 0{\cdot}0104, \qquad \tfrac{1}{3}A_3 = -C_3 = 0{\cdot}0016.$$

The numerical values confirm the impression that while the β-coefficients diminish indefinitely as the height of the wave becomes smaller, it is more difficult to obtain their values by the method of successive approximation used in dealing with the infinite set of equations for them.

The behaviour of the β-coefficients is made clearer by studying the leading terms B_{01}, B_{03}, ..., on the right-hand side of equations (15). Over the whole range for α, from zero to infinity, B_{01} only varies from about 1·24 to 1; consequently, from the first equation, g is never much different from unity. From the remaining equations, we see that β_1, β_2, β_3, ..., form a parallel series to B_{03}, B_{05}, B_{07}, ..., taken in order.

It is only for the highest wave ($\alpha = 0$) that all the terms of the latter series are positive and decrease steadily to zero from the first term; for other values, the series is not quite so simple in form, although in all cases the terms converge ultimately to zero. The character of these terms is best illustrated by numerical examples, such as are given in the following Table:

$e^{-2\alpha}$.	B_{01}.	B_{03}.	B_{06}.	B_{07}.	B_{09}.
1·0	1·2405	0·124	0·0443	0·0225	0·0136
0·9	1·111	0·083	−0·007	−0·013	−0·010
0·75	1·0806	−0·0167	−0·0157	−0·0126	−0·0071
0·3	1·008	−0·0022	−0·0042	−0·0013	−0·0004

The quantities $B_{0,2n+1}$, and, in fact, all the coefficients $B_{r,2n+1}$, can be studied algebraically from the relations (14). The algebraic solution of equations (15), together with a formal study of convergence, would be of great interest; meantime the numerical illustrations given in the foregoing discussion may serve to show the possibility of a general scheme which includes waves of any permissible height.

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Wave Resistance: Some Cases of Three-dimensional Fluid Motion.

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1. Calculations of wave resistance, corresponding to a pressure system travelling over the surface, have hitherto been limited to two-dimensional fluid motion; in those cases, the distribution of pressure on the surface is one-dimensional, and the regular waves produced have straight, parallel crests. The object of the following paper is to work out some cases when the surface pressure is two-dimensional and the wave pattern is like that produced by a ship. A certain pressure system symmetrical about a point is first examined, and more general distributions are obtained by superposition. By combining two simple systems of equal magnitude, one in rear of the other, we obtain results which show interesting interference effects. In similar calculations with line pressure systems, at certain speeds the waves due to one system cancel out those due to the other, and the wave resistance is zero; the corresponding ideal form of ship has been called a wave-free pontoon. Such cases of perfect interference do not occur in three-dimensional problems; the graph showing the variation of wave resistance with velocity has the humps and hollows which are characteristic of the resistance curves of ship models.

Although the main object is to show how to calculate the wave resistance for assigned surface pressures of considerable generality, it is of interest to interpret some of the results in terms of a certain related problem. With certain limitations, the waves produced by a travelling surface pressure are such as would be caused by a submerged body of suitable form. The expression for the wave resistance of a submerged sphere, given in a previous paper, is confirmed by the following analysis. It is also shown how to extend the method to a submerged body whose form is derived from stream lines obtained by combining sources and sinks with a uniform stream; in particular, an expression is given for the wave resistance of a prolate spheroid moving in the direction of its axis.

2. Take axes Ox and Oy in the undisturbed horizontal surface of deep water, and Oz vertically upwards, and let ζ be the surface elevation. For an initial impulse symmetrical about the origin, that is for initial data

$$\rho\phi_0 = \mathrm{F}(\varpi); \qquad \zeta = 0;$$

where $\varpi^2 = x^2+y^2$, the velocity potential of the subsequent fluid motion, and the surface elevation are given by*

$$\phi = \frac{1}{\rho}\int_0^\infty f(\kappa)\, e^{\kappa z}\, J_0(\kappa\varpi)\cos(\kappa V t)\,\kappa d\kappa, \tag{1}$$

$$\zeta = -\frac{1}{g\rho}\int_0^\infty f(\kappa)\, J_0(\kappa\varpi)\sin(\kappa V t)\,\kappa^2 V d\kappa, \tag{2}$$

where $V^2 = g/\kappa$, and

$$f(\kappa) = \int_0^\infty F(\alpha)\, J_0(\kappa\alpha)\,\alpha\, d\alpha. \tag{3}$$

We have assumed that it is permissible to use the integral theorem

$$F(\varpi) = \int_0^\infty J_0(\kappa\varpi)\,\kappa\, d\kappa\int_0^\infty F(\alpha)\, J_0(\kappa\alpha)\,\alpha\, d\alpha. \tag{4}$$

We obtain the effect of a pressure system moving over the surface with uniform velocity c in the direction $-x$ by integrating (1) and (2) after suitable modifications. We replace t by $t-\tau$ and x by $x-c\tau$, and integrate with respect to τ over the time during which the system has been in motion. We shall limit the present discussion to the case when the system has been in motion for a long time, so that if we take an origin moving with the system a steady relative condition has been attained. In this case, with a moving origin O, we replace x by $x+cu$ and t by u in (1) and (2), and obtain the required results†

$$\rho\phi = \int_0^\infty e^{-\frac{1}{2}\mu u}\, du\int_0^\infty f(\kappa)\, e^{\kappa z}\, J_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}]\cos(\kappa V u)\,\kappa d\kappa, \tag{5}$$

$$g\rho\zeta = -\int_0^\infty e^{-\frac{1}{2}\mu u}\, du\int_0^\infty f(\kappa)\, J_0[\kappa\sqrt{\{(x+cu)^2+y^2\}}]\sin(\kappa V u)\,\kappa^2 V d\kappa, \tag{6}$$

where $f(\kappa)$ is obtained from the assigned pressure distribution $p = F(\varpi)$ by means of (3).

The introduction of the factor $\exp(-\mu u/2)$ is familiar in these problems and needs only a brief explanation. It may be regarded as an artifice to keep the integrals determinate, it being understood that ultimately μ is to be made infinitesimal. Or, again, it ensures that the solution is the fluid motion which would establish itself eventually under the action of dissipative forces, however small.

In the steady motion with which we are concerned, we may imagine a rigid cover fitting the water surface everywhere and moving forward with uniform velocity c. The assigned pressure system $F(\varpi)$ is applied to the

* Lamb, 'Hydrodynamics,' 6th edn. (1932), p. 432.
† 'Roy. Soc. Proc.,' A, vol. 81, p. 417 (1908).

water surface by means of this cover; hence the corresponding wave resistance is simply the total resolved pressure in the direction Ox.* With the usual limitation that the slope of the surface is everywhere small, this leads to

$$R = \int F(\varpi) \frac{\partial \zeta}{\partial x} dS, \tag{7}$$

taken over the whole surface.

The evaluation of the steady wave resistance for an assigned pressure $F(\varpi)$ is to be carried out by means of (3), (5), (6), and (7). However, we may obtain simpler expressions before applying them to particular cases.

3. For this purpose we analyse the wave disturbance (5) into simple constituents, in fact into one-dimensional disturbances ranged at all possible angles round Ox, the line of advance. We have

$$\pi J_0 [\kappa\{(x+cu)^2+y^2\}^{1/2}] = \int_0^{\pi} e^{i\kappa(x+cu)\cos\phi} \cos(\kappa y \sin\phi)\, d\phi. \tag{8}$$

Substitute in (6) and we can now carry out the integration with respect to u; for we have

$$2\int_0^{\infty} e^{-\frac{1}{2}\mu u} e^{i\kappa cu\cos\phi} \sin(\kappa V u)\, du$$
$$= (\kappa c\cos\phi + \kappa V + \tfrac{1}{2}\mu i)^{-1} - (\kappa c\cos\phi - \kappa V + \tfrac{1}{2}\mu i)^{-1}. \tag{9}$$

We simplify this expression further by using the fact that μ was introduced only to keep the integrals determinate, and is eventually to be made infinitesimal; we can therefore reject terms in μ which are superfluous for this purpose. The process receives its justification in the course of the analysis. This being understood, we can use, instead of the right-hand side of (9), the expression

$$-2(V/c^2)\sec^2\phi / \{\kappa - \kappa_0 \sec^2\phi + i(\mu/c)\sec\phi\}, \tag{10}$$

where $\kappa_0 = g/c^2$. Using these results in (6), and making a slight transformation, we can express the surface elevation in the form

$$\zeta = \frac{\kappa_0}{2\pi g\rho} \int_{-\pi/2}^{\pi/2} \sec^2\phi\, d\phi \int_0^{\infty} \kappa f(\kappa) \left\{ \frac{e^{i\kappa(x\cos\phi + y\sin\phi)}}{\kappa - \kappa_0\sec^2\phi + i(\mu/c)\sec\phi} \right.$$
$$\left. + \frac{e^{-i\kappa(x\cos\phi + y\sin\phi)}}{\kappa - \kappa_0\sec^2\phi - i(\mu/c)\sec\phi} \right\} d\kappa. \tag{11}$$

In (11) we have the surface elevation analysed into plane wave constituents, each element moving in a line making an angle ϕ with Ox. Carrying out the integration with respect to κ, we can express each constituent

* 'Roy. Soc. Proc.,' A, vol. 93, p. 244 (1917).

in terms of a simple harmonic wave in rear of the line $x \cos \phi + y \sin \phi = 0$, together with a disturbance symmetrical with respect to this line. We might continue the discussion for general types of pressure distribution, provided the functions are such that certain transformations may be used; however, it is simpler to study in detail a few cases for which the conditions are all satisfied.

4. At this stage it is convenient to specify the system

$$p = \mathrm{F}(\varpi) = \mathrm{A}f/(f^2+\varpi^2)^{3/2}, \tag{12}$$

where A and f are constants. Using (3) for this case, we have $f(\kappa) = \mathrm{A}e^{-\kappa f}$. Returning to the expression (11) for the elevation, we consider the element making an angle ϕ with Ox. We change to axes Ox', Oy' given by $x' = x \cos \phi + y \sin \phi$, $y' = y \cos \phi - x \sin \phi$. The integral with respect to κ then becomes

$$\mathrm{A}\int_0^\infty \kappa e^{-\kappa f}\left\{\frac{e^{i\kappa x'}}{\kappa-\kappa_0 \sec^2\phi + i(\mu/c)\sec\phi} + \frac{e^{-i\kappa x'}}{\kappa-\kappa_0\sec^2\phi - i(\mu/c)\sec\phi}\right\}d\kappa. \tag{13}$$

This integral can be transformed by contour integration; as it is of a type familiar in plane wave problems* we write down the results when these have been simplified by making μ zero after the transformation has been carried out. We have for the value of (13).

$$4\pi\kappa_0\mathrm{A}\sec^2\phi\, e^{-\kappa_0 f\sec^2\phi}\sin(\kappa_0 x'\sec^2\phi)$$
$$+2\mathrm{A}\int_0^\infty \frac{\kappa_0\sec^2\phi\cos fm + m\sin fm}{m^2+\kappa_0^2\sec^4\phi}e^{mx'}\,m\,dm, \text{ for } x'<0;$$
$$2\mathrm{A}\int_0^\infty \frac{\kappa_0\sec^2\phi\cos fm + m\sin fm}{m^2+\kappa_0^2\sec^4\phi}e^{-mx'}\,m\,dm, \text{ for } x'>0. \tag{14}$$

From (11) and (14) we could now write down the elevation ζ as the sum of the constituents for all values of ϕ in the range from $-\pi/2$ to $\pi/2$. The first term in (14) represents simple waves in the rear of the corresponding wave front $x \cos \phi + y \sin \phi = 0$; hence the integration of this term would only extend over elements for which the assigned point (x, y) was in the rear of the wave front. The other terms in (14) represent a disturbance symmetrical with respect to the wave front, and diminishing with increasing distance from it. We shall not write down the expressions, as we do not intend to examine the wave pattern in detail. From the definition in (7), it follows that we can evaluate the wave resistance R by considering first a simple constituent of the elevation and then summing with respect to ϕ.

Since the pressure system is symmetrical with respect to the origin, the symmetrical local disturbance in (14) gives no resultant contribution to R;

* Compare, for example, 'Roy. Soc. Proc.,' A, vol. 93, p. 524 (1917).

also the part due to the regular waves in rear of the wave front $x \cos\phi + y \sin\phi = 0$ is given by

$$4\pi\kappa_0^2 \mathrm{A} \sec^3\phi\, e^{-\kappa_0 f \sec^2\phi} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{0} \frac{\mathrm{A} f \cos(\kappa_0 x' \sec^2\phi)}{(x'^2+y'^2+f^2)^{3/2}} dx'$$
$$= 4\pi^2 \mathrm{A}^2 \kappa_0^2 \sec^3\phi\, e^{-2\kappa_0 f \sec^2\phi}. \quad (15)$$

Collecting these results we have, from (11) and (15),

$$\mathrm{R} = (4\pi/g\rho)\, \mathrm{A}^2 \kappa_0^3 \int_0^{\pi/2} \sec^5\phi\, e^{-2\kappa_0 f \sec^2\phi}\, d\phi. \quad (16)$$

We may express R in terms of known functions in two convenient forms. If $\mathrm{W}_{k,m}(\alpha)$ is the confluent hypergeometric function defined, under certain conditions, by*

$$\mathrm{W}_{k,m}(\alpha) = \frac{e^{-\alpha/2}\alpha^k}{\Gamma(\frac{1}{2}-k+m)} \int_0^{\infty} u^{-k+m-\frac{1}{2}} (1+u/\alpha)^{k+m-\frac{1}{2}} e^{-u}\, du, \quad (17)$$

and if $\mathrm{K}_n(\alpha)$ is the Bessel function for which†

$$\mathrm{K}_n(\alpha) = (-1)^n \int_0^{\infty} e^{-\alpha\cosh\psi} \cosh n\psi\, d\psi, \quad (18)$$

we find, after some reduction, that

$$\mathrm{R} = (\pi^{3/2}/4g\rho f^3)\, \mathrm{A}^2 \alpha^{3/2} e^{-\alpha/2}\, \mathrm{W}_{1,1}(\alpha) \quad (19)$$
$$= (\pi/8g\rho f^3)\, \mathrm{A}^2 \alpha^3 e^{-\alpha/2} \{\mathrm{K}_0(\alpha/2) - (1+1/\alpha)\, \mathrm{K}_1(\alpha/2)\}, \quad (20)$$

where $\alpha = 2\kappa_0 f = 2gf/c^2$.

In a previous paper,‡ the same function of velocity, except for the constant factors, was found for the wave resistance of a submerged sphere; the result was given in the form (19), and a graph was drawn to show R as a function of c. The resistance rises to a maximum in the neighbourhood of $c = \sqrt{(gf)}$, and then falls asymptotically to zero.

Although there are few tables available for the functions K_n in general,† K_0 and K_1 are given in 'Funktionentafeln' (Jahnke u. Emde) under the form of $(i\pi/2)\, \mathrm{H}_0^{(1)}(ix)$ and $(\pi/2)\, \mathrm{H}_1^{(1)}(ix)$ respectively.

5. Reference has been made to the wave resistance of a sphere submerged at a depth f, large compared with the radius a: this was calculated directly as the resultant horizontal pressure on the sphere. The connection with the present analysis is easily shown.

In the paper referred to, the approximate solution for a submerged body was found directly, following Prof. Lamb's method for a cylinder. It is con-

* Whittaker and Watson, 'Modern Analysis,' p. 334.

† Grey and Mathews, 'Bessel Functions,' p. 90.

‡ 'Roy. Soc. Proc.,' A, vol. 93, p. 530 (1917).

(Note by Editor: These functions have now been tabulated in G. N. Watson, Theory of Bessel Functions (pub. C.U.P., 1922).

venient to repeat here the expressions for the velocity potential and surface elevation due to a cylinder and to a sphere, putting them into the same notation for purposes of comparison. We have, for a cylinder

$$\phi = \frac{ca^2 x}{x^2+(z+f)^2} - \frac{ca^2 x}{x^2+(z+f)^2}$$
$$+2ca^2\int_0^\infty e^{-\mu u/2}\,du\int_0^\infty e^{-\kappa(f-z)}\sin\kappa(x+cu)\sin(\kappa V u)\,\kappa V\,d\kappa, \quad (21)$$

$$\zeta = 2a^2 f/(x^2+f^2) - 2a^2\int_0^\infty e^{-\mu u/2}\,du\int_0^\infty e^{-\kappa f}\cos\kappa(x+cu)\sin(\kappa V u)\,\kappa V\,d\kappa, \quad (22)$$

and for a sphere

$$\phi = \frac{ca^3 x}{2\{x^2+y^2+(z+f)^2\}^{3/2}} - \frac{ca^3 x}{2\{x^2+y^2+(z-f)^2\}^{3/2}}$$
$$-ca^3\frac{\partial}{\partial x}\int_0^\infty e^{-\mu u/2}\,du\int_0^\infty e^{-\kappa(f-z)}J_0[\kappa\surd\{(x+cu)^2+y^2\}]\sin(\kappa V u)\,\kappa V\,d\kappa. \quad (23)$$

$$\zeta = a^3 f/(x^2+y^2+f^2)^{3/2}$$
$$-a^3\int_0^\infty e^{-\mu u/2}\,du\int_0^\infty e^{-\kappa f}J_0[\kappa\surd\{(x+cu)^2+y^2\}]\sin(\kappa V u)\,\kappa^2 V\,d\kappa. \quad (24)$$

These expressions satisfy the conditions at the free surface, namely, $c\partial\phi/\partial x+g\zeta = 0$ and $\partial\phi/\partial z = c\partial\zeta/\partial x$, when μ is made infinitesimal. Opportunity has been taken to correct an obvious mistake in sign in the expressions for the sphere; in the former paper, the last terms in (23) and (24) were given as positive instead of negative.

Returning to the comparison with § 4, consider the expression (24) for the surface elevation due to a submerged sphere. The first part represents a disturbance symmetrical about the origin, due to a doublet at the centre of the sphere, together with an equal opposite doublet at a point a height f above the free surface. Compare now the second term in (24) with the surface elevation given by (6) when the pressure system is (12), so that $f(\kappa) = Ae^{-\kappa f}$. The two expressions are identical, with a suitable relation between the constants; we must have $a^3 = A/g\rho$, or the corresponding moment of the doublet is $Ac/2g\rho$. We have then two related problems. For the submerged sphere the pressure is constant at the free surface, and the surface elevation consists of the two parts in (24); the wave resistance depends upon the supply of energy needed to maintain the waves contained in the second part of (24), and this energy is supplied through the work of the pressure at the surface of the sphere. On the other hand, for the travelling surface pressure,

$$p = g\rho a^3 f/(f^2+\varpi^2)^{3/2}, \quad (25)$$

the surface elevation is the same as the second part of (24); hence the same supply of energy is needed, and is obtained in this case from the work of the applied pressure. Thus we may assume that the wave resistance is the same in the two problems.

From (19) and (25) we have

$$R = \tfrac{1}{4}\pi^{3/2} g\rho f^{-3} a^3 \alpha^{3/2} e^{-\alpha/2} W_{1,1}(\alpha), \tag{26}$$

which agrees with the value for the sphere given in the previous paper. The connection between the wave patterns of a submerged body and of a certain surface pressure has been pointed out by Dr. G. Green in a recent paper,* in which the correspondence is developed from a different point of view. In the following analysis we deal only with combinations of simple pressure systems (12), and the corresponding submerged body can be found from a similar combination of doublets, as in the preceding case.

6. The foregoing results can be generalised for other symmetrical forms of local pressure distribution, provided transformations such as are used in (4) and (14) are applicable. Assuming this, it appears, from the analysis of § 3 and § 4, that for a pressure system $p = F(\varpi)$ we have

$$R = (4\pi/g\rho)\,\kappa_0^3 \int_0^{\pi/2} \sec^5\phi\,\{f(\kappa_0 \sec^2\phi)\}^2\,d\phi, \tag{27}$$

where $f(\kappa)$ is given by (3).

7. Some points of interest in the theory of wave resistance can be illustrated by combinations of the simple type (12). We consider first two equal systems, at a distance $2h$ apart, and advancing in the direction of the line of centres; that is,

$$p = Af/(f^2+\varpi_1^2)^{3/2} + Af/(f^2+\varpi_2^2)^{3/2}, \tag{28}$$

where $\varpi_1^2 = (x-h)^2+y^2$ and $\varpi_2^2 = (x+h)^2+y^2$.

Writing, for the moment, p_1 and p_2 for the two component systems, and ζ_1, ζ_2 for the surface elevations which would be caused by these systems acting separately, the waves due to the combination are given by $\zeta_1+\zeta_2$, since we neglect, as usual, the squares of the fluid velocities. It follows from the definition in (7) that the wave resistance is the sum of four parts, R_1, R_2, R_{12}, and R_{21}. Here R_1 is the resistance due to the pressure p_1 acting on the waves produced by p_1, R_{12} is that due to p_1 operating on the waves caused by p_2, and similarly for R_2 and R_{21}.

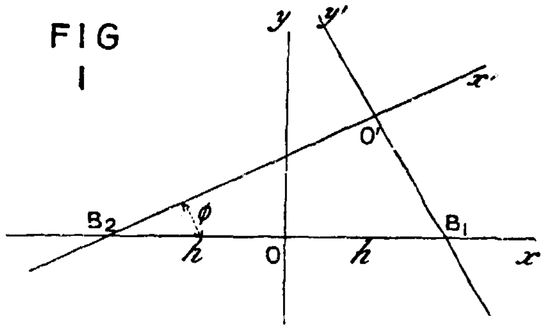
It follows from § 4, that

$$R_1 = R_2 = (4\pi/g\rho)\,A^2\kappa_0^3 \int_0^{\pi/2} \sec^5\phi\, e^{-2\kappa_0 f \sec^2\phi}\,d\phi, \tag{29}$$

* G. Green, 'Phil. Mag.,' vol. 36, p. 48 (1918).

The terms R_{12} and R_{21} represent the interference effects. Let B_1 and B_2 be the centres of the two systems p_1 and p_2. To calculate R_{21}, consider a constituent plane wave-front through B_1; take this line as a new axis $O'y'$, and a perpendicular line through B_2 for the axis $O'x'$, as in fig. 1.

FIG 1



Then, corresponding to the expression (15) in § 4, we have as an element of R_{21} the quantity

$$4\pi\kappa_0^2 A \sec^3\phi\, e^{-\kappa_0 f \sec^2\phi} \int_{-\infty}^{\infty} dy' \int_{-\infty}^{0} \frac{f A \cos(\kappa_0 x' \sec^2\phi)\, dx'}{\{(x'+2h\cos\phi)^2+y'^2+f^2\}^{3/2}}. \qquad (30)$$

The similar element in the value of R_{12} is the expression (30) when we have replaced $x'+2h\cos\phi$ by $x'-2h\cos\phi$. Adding the two elements and carrying out the integration with respect to y', we have, as an element of $R_{12}+R_{21}$,

$$8\pi f\kappa_0^2 A^2 \sec^3\phi\, e^{-\kappa_0 f\sec^2\phi} \int_{-\infty}^{\infty} \frac{\cos(\kappa_0 x' \sec^2\phi)\, dx'}{(x'+2h\cos\phi)^2+f^2}$$
$$= 8\pi^2\kappa_0^2 A^2 \sec^3\phi\, e^{-2\kappa_0 f\sec^2\phi} \cos(2\kappa_0 h\sec\phi). \qquad (31)$$

Replacing from (11) the proper factor and integrating with respect to ϕ, we have

$$R_{12}+R_{12} = (8\pi/g\rho)\,\kappa_0^3 A^2 \int_0^{\pi/2} \sec^5\phi\, e^{-2\kappa_0 f\sec^2\phi} \cos(2\kappa_0 h\sec\phi)\, d\phi. \qquad (32)$$

Finally, from (29) and (32), the total wave resistance R is given by

$$R = (16\pi/g\rho)\, A^2\kappa_0^3 \int_0^{\pi/2} \sec^5\phi\, e^{-2\kappa_0 f\sec^2\phi} \cos^2(\kappa_0 h\sec\phi)\, d\phi. \qquad (33)$$

We can express R in series of known functions by expanding $\cos(2\kappa_0 h\sec\phi)$ either in powers of $\kappa_0 h$, or in Bessel functions $J_n(2\kappa_0 h)$; however, as these series involve either $W_{k,m}(2\kappa_0 f)$ or $K_n(\kappa_0 f)$,† they are of no use for numerical calculations.

It is not difficult to calculate numerical values directly from the integral (33) for given values of the constants. To obtain a graph showing the

† See note by Editor on page 150.

variation of the resistance R with the velocity c, the following method is sufficiently accurate, at least for illustrating the main features.

Take as a definite example, $h = 2f$; then, writing α for $2gf/c^2$, we require to calculate the value of $\alpha^3 \int_0^{\pi/2} \sec^5\phi\, e^{-\alpha\sec^2\phi} \cos^2(\alpha \sec \phi)\, d\phi$ for various values of α.

The integrand can be obtained without much trouble, and it was found sufficient to calculate its value at intervals of 10° throughout the range from 0 to $\pi/2$; the mean value was found from half the sum of the initial and final values together with the sum of the intermediate ones. In the course of these calculations, we have material for obtaining the value of

$$\alpha^3 \int_0^{\pi/2} \sec^5\phi\, e^{-\alpha\sec^2\phi}\, d\phi$$

by the same method; but this integral is equal to

$$\tfrac{1}{4}\alpha^3 e^{-\alpha/2}\{K_0(\alpha/2)-(1+1/\alpha)K_1(\alpha/2)\},$$

and we can find its value also from the tables of K_0 and K_1 mentioned in § 4. By comparing results we obtain some idea of the accuracy of this method of numerical integration. The calculations can be lightened for present purposes by choosing, from general principles, values of α which correspond to important points on the graph.

By this method we obtain values of R for different values of c, for this particular case. The result is shown in the full curve in fig. 2; the scale for R is arbitrary, the unit for c is the velocity $\sqrt{(gf)}$. The dotted curve is a mean curve, and is equal to $R_1 + R_2$ in the notation of this section; that is it represents the sum of the resistances due to the two systems, ignoring any interference effects.

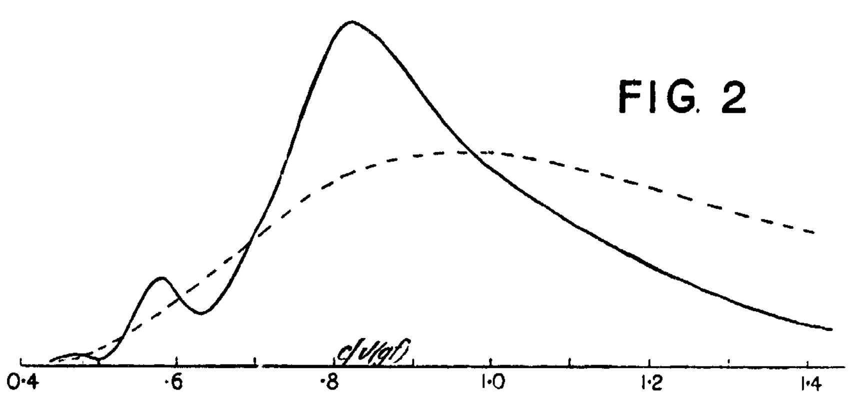


FIG. 2

The graph is of interest in its exhibition of the typical humps and hollows, occurring in general when $2\pi c^2/g$ is a sub-multiple of $2h$. The

prominence of the interference effects depends upon the relative magnitudes of the constants f and h; the example we have chosen shows a pronounced effect due to the final maximum of interference being near the maximum of the mean curve.

8. We may note briefly the interpretation in terms of a submerged body The surface of the body is one formed of stream-lines due to the two equal doublets in a uniform stream; the axes of the doublets are in the same horizontal line at a depth f, which must be large compared with, at least the vertical dimensions of the body. For instance, with suitable relations between the constants, the result would give the wave resistance of two small bodies, of nearly spherical shape, one behind the other at a distance large compared with their dimensions.

9. By combining simple symmetrical pressure systems, we may generalise the previous results; this seems an easier process than the direct discussion of unsymmetrical systems. We shall assume that the component simple systems are all of the type (12) and have the constant f of the same value, and that the centres of the systems all lie on the axis Ox.

In the first place we must extend the analysis of §8 to two components of unequal magnitudes A and B, with their centres at the points $(h, 0)$ and $(k, 0)$ respectively. From the argument expressed in (29)-(32), it is easily shown that the value (33) for the wave resistance must be altered by replacing $A^2 \cos^2 (\kappa_0 h \sec \phi)$ by

$$\tfrac{1}{4}\left[A^2+B^2+2AB\cos\{\kappa_0(h-k)\sec\phi\}\right]. \tag{34}$$

Suppose further that the pressure system is given as a continuous line distribution of components along Ox in a range from h_1 to h_2, the magnitude of the element with its centre at $(x, 0)$ being proportional to some function $\psi(x)$; in other words, suppose the surface pressure is given by

$$p = Af\int_{h_1}^{h_2}\frac{\psi(h)\,dh}{\{(x-h)^2+y^2+f^2\}^{3/2}}, \tag{35}$$

the function $\psi(h)$ being such that the transformations used in the preceding analysis are permissible. For the system (35) we have to sum (34) for all possible pairs of elements; this is performed by taking the double summation

$$\tfrac{1}{4}\int_{h_1}^{h_2}\psi(h)\,dh\int_{h_1}^{h_2}\psi(k)\cos\{\kappa_0(h-k)\sec\phi\}\,dk. \tag{36}$$

The wave resistance for the system (32) can be completed now from (33) and (36); we have

$$R = (4\pi/g\rho)\,\kappa_0{}^3A^2$$
$$\int_{h_1}^{h_2}\psi(h)\,dh\int_{h_1}^{h_2}\psi(k)\,dk\int_0^{\pi/2}\sec^5\phi\, e^{-2\kappa_0 f\sec^2\phi}\cos\{\kappa_0(h-k)\sec\phi\}\,d\phi. \tag{37}$$

10. Consider as an example the case when $\psi(h)$ is constant, so that the surface pressure is

$$p/\mathrm{A}f = \int_{-h}^{h} \frac{dh}{\{(x-h)^2+y^2+f^2\}^{3/2}}$$

$$= -\frac{x-h}{(y^2+f^2)\{(x-h)^2+y^2+f^2\}^{1/2}} + \frac{x+h}{(y^2+f^2)\{(x+h)^2+y^2+f^2\}^{1/2}}. \quad (38)$$

This may be regarded as the combination of two equal systems of opposite sign, with their centres at the points $(h, 0)$ and $(-h, 0)$ but not symmetrical round these points.

In this case, after carrying out the integrations with respect to h and k (37) gives

$$\mathrm{R} = (16\pi/g\rho)\,\mathrm{A}^2\kappa_0 \int_0^{\pi/2} \sec^3\phi\, e^{-2\kappa_0 f \sec^2\phi} \sin^2(\kappa_0 h \sec\phi)\, d\phi. \quad (39)$$

The integral may be treated similarly to (33). One of the main differences lies in the factor $\sin^2(\kappa_0 h \sec\phi)$ instead of $\cos^2(\kappa_0 h \sec\phi)$; this is because we have now two equal positive and negative systems instead of two positive systems, and in consequence the series of humps and hollows on the resistance curve will be interchanged.

We have chosen this case partly because of the corresponding problem in the motion of a submerged body at depth f. Integrating a line of doublets of constant strength results in a simple source at one end of the line and an equal sink at the other. Hence, the submerged body is one of the oval-shaped surfaces of revolution formed by combining a source and sink with a uniform stream; it follows that, as in § 5, the strength of the source is $\mathrm{A}c/2g\rho$. It may be noted that the coefficient A in (39) has different dimensions from that in (33), agreeing with its introduction in (38). By making h small in (39) we regain the former result for a sphere.

11. If a prolate spheroid of semi-axis a and eccentricity e is moving in an infinite liquid with velocity c in the direction of its axis of symmetry, it can be shown that the velocity potential may be written in the form

$$\phi = \mathrm{A}c \int_{-ae}^{ae} \frac{(a^2e^2-h^2)(x-h)\,dh}{\{(x-h)^2+y^2+z^2\}^{3/2}}, \quad (40)$$

where $\mathrm{A} = 1/[4e/(1-e^2) - 2\log\{(1+e)/(1-e)\}]$, and where we have, for the moment, taken Ox along the axis of symmetry of the spheroid. This expresses ϕ as due to a line of doublets ranged along the axis between the two foci. Hence the surface pressure corresponding to the motion of the spheroid with its axis at depth f is

$$p = 2g\rho f\mathrm{A} \int_{-ae}^{ae} \frac{(a^2e^2-h^2)\,dh}{\{(x-h)^2+y^2+f^2\}^{3/2}}, \quad (41)$$

reverting to axes with the origin in the free surface. It should be noticed that, as in § 5, the surface pressure (41) does not give the same surface elevation as the moving spheroid; the surface condition in the latter case is that the pressure should be constant at the free surface. But (41) does give the same wave formation as the spheroid, and that is the part of the surface effect upon which the wave resistance depends. The complete surface elevation can be easily written down by direct methods as in the case of the submerged sphere.

Using (41) now as an example of (35), we find the wave resistance of the spheroid from (37); after integrating with respect to h and k, the result is

$$R = 128\pi^3 g\rho a^3 e^3 A^2 \int_0^{\pi/2} \sec^2\phi\, e^{-2\kappa_0 f \sec^2\phi} \{J_{3/2}(\kappa_0 ae \sec\phi)\}^2 d\phi. \qquad (42)$$

It can be verified that this gives the result for the sphere by making e zero.

For a given relation between f and ae, the value of R can be obtained approximately by the numerical methods used in the previous examples; judging from rough calculations, it appears that the resistance curve does not show prominent humps and hollows. This might be anticipated from the surface pressure (41), which can be evaluated in simple form; if we represent the pressure distribution by a surface with p, x, and y as co-ordinates, then (41) gives a single oval-shaped peak with its longer axis in the direction Ox. On the other hand, the pressure distribution (28) represents two distinct peaks. We may compare in this respect the behaviour of ships' models; it depends upon the shape of bow and stern, and the relation between them, whether the resistance curve has marked interference effects or is a continuously ascending curve.

12. We have limited the previous cases to combinations of simple pressure systems ranged along the axis Ox. The method can obviously be extended to systems with their centres on Oy; or again, for systems situated in the plane xy, a four-fold summation in the manner of (36) would give further generality. For the corresponding problem of the motion of a submerged body, one could obtain the wave resistance of any body whose surface is formed of stream lines due to the combination of sources and sinks with a uniform stream.

TURBULENT FLUID MOTION AND SKIN FRICTION.

By Professor T. H. Havelock, F.R.S.

[Read at the Spring Meetings of the Sixty-first Session of the Institution of Naval Architects, March 26, 1920.]

Introduction.

1. It is generally admitted that our knowledge of the laws of skin friction for a solid moving through a fluid is not very satisfactory. This may be ascribed to two main reasons: in the first place the inherent difficulties of the theory of turbulent fluid motion are great even in the simplest cases, and in the second place most of the experimental data which are available have been gathered, not with the primary object of building up a consistent theory, but with more immediately practical aims in view.

Although no general investigation is attempted in the following notes, it is hoped that they may be of interest as a critical discussion of certain aspects of the problem. The work may be summarized briefly as follows :—

(1) An examination of experimental results with a view to defining or estimating the (apparent) velocity of slip of a fluid in turbulent motion past a solid.

(2) The expression of the frictional force per unit area at any point of a plane surface in the form $\kappa \rho v^2$, where v is the relative velocity at the point : determination of the value of κ from experimental results.

(3) The calculation of the total frictional resistance in the case of a plank for which the distribution of velocity is known : remarks on the distribution of velocity for a long plank.

(4) Two numerical calculations to illustrate the assumptions involved in applying a similar method to curved surfaces.

(5) Connection with the law of similarity : the effect of the ratio of breadth to length in the case of planks ; remarks on the extension to long planks and high velocities ; general problem of ship resistance.

Relative Surface Velocity.

2. When a liquid flows in steady turbulent motion through a pipe it is usual to express the resistance of the wall in terms of the mean velocity over the cross-section, because it can be defined precisely and measured accurately. Further, in any theoretical study of the motion, it seems necessary to assume that the fluid velocity at the wall is zero, there being no slipping of the layer actually in contact with the wall. However, in many cases it is found that the velocity varies little over a large part of the cross-section and is an appreciable fraction of the mean velocity at points very near the wall ; this occurs when the turbulent *régime* is well established, either because of high velocities or of large diameter of the pipe. It may be then, for some purposes, a matter of practical convenience to treat the motion as if there were a velocity of slip at the wall. The

magnitudes involved may be illustrated by some numerical cases. Taking indirect calculations first, we may quote an instance from Lamb's *Hydrodynamics* (6th edn., p. 666). Assume that the resistance per unit area of the wall of the pipe is given by $\kappa \rho v_0^2$, where ρ is the density and v_0 the mean velocity of the liquid. Also suppose the velocity to be approximately v_0 over the cross-section, except in a thin layer of thickness l in which there is laminar flow. In order to obtain the same resistance per unit area, we must have $\mu v_0/l = \kappa \rho v_0^2$, or $l = \nu/\kappa v_0$, where μ is the viscosity and ν the kinematical viscosity. For water moving with a mean velocity of 300 cm./sec., this gives $l = 0 \cdot 024$ cm.

For the cognate problem of the motion of a solid through a liquid, take an example from Froude's data for planks. The resistance of a 2-ft. plank at 600 ft./min. is given as 0·41 lb. per sq. ft.; the thickness of the equivalent layer for laminar motion giving the same resistance is found from $\mu v/l = 0 \cdot 41$, or $l = 0 \cdot 007$ in., approximately.

But these are indirect estimates, and we turn now to experimental determinations of the velocity. Here the velocity is obtained by means of a Pitot tube, and it is obvious that the nearest point to the wall at which an experimental value can be found depends upon the dimensions of the Pitot tube. For the motion of a plank through water we have Calvert's measurements of frictional wake.* In this case the Pitot tube was one-eighth of an inch in diameter. It was found that the relative velocity at the surface of the plank decreased from full speed at the front end to about half that speed at the aft end of a 28-ft. plank moving at about 400 ft./min. For turbulent flow through pipes, passing over the earlier work of Bazin and others, we may take an example from measurements by Stanton.† The Pitot tube was of rectangular section, the external dimension in the direction of the radius of the pipe being 0·33 mm. With a smooth pipe of 2·465 cm. radius, the velocity at the axis being 1,525 cm./sec., the velocity at 0·025 cm. from the wall is given as 592 cm./sec. Further, the mean velocity is about 0·81 of the velocity at the axis.‡ Hence we may deduce that the (apparent) velocity at the wall is 0·475 of the mean velocity. A similar result is obtained from other cases given in the papers quoted, the value of $V d/\nu$ being in the neighbourhood of 50,000.

We shall assume that we can refer to a relative surface velocity which is sufficiently definite for certain purposes, the limitations being indicated by the numerical examples which have been given.

Plane Surfaces.

3. We wish to see if the frictional force per unit area on any plane element of surface can be expressed by $\kappa \rho v^2$, where v is the relative velocity of the fluid and wall at the point, ρ is the density of the fluid, and κ is a non-dimensional coefficient of roughness. One of the earliest attempts to analyse turbulent fluid motion, by Boussinesq, involved a surface friction of this kind, together with a constant effective coefficient of eddy viscosity, or of mechanical viscosity as it was called by Osborne Reynolds. Experimental results on flow through pipes can be fitted more or less by a scheme of this kind, but it is generally recognized now as only an approximate statement. In the first place the mean friction on the walls is not simply proportional to (velocity)^2, but depends also on the diameter; so that the friction on an element of the wall may include a term involving its curvature. Further, the effective eddy viscosity is not found to be constant over the cross-section, though it varies little except near the walls. A similar theory has been applied recently by G. I. Taylor to the turbulent motion of the atmosphere and the skin friction of the wind on the earth's surface.

Rankine, in his method of augmented surface, assumed a skin friction proportional

* C. A. Calvert, Trans. I.N.A., Vol. XXXIV, p. 61, 1893.
† T. E. Stanton, *Proc. Roy. Soc.*, A, 85, p. 366, 1911.
‡ Stanton and Pannell, *Phil. Trans.*, A, 214, p. 205, 1914.

to (velocity)^2; but the working out of the idea involved various assumptions which are no longer regarded as legitimate.

In these notes, the scope is much more limited. The method is applied, in the first instance, only to plane surfaces; and, without further theoretical elaboration, some experimental results are examined from this point of view

To obtain a value of the coefficient κ for smooth surfaces, take first some of the earlier data: Bazin's results for water flowing in open smooth canals of great breadth compared with the depth. These have been expressed in various empirical formulæ; we shall quote one numerical case.* If R is the skin friction per unit area. V the mean velocity, v_m the velocity at the open surface and v the (apparent) velocity at the bottom of the canal, we are given

$$v_m = \mathrm{V}\,(1 + 1\cdot 81\sqrt{\zeta})\,;\quad v = \mathrm{V}\,(1 - 3\cdot 62\sqrt{\zeta})$$

where $\zeta = 2\,\mathrm{R}/\rho\,\mathrm{V}^2$. With a mean velocity V = 142·9 cm./sec., and $\zeta = 0\cdot 0044$, this gives

$$\mathrm{R} = 0\cdot 0022\,\rho\,\mathrm{V}^2 = 0\cdot 0038\,\rho\,v^2$$

However, we have a more accurate expression of recent work in Lees' formula for turbulent flow in smooth pipes,† namely :—

$$\mathrm{R} = \rho\,\mathrm{V}^2\{0\cdot 0009 + 0\cdot 0765\,(\nu/\mathrm{V}\,d)^{0\cdot 35}\}$$

This formula includes the results of Stanton and Pannell quoted in the previous section; we may therefore use for the relation between the velocity v at the wall and the mean velocity V the equation $v = 0\cdot 475$ V. Further, if we assume the formula to hold when the diameter d of the pipe is made very large, we deduce an expression for a plane surface in the required form, namely :—

$$\mathrm{R} = 0\cdot 004\,\rho\,v^2$$

We shall use this expression to estimate the frictional resistance of a smooth plane surface, v being the relative velocity at the surface.

4. In order to apply this method, it is necessary to know the distribution of velocity over the surface. Unfortunately there are very few determinations available for this purpose, although no doubt others may have been made in recent years. The only direct observations which have been published appear to be those of Calvert, given in his paper on the measurement of wake currents to which reference has already been made.

A plank, 28 ft. long and coated with black varnish, was drawn along the surface of water and measurements were made with (Pitot) tubes projecting beneath the underside of the plank. "The speeds recorded at distances of 1 ft., 7 ft., 14 ft., 21 ft., and 28 ft. from the leading end were respectively 16 per cent., 37 per cent., 45 per cent., 48 per cent., and 50 per cent. of the velocity of the plank; and these proportions appear to be maintained at all speeds between 200 and 400 ft. per minute, the latter being the highest speed that the arrangements would allow."

The relative velocities at these points are thus, respectively, 0·84, 0·63, 0·55, 0·52, and 0·5 of the velocity of the plank. The width of the plank is not stated, and we must assume the effect of the finite width upon the distribution of velocity to be small. Summing up the friction along the plank, supposed of unit width, we have :—

$$\text{Total skin friction} = \int_0^{28} 0\cdot 004\,\rho\,v^2\,d\,l$$

From Calvert's observations we may draw a fair curve showing the variation of v^2/V^2 along the plank, where V is the velocity of the plank; it is shown in curve A of Fig. 1.

* Data from Von Mises, *Elem. der Tech. Hydromech.*, teil 1, p. 97.
† C.H. Lees, *Proc. Roy. Soc.*, A, 91, p. 49, 1914.

The integral can now be evaluated approximately from the graph: applying Simpson's rule with intervals of 1 ft., the integral of v^2/V^2 along the plank comes to $10 \cdot 2$. With a velocity $V = 400$ ft./min. this gives a total resistance of $3 \cdot 51$ lb.

The resistance of the plank was not measured by Calvert. However, we may obtain another estimate from W. Froude's results (*Brit. Assoc. Reports*, 1874). Using Plate II. of that report, we can read off from the curves the resistance of a 28-ft. varnished plank at 400 ft./min.; it is $3 \cdot 51$ lb., as nearly as can be estimated. Naturally one need attach no importance to the coincidence; except that with a constant coefficient $\kappa = 0 \cdot 004$ and taking account of the actual distribution of surface velocity, the value of the total friction is in agreement with direct measurements in similar cases.

5. It must not be supposed that this method means that the total skin friction is proportional to the square of the velocity V of the body. From the theory of physical dimensions applied to similar bodies we have :—

$$R = \rho V^2 f(V l/\nu)$$

On the present statement, the only difference is that it is the relative surface velocity which is some undetermined function; for instance, in the graph of Fig. 1, if x is the distance from the leading end the graph must satisfy an equation of the form :—

$$v^2/V^2 = F(x/l, \quad V l/\nu)$$

After integrating along the plank, we obtain then R in the general functional form given above.

6. Assuming the value $0 \cdot 004$ for κ for smooth planks we may deduce some information as to the fall of surface velocity, for the mean resistance per unit area divided by $\kappa \rho$ gives the average value of v^2 over the surface.

Taking Zahm's experiments * on varnished planks in air, using a suitable value of ρ and taking the results as they are given in the table for the resistance of planks of various lengths at 10 ft./sec., we obtain the following :—

Length	2	4	8	12	16
Average v^2/V^2	0·574	0·543	0·516	0·497	0·49

From the similar tables of W. Froude for planks in water at 10 ft./sec., we find—

Length	2	8	20	50
Average v^2/V^2	0·529	0·419	0·359	0·316

There is a much quicker fall in water than in air, but of course the $V l/\nu$ values do not correspond in the two sets. Froude gives a column which is said to be the resistance per square foot of the last foot of plank; this is, one may suppose, obtained as the difference in resistance of two planks differing in length by 1 ft., and it obviously assumes that the addition of 1 ft. to the rear of a plank does not alter appreciably the distribution of velocity over the rest of the plank. Taking the figures as they stand, we may deduce the average value of v^2 over the last foot of plank for various lengths; they give :—

Length	2	8	20	50
Average	0·503	0·340	0·309	0·291

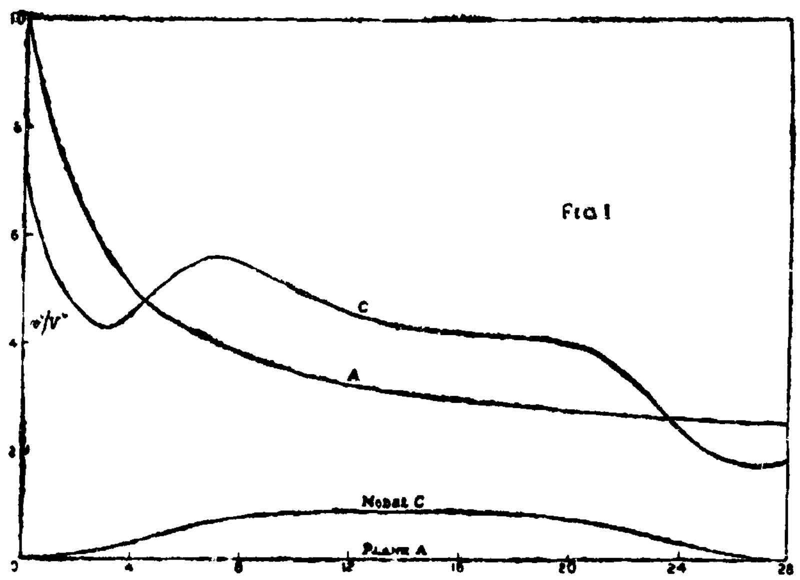
the second row being the average value of v^2/V^2 over the last foot. Taking the square root, we may estimate the relative velocity at the end of a 50-ft. plank moving at 10 ft./sec. as about $0 \cdot 54$ of the velocity of the plank; and this estimate will be on the high side. It may be compared with the value $0 \cdot 475$ which we found for the similar ratio in flow through pipes when the steady state has been reached.

* A. F. Zahm, *Phil. Mag.*, 8, p. 58, 1904.

Curved Surfaces.

7. If a body is moving through a liquid we may suppose the force on an element dS of the surface to be resolved into a normal pressure and a frictional force $R\,dS$; the latter will be in a direction opposite to the relative velocity and, if we suppose it to make an angle θ with the direction of motion of the body, we may define the skin friction as $\int R\,dS\cos\theta$, taken over the wetted surface.

For plane surfaces we have shown that there is some justification for taking R equal to $\kappa\rho v^2$, where v is the relative velocity; in the general case one would probably have an additional term involving the curvature of each point. Consider first the case of "two-dimensional" flow, when the longitudinal cross-section of the body is of ship-shape form. Here each element is curved in the line of motion, and if the curvature is small and we assume $R = \kappa\rho v^2$, the effect of the curvature is to be found in the distribution of velocity. The effect of this kind of curvature has been discussed by Mr. G. S. Baker by estimating



the distribution of velocity in stream-line motion. It should be noted that it is not the same as the effect of the shape of midship section, for there the curvature is at right angles to the line of flow. Naturally in three-dimensional flow both effects are superposed, and cannot be disentangled. No experimental determinations of surface velocity appear to have been published, at least for ship forms in water The extension from plane to curved surfaces is thus to a large extent speculative; however, as the extension has been made already in other methods, two numerical examples are given here to illustrate the various assumptions.

8. For two-dimensional motion, suppose that the model is 28 ft. long, as for Calvert's plank, with a longitudinal section shown, as to the upper half only, in model C of Fig. 1. This is a form for which Baker and Kent * have calculated the pressure distribution in stream-line motion; from the curves given in that paper we can draw a curve of the distribution of v^2/V^2 in stream-line motion, v being the relative surface velocity and V the velocity of the model. Now, as an arbitrary assumption, suppose that in turbulent flow v^2 diminishes for the model according to the same law as for the 28-ft. plank; that is, we take a reduction factor at each point from the curve A of Fig. 1. We obtain thus the curve C of Fig. 1 as an estimated distribution of relative velocity, or rather it

* G. S. Baker and J. L. Kent, Trans. I.N.A., Vol. L., Pt. II., p 37, 1913.

shows the values of v^2/V^2 for the model. Also the total skin friction, per unit breadth

$$= \int R \, d S \cos \theta = \int \kappa \rho v^2 d l$$

taken along the straight axis of the model. Estimating the area under the curve C, and the curved length of Model C, we can calculate the mean resistance per unit area. It appears that the model has a mean resistance per unit area about 11 per cent. greater than that of a plank of the same length.

9. For a three-dimensional case we take similar preliminary data from a paper by Mr. D. W. Taylor * on solid stream forms. We carry out the same process as in the previous section, and it is unnecessary to reproduce the corresponding curves. The only difference arises from the fact that the solid is one of revolution with pointed ends; consequently the element of area approaches zero at the two ends. If y is the ordinate of the ship form at any point on the axis, we have to graph the values of $y v^2$ on the straight axis of symmetry as a base, instead of simply v^2 as in the two-dimensional problem. As far as the numerical approximation has been carried, it appears that the mean resistance per unit area for this model is about equal to, or slightly less than, that of a plank of the same length.

10. The resistance of a small appendage on the surface of a ship must depend chiefly upon the relative surface velocity in its neighbourhood. It is appropriate to refer here to some experiments by Mr. Baker † to determine the added resistance due to local roughness of a model. If the rough area were small enough relatively so as not to affect appreciably the flow over the rest of the model, and if the slope of the surface and the direction of flow were known, it might be possible to deduce information about the velocity distribution; however, one cannot analyse in this way the results to which reference has been made.

In regard to skin friction for curved surfaces especially, one may venture to quote and endorse a remark made by Professor Lees ‡: "It is of prime importance that further measurements should be made on bodies which lend themselves to simple theoretical treatment in order to build up a satisfactory theory."

Law of Similarity for Planks.

11. The law of similarity in its usual form :—

$$R = \rho V^2 f (V l/\nu)$$

applies to bodies which are geometrically similar in form, and are similar as regards scale of roughness. In experiments with planks we may perhaps neglect the thickness and suppose the motion to be in two dimensions only; but the planks will not be similar unless the ratio of breadth to length is constant. In other words, the general formula from physical dimensions is :—

$$R = \rho V^2 f (b/l, \ V l/\nu)$$

where the undetermined function depends upon two quantities, the ratios b/l and $V l/\nu$.

In most experiments the ratio b/l has not been kept constant, but the planks have been of constant breadth and varying length. Consider, for example, Zahm's results,§ which he expressed in the empirical formula :—

$$R = k \, l^{-0 \cdot 07} V^{1 \cdot 85}$$

* D. W. Taylor, Trans. I.N.A., Vol. XXXVI., p. 234, 1895.
† G. S. Baker, *Trans. North-East Coast Inst.*, Vol. XXXII., p. 50, 1915.
‡ C. H. Lees, Trans. I.N.A., Vol. LVIII., p. 64, 1916.
§ Zahm, loc. cit.

It is usual, following Lord Rayleigh, to correct this to satisfy the law of similarity and to write :—

$$R = \text{Const.} \times \rho\, V^2\, (\nu/V\, l)^{0\cdot15}$$

It is probably true that the experiments are not sufficient to decide between these two forms. The present point is that without altering the empirical law as regards l and V, the formula can be made to satisfy the dimensional equation by writing it, for instance, in the form :—

$$R = \text{Const.} \times \rho\, V^2 \times (l/b)^{0\cdot08} \times (\nu/Vl)^{0\cdot15}$$

Similar remarks may be applied to Froude's experiments with planks in water. For instance, with planks coated with fine, medium, or coarse sand the resistance is proportional to the square of the speed. Hence in these cases the quantity $R/\rho\, V^2$ is a function of the ratio b/l and of the coefficient κ, which may be called the ratio of roughness; but it is not possible to separate the two effects in the results.

12. Consider the distribution of relative surface velocity from front to rear of a long plank. Neglecting the disturbance of the edges, we may divide the distribution roughly into three stages; firstly, one in which the velocity falls rather rapidly, then a long stretch in which it is practically constant, and finally a relatively short stage in which the influence of the end is appreciable. For a very long plank in which the middle stage predominates, the mean resistance per unit area will approximate to $\kappa\, \rho\, v^2$, where v is the steady value of the surface velocity. On the other hand, for shorter planks a two-term formula may be sufficient, which may possibly be of the type $\rho\, V^2\{A + B\, (\nu/V\, l)^n\}$.

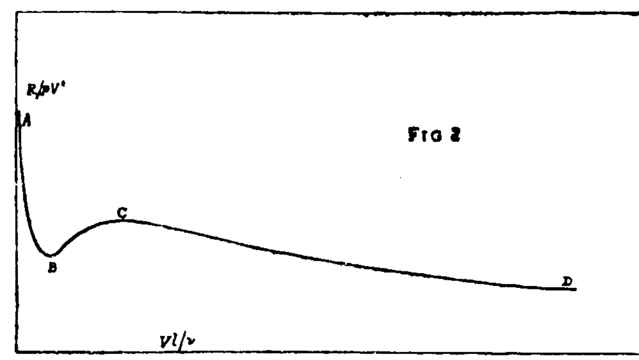


FIG 2

Again, if the breadth is taken into account such a formula would be incomplete. Here in the extreme case of a long plank of finite breadth, the analogy of steady flow through a pipe is suggested; and the mean resistance should approximate to a two-term formula of the type just given, with the length l replaced by the breadth d. This is the argument which has been worked out by Professor Lees in the paper * already quoted; in that analysis d is taken as the diameter of an equivalent circular cylinder and deduced by a certain method from the dimensions of the plank.

13. On the analogy of the law of similarity for flow in pipes, Mr. Baker † has collected results on planks and models into one diagram in which $R/\rho\, V^2$ is graphed on a base $V\, l/\nu$. We have seen that certain reservations are necessary in grouping the data from planks in this way; but the general trend of the curves obtained is very suggestive. Fig. 2 shows the main points in a diagrammatic sketch, not drawn to scale, but based on the paper quoted.

The stage A B represents simple viscous fluid motion when R is proportional to V. B C is an unstable condition when the flow may be partly simple and partly turbulent; after C the latter *régime* becomes permanently established. If the resistance R is represented by a single-term formula $f\, V^n$, it is clear that the best single power is V^2 in the neighbourhood of the points B and C. It may be noted that Froude gives V^2 for short smooth planks of 2 ft. in length, and it may be presumed that the region near C was then under observation. As the length is increased, the best single power decreases to, say, $V^{1\cdot85}$ near D, if we take this point to represent the limit of available data. Froude's

* C. H. Lees, loc. cit.

† G. S. Baker, *Trans. North-East Coast Inst.*, loc. cit.

extension to very long planks is equivalent to extending the curve beyond D so that it approaches the base-line ultimately. On the other hand, the analogy with the problem of flow through pipes suggests that the curve approximates ultimately to a line at a finite distance above the base-line. In the latter case, the best single power must increase again at some stage and ultimately approach V^2 again. However, it is generally recognized that all that can be said is that any reasonable extension of the curve beyond D must lie within certain limits, that in fact being the statement made by W. Froude* in this respect; we are not able yet to decide between alternative methods.

14. In conclusion a few remarks may be made on the general problem of ship resistance. It is usual to divide up the total resistance into three parts: frictional, eddy-making, and wave-making resistance. An alternative method is to think of the direct action upon each element of the wetted surface; this action may be resolved into a normal pressure p and a tangential force R at each point. The integrated effect of R gives the total skin friction, while the resultant of the pressure distribution may be called the body or form resistance. In the ship problem it is assumed that the latter corresponds in the main to the wave resistance, together with that due to eddy-making of the more obvious kind; however, in general, the distribution of normal pressure and of tangential force will be interdependent and will each be affected by all the circumstances of the motion. It would be of interest to have some case analysed in this way, with the pressure distribution determined experimentally. This method has been adopted in the corresponding problem in aeronautics, which is simpler in some respects. For an airship envelope, in the form of a surface of revolution, the pressure distribution can be found experimentally; the difference between the resultant and the total resistance then gives the skin friction.† If there were, for the same case, experimental determinations of the distribution of velocity over the envelope, it would be possible to compare the total skin friction with the resultant of a distribution of tangential force $k \rho v^2$ taken over the surface. Results for submerged bodies in water might be deduced from those in air by the law of similarity; but it would be preferable if direct results could be obtained, experimentally, for the distributions of normal pressure and of velocity for simple forms intermediate between the plank and the ordinary type of ship model.

* W. Froude, *Brit. Assoc. Reports*, 1874, p. 255.
† Cf. L. Bairstow, *Applied Aerodynamics*, p. 357.

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The Stability of Fluid Motion.

By T. H. HAVELOCK, F.R.S.

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1. The following notes on the stability of fluid motion arose from a desire to use the energy method, introduced by Reynolds and modified by Orr, as a measure of the comparative degree of stability of various types of flow under different boundary conditions. A few examples are worked out to illustrate this point of view: in § 5 a case which resembles the flow of a stream with a free surface; in § 7 flow which approximates to a uniform stream between fixed walls without slipping at the walls; in §§ 6, 8 motion with other boundary conditions. Before proceeding to these, it seems desirable to give a short account of the method in the form in which it is used later, together with some remarks on its relation to the classical method of small vibrations.

2. We shall consider only two-dimensional motion of an incompressible viscous fluid limited by the planes $y = \pm a$. Let the steady state under an assigned forcive and given boundary conditions be specified by a velocity, U, parallel to the axis of x. Let the disturbed state have velocity components $(\mathrm{U}+u,\, v)$ and let the additional pressure be p. Then, by taking the difference of the two sets of hydrodynamical equations for the two states and neglecting squares and products of the additional velocities, we have

$$\frac{\partial u}{\partial t}+\mathrm{U}\frac{\partial u}{\partial x}+v\frac{\partial \mathrm{U}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\nabla^2 u,$$

$$\frac{\partial v}{\partial t}+\mathrm{U}\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\nabla^2 v, \qquad (1)$$

together with the equation of continuity.

It is convenient to introduce non-dimensional variables given by

$$x = a\xi; \qquad y = a\eta; \qquad a\tau = \overline{\mathrm{U}}t;$$

where $\overline{\mathrm{U}}$ is the mean velocity over the cross-section in the steady state. Further, we write $\overline{\mathrm{U}}\mathrm{U}$ instead of U, and take the current function of the additional velocity to be $\overline{\mathrm{U}}a\psi$. Eliminating p from the two equations (1), we obtain

$$\mathrm{R}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\xi}\right)\nabla^2\psi-\mathrm{R}\mathrm{U}''\frac{\partial\psi}{\partial\xi} = 2\nabla^4\psi, \qquad (2)$$

where U″ is written for $d^2\mathrm{U}/d\eta^2$, and R is Reynolds' number $2a\overline{\mathrm{U}}/\nu$. There are in addition the appropriate boundary conditions for the disturbing function, ψ. The classical method of examining the stability of a given

distribution U consists in assuming a solution of (2) of the form $\exp.\{i(n\tau+p\xi)\}f_n(\eta)$. For any arbitrary real value of p, the corresponding possible forms of $f_n(\eta)$ and values of n are found from (2) together with the boundary conditions. The distribution U may be said to be thoroughly stable if every possible value of n has a positive imaginary part, and if this holds for all positive values of R.

The usual boundary conditions, which we shall assume in the first place, are $u = 0$, $v = 0$, or

$$\psi = 0\,; \qquad \partial\psi/\partial\eta = 0: \qquad \eta = \pm 1. \tag{3}$$

From the work of Kelvin, Rayleigh, Orr, Hopf, and others, it may be taken that the simple shearing motion, $U = 1+\eta$, is thoroughly stable in this sense; and probably a similar conclusion holds for motion under a constant force or pressure gradient, namely $U = \frac{3}{2}(1-\eta^2)$.

There are various possible explanations of the well-known divergence between these results and the behaviour of actual fluids. In the first place, it is obvious that the physical properties, whether of the fluid or of the walls, are inadequately specified in the mathematical statement of the problem. But, apart from this, the disturbances have been supposed small, and second order terms neglected. Again, in a system of this type, a disturbance may be small initially and may converge ultimately to zero, but may be very large at intermediate times, and may thus give rise to practical instability.

The energy method of Reynolds is in a different category from these in that it takes the mathematical problem as it stands and does not necessarily involve the actual magnitude of the disturbance; in fact, it forms a new criterion or measure of degree of stability. The energy of the disturbance being defined by

$$E = \tfrac{1}{2}\rho a^2 \bar{U}^2 \iint \left\{ \left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 \right\} d\xi\, d\eta, \tag{4}$$

we have from (2) and (3), after integrating by parts,

$$\frac{dE}{dt} = \tfrac{1}{2}\mu \bar{U}^2 \left[R \iint U' \frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\, d\xi\, d\eta - 2\iint (\nabla^2\psi)^2\, d\xi\, d\eta \right]. \tag{5}$$

Here dE/dt means the rate of increase of E in a region whose end boundaries move with the steady velocity U. We may replace this by $\partial E/\partial t$ for a region with fixed ends, and we shall then have additional terms on the right of (5) denoting flux of energy across these ends. The latter terms may be omitted under conditions which cover the usual cases: namely, either the disturbance is periodic in ξ, or it is limited or localised so that ψ and its derivatives converge sufficiently rapidly to zero for $\xi = \pm\infty$. We shall assume such conditions to hold in what follows, and references to boundary conditions mean those which hold at the planes $\eta = \pm 1$.

Reynolds' method of using (5) to determine a criterion of stability consisted in assuming a suitable form for ψ and finding the least value of R for which the right-hand side of (5) is zero. It is usually stated that this method assumes turbulent motion to be already in existence, and it then gives a criterion to show whether the turbulence is increasing or decreasing momentarily; but this is somewhat misleading without defining what is meant by turbulent motion. Equation (5), as stated above, applies to any small arbitrary disturbance, neglecting terms of the second order, as in the ordinary method of small vibrations; further, U is a laminar fluid motion satisfying the usual hydrodynamical equations under the given conditions.

On the other hand, Reynolds defined U as the mean velocity at each point, taken over a small region or during a short time, and this principal or mean motion need not satisfy the ordinary equations. The extra velocities u and v then play a double part, in that they specify the disturbance, and at the same time give a measure of the turbulence; they must satisfy certain conditions as to their mean values, and then equation (5) holds in the same form when mean values are used. However, in applying it to find the criterion for flow under a constant pressure gradient, Reynolds, and Sharpe following him, did, in fact, take U to be the usual form, $C(a^2-y^2)$, for steady laminar flow. But in turbulent flow, although the variations of velocity at any point are small, yet they may cause the gradient of the mean velocity to differ appreciably from its value in laminar flow, as is obvious from a comparison of the curves of distribution of velocity across a pipe in regular and in turbulent flow.

However, it is unnecessary to dwell on this distinction, as it has been pointed out clearly by Lorentz* and other writers; further, we shall consider here only small disturbances.

3. Under these circumstances, the energy method has been given a precise and definite meaning by Orr† from the following considerations:—

If the right-hand side of (5) is positive, the energy of the disturbance is momentarily increasing. But, for a given velocity distribution, U, it may be impossible to find any function, ψ, satisfying the boundary conditions, such that that expression is positive, unless R exceeds a certain value. If such be the case, this least value of R is a critical value of definite significance. The corresponding critical disturbance is found by taking the variation of the equation

$$R\iint U'\frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta}\,d\xi\,d\eta-2\iint(\nabla^2\psi)^2\,d\xi\,d\eta = 0, \qquad (6)$$

subject to $\delta R = 0$.

* H. A. Lorentz, 'Abhandlungen über Theor. Phys.,' vol. 1, p. 43.

† W. McF. Orr, 'Proc. Roy. Irish Acad.,' vol. 27, p. 9 (1907).

Carrying out the variation, and using the boundary conditions (3), we obtain

$$4\nabla^4\psi + 2\mathrm{RU}'\frac{\partial^2\psi}{\partial\xi\,\partial\eta} + \mathrm{RU}''\frac{\partial\psi}{\partial\xi} = 0. \tag{7}$$

To find the critical value of R, we assume first that ξ occurs in ψ as a factor exp. $ip\xi$, and then solve (7); using the boundary conditions, we have an equation from which we can find the least value of R for a given value of p, and finally we take the minimum value of R with respect to p.

The process has been expressed in a different form by Hamel.* Using the corresponding Green's function for the equation $\nabla^4\psi = 0$, the equation (7) may be replaced by a linear integral equation for ψ, of which the required value of R is the lowest characteristic number.

Returning to equation (5), if $d\mathrm{E}/dt$ is positive for any assigned initial disturbance, it does not follow that the motion is unstable in the ordinary sense. But, if there exists an absolute minimum for R in the manner explained above, it follows that, when R is less than this value, $d\mathrm{E}/dt$ is negative for every initial disturbance, and must always remain negative. Thus the system has at least a much higher degree of stability for such values of R compared with those greater than the critical minimum. Obviously, this method does not produce any new information which is not implicit in the ordinary equations, such as equations (2) and (3); but it presents part of that information in a different form, so that the critical minimum of R may be used as a measure of the degree of stability of various distributions of velocity under different boundary conditions.

4. It is convenient to classify the boundary conditions under which the energy equation (5) is valid. For this purpose we use an alternative form derived directly from equations (1), with the ordinary notation

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x}; \quad p_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right); \quad p_{yy} = -p + 2\mu\frac{\partial v}{\partial y}. \tag{8}$$

We have

$$\frac{d\mathrm{E}}{dt} = \int\{u(lp_{xx} + mp_{xy}) + v(lp_{xy} + mp_{yy})\}\,ds - \rho\iint uv\frac{d\mathrm{U}}{dy}\,dx\,dy$$
$$+\iint\left\{p_{xx}\frac{\partial u}{\partial x} + p_{yy}\frac{\partial v}{\partial y} + p_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}dx\,dy, \tag{9}$$

where ds is a line element of the boundary and (l, m) the normal.

We have specified the conditions at the end boundaries, and we are concerned now with the planes $y = \pm a$. It follows that we get the energy

* G. Hamel, 'Gött. Nachr., Math. Phys. Klasse,' 1911, p. 261.

equation (5), without any surface integrals expressing transfer of energy across the boundaries, with the following combinations:

(i) $u = 0$, $v = 0$; (ii) $u = 0$, $p_{yy} = 0$; (iii) $v = 0$, $p_{xy} = 0$; (iv) $p_{xy} = 0$, $p_{yy} = 0$.

We may also verify that, under these conditions, the variation of (6) leads to the same differential equation (7).

5. Most of the fluid motions whose stability has been examined, come under case (i) of the above. A different case of special interest is a stream with a free upper surface, the conditions at the upper surface being as in (iv). These conditions, however, do not lead to simple expressions in terms of the disturbing function, ψ; moreover it is not permissible to regard the upper free surface as rigorously plane. We therefore, following Kelvin,* replace the problem by one which is very nearly the same but is more easily specified; it may be described as a broad river flowing over a perfectly smooth inclined plane bed, the upper surface being fitted by a parallel plane cover moving with the water in contact with it. The conditions at the upper surface then come under case (iii) of the previous section.

We take the origin in the upper surface in this case, so that a is the depth of the stream and R is $a\mathrm{U}/\nu$. The steady state is given by

$$\mathrm{U} = \tfrac{3}{2}(1-\eta^2). \tag{10}$$

Using this in (7) and assuming ψ to be proportional to $e^{ip\xi}$, the differential equation becomes

$$\left(\frac{d^2}{d\alpha^2}-1\right)^2\psi - k\left(2\alpha\frac{d\psi}{d\alpha}+\psi\right) = 0, \tag{11}$$

where $\alpha = p\eta$, and $k = 3i\mathrm{R}/2p^3$.

The boundary conditions are $u = 0$, $v = 0$ at the bed of the stream, and $v = 0$, $p_{xy} = 0$ at the upper surface; these reduce to

$$\begin{aligned} \psi &= 0,\ d^2\psi/d\alpha^2 = 0; \qquad \alpha = 0; \\ \psi &= 0,\ d\psi/d\alpha = 0; \qquad \alpha = p. \end{aligned} \tag{12}$$

Equation (11) was solved by Orr for flow between two fixed planes with u and v zero at both boundaries, and it was found necessary to consider only solutions in even powers of α. We shall require here the corresponding solutions in odd powers. Writing a solution in the form

$$\psi = \Sigma\, \mathrm{A}_n\alpha^n/n!$$

we have the sequence relation

$$\mathrm{A}_{n+4} - 2\mathrm{A}_{n+2} + \{1-(2n+1)k\}\mathrm{A}_n = 0. \tag{13}$$

* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 330.

Denoting by ψ_0, ψ_1, ψ_2, ψ_3 the solutions beginning with 1, α, α^2, α^3 respectively, it follows from (12) that the boundary conditions lead to

$$\psi_1 d\psi_3/d\alpha - \psi_3 d\psi_1/d\alpha = 0 \tag{14}$$

where α has to be replaced by p.

Calculating the coefficients far enough to give sufficient accuracy for our purpose, we have

$$\begin{aligned}\psi_1 = {} & \alpha + 2\alpha^3/3! + (3+3k)\alpha^5/5! + (4+20k)\alpha^7/7! \\ & + (5+70k+33k^2)\alpha^9/9! + (6+180k+366k^2)\alpha^{11}/11! \\ & + (7+385k+2029k^2+627k^3)\alpha^{13}/13! + (8+728k+7832k^2+9672k^3)\alpha^{15}/15! \\ & + (9+1260k+24030k^2+73500k^3+16929k^4)\alpha^{17}/17! + \ldots, \\ \psi_3 = {} & \alpha^3/3! + 2\alpha^5/5! + (3+7k)\alpha^7/7! + (4+36k)\alpha^9/9! \\ & + (5+110k+105k^2)\alpha^{11}/11! + (6+260k+894k^2)\alpha^{13}/13! \\ & + (7+525k+4213k^2+2415k^3)\alpha^{15}/15! + (8+952k+14552k^2 \\ & \qquad + 28968k^3)\alpha^{17}/17! + \ldots\end{aligned}$$

Forming equation (14) we have

$$\begin{aligned} & 2/3! + 8p^2/5! + 32p^4/7! + 128p^6/9! + (512+192k^2)p^8/11! \\ & + (2048+2244k^2)p^{10}/13! + (8192+19456k^2)p^{12}/15! \\ & + (32768+139264k^2)p^{14}/17! + (131072+901120k^2 \\ & \qquad + 129024k^4)p^{16}/19! + \ldots = 0. \end{aligned} \tag{15}$$

Only even powers of k appear in this equation, thus giving a check upon the arithmetic; further, the terms independent of k may be summed. Taking the least root of (15) as an equation for k^2, we have approximately

$$R^2 = \frac{\sinh 2p - 2p}{9p^5\left(\frac{192}{11!} + \frac{2244p^2}{13!} + \frac{19456p^4}{15!} + \frac{139264p^6}{17!} + \frac{901120p^8}{19!} + \ldots\right)}. \tag{16}$$

Instead of forming an equation for the minimum value of R, it is simpler to find it by trial. We find, with sufficient accuracy, that it occurs near $p^2 = 11$, and then, approximately,

$$R = 96. \tag{17}$$

The corresponding value, found by Orr, for flow under similar conditions but with a fixed plane at the upper surface, is 117. We conclude then that flow in an open canal has a lower degree of stability than flow between fixed planes.

Turning to experimental results, the number usually quoted for flow through a tube is 2000 approximately. This was obtained chiefly from

experiments with smooth glass tubes; a much lower number, of the order of 400, has been found from metal tubes. The only available direct results for flow in an open stream appear to be those given by Hopf,* who found R to be of the order of 300. These results agree in character with the theoretical calculations, which is all that could be expected.

It is of interest to note that this appears to contradict a statement by Reynolds† in one of his earlier papers. He classes separately circumstances conducive to steady motion and those conducive to unsteady motion: among the former a free surface, and in the latter solid bounding walls. However, this opinion seems to be based on visual observation of eddies caused by the wind beneath the oiled surface of water. "At a sufficient distance from the windward edge of an oil-calmed surface there are always eddies beneath the surface, even when the wind is light. . . . Without oil I was unable to perceive any indication of eddies."

This introduces a different property of a boundary surface, namely, that of initiating disturbances. The mathematical statement ignores this property and specifies only control of the velocity functions: the disturbances are supposed to be initiated by some extraneous agency, and it is tacitly assumed that all types of disturbance are equally probable. It may be, for instance, that the theoretical results for flow through pipes should be compared with experiments on rough pipes rather than those with perfectly smooth walls. However, we may conclude that a solid boundary is conducive to stability in so far as it ensures that there is no slipping of the fluid in contact with it.

6. In determining the minimum value of R from the differential equation (7), there are only two factors: the distribution of steady velocity, U, and the boundary conditions for the disturbance. The comparison in the previous section, between an open stream and flow between fixed walls, involved changes in both these factors. We may separate the effect of the boundary conditions by assuming the same value of U as in (10), but expressing the property of the supposed moving plane in contact with the upper surface by $u = 0$, $v = 0$, instead of by $v = 0$, $p_{xy} = 0$. To anticipate the argument of the next sections, we should expect a value of R intermediate between 96 and 117.

We have the same equation (11) for ψ, together with $\psi = 0$, $d\psi/d\alpha = 0$ at $\alpha = 0$, and $\alpha = p$. It follows that only the solutions ψ_2 and ψ_3 are involved, and we have

$$\psi_2\, d\psi_3/d\alpha - \psi_3\, d\psi_2/d\alpha = 0. \qquad (18)$$

* L. Hopf, 'Ann. der Phys.,' vol. 32, p. 777 (1910).

† O. Reynolds, 'Scientific Papers,' vol. 2, pp. 57, 59. See also A. H. Gibson, 'Phil. Mag.,' vol. 25, p. 81 (1913).

when $\alpha = p$. The series for ψ_3 is given in §5; also we have

$$\psi_2 = \alpha^2/2! + 2\alpha^4/4! + (3+5k)\alpha^6/6! + (4+28k)\alpha^8/8!$$
$$+ (5+90k+65k^2)\alpha^{10}/10! + (6+220k+606k^2)\alpha^{12}/12!$$
$$+ (7+455k+3037k^2+1365k^3)\alpha^{14}/14! + (8+840k+10968k^2$$
$$+ 17880k^3)\alpha^{16}/16! + \ldots$$

The boundary equation (18) leads to

$$2/4! + 8p^2/6! + 32p^4/8! + 128p^6/10! + (512+280k^2)p^8/12!$$
$$+ (2048+3136k^2)p^{10}/14! + (8192+25216k^2)p^{12}/16!$$
$$+ (32768+174080k^2)p^{14}/18! + \ldots = 0. \quad (19)$$

The minimum value of R seems to occur for about $p^2 = 12$, though it is not a sharply defined minimum; however, with a similar approximation as in previous cases, we find the critical minimum of R to be 110.

7. It is well known that, when fluid motion through a tube has changed from laminar to turbulent flow, the distribution of mean velocity over the cross-section alters so that the velocity becomes more nearly uniform over the greater part of the section, while falling to zero at the walls. This suggests a study of the comparative stability when the distribution of velocity alters in this manner, the boundary conditions being unchanged.

However, it must be noted that we assume the distribution to be a steady state which has been acquired under a law of force, which may be determined from the hydrodynamical equations, so as to give the required form for U.

A simple form, which illustrates the points in question, is

$$U = (1+1/2n)(1-\eta^{2n}). \quad (20)$$

As n is made larger, the velocity approximates more closely to the mean velocity, U, over the greater part of the cross-section, while remaining zero at the walls. The corresponding law of force is, in the usual notation,

$$X = \nu(4n^2-1)(U/a^2)\eta^{2n-2}. \quad (21)$$

The greater the value of n, the more is the field of force concentrated near the walls, quite apart from the value of the viscosity. The flow approximates to a uniform stream, but retaining the condition of zero velocity at the walls.

The usual case of flow under a uniform field of force is given by $n = 1$. It is sufficient for comparison to work out another numerical case, say $n = 2$. We have then

$$U = \tfrac{5}{4}(1-\eta^4). \quad (22)$$

Equation (7) becomes

$$\left(\frac{d^2}{d\alpha^2}-1\right)^2\psi-2k\left(2\alpha^3\frac{d\psi}{d\alpha}+3\alpha^2\psi\right)=0, \tag{23}$$

where $k = 5i\mathrm{R}/8p^5$.

The boundary conditions are

$$\psi = 0\,; \qquad d\psi/d\alpha = 0\,; \qquad \alpha = \pm p.$$

Solving (23) by a power series $\Sigma \mathrm{A}_n\alpha^n/n\,!$, we have

$$\mathrm{A}_{n+6} = 2\mathrm{A}_{n+4}-\mathrm{A}_{n+2}+2k(n+1)(n+2)(2n+3)\,\mathrm{A}_n. \tag{24}$$

As in the simpler cases, it is sufficient to choose fundamental solutions involving only even powers of α; denoting these by ψ_0 and ψ_2 we have

$$\begin{aligned}\psi_0 = {} & 1+\alpha^2/2\,!+\alpha^4/4\,!+(1+12k)\,\alpha^6/6\,!+(1+192k)\,\alpha^8/8\,!\\ & +(1+1032k)\,\alpha^{10}/10\,!+(1+3552k+20160k^2)\,\alpha^{12}/12\,!+(1+9492k\\ & +696960k^2)\,\alpha^{14}/14\,!+(1+21504k+8162256k^2)\,\alpha^{16}/16\,!+\ldots\end{aligned}$$

$$\begin{aligned}\psi_2 = {} & \alpha^2/2\,!+2\alpha^4/4\,!+3\alpha^6/6\,!+(4+168k)\,\alpha^8/8\,!+(5+1656k)\,\alpha^{10}/10\,!\\ & +(6+8184k)\,\alpha^{12}/12\,!+(7+28392k+574560k^2)\,\alpha^{14}/14\,!\\ & +(8+78960k+11204352k^2)\,\alpha^{16}/16\,!+(9+188496k\\ & +102266496k^2)\,\alpha^{18}/18\,!+\ldots.\end{aligned}$$

From the boundary condition

$$\psi_0 d\psi_2/d\alpha-\psi_2 d\psi_0/d\alpha = 0,$$

we obtain the equation

$$\begin{aligned} & p+2p^3/3\,!+8p^5/5\,!+32p^7/7\,!+128p^9/9\,!+512p^{11}/11\,!\\ & +(2048+129024k^2)\,p^{13}/13\,!+(8192+3280896k^2)\,p^{15}/15\,!\\ & +(32768+7753296k^2)\,p^{17}/17\,!+\ldots = 0. \end{aligned} \tag{25}$$

Using this as an equation for R, we find by trial that the minimum value occurs near $p^2 = 3$; and the critical minimum value of R is 280 approximately.

The corresponding value for the ordinary parabolic distribution ($n = 1$) is 117. Thus, the critical value of R increases as the flow approximates more closely to a uniform stream, without slipping at the walls; and, in this sense, the motion becomes increasingly stable.

8. It has been stated that, under the boundary conditions $u = 0$, $v = 0$, there is thorough stability, in the ordinary sense, for simple shearing motion and probably also for laminar flow between fixed planes. In view of the behaviour of actual fluids in similar conditions, another suggestion has been put forward by Hopf.* He proposes to express the influence of a wall by making the extra normal pressure, due to the disturbance, constant at the

* L. Hopf, 'Ann. der Phys.,' vol. 59, p. 538 (1919).

well, together with no tangential slipping; in fact, his boundary conditions come under case (ii) of § 4, namely $u = 0$, $p_{yy} = 0$. With these assumptions, he applies the method of small vibrations to simple shearing motion between a fixed plane and a parallel moving plane. It appears that the motion is unstable for disturbances whose wave-length exceeds a certain value; for smaller wave-lengths it is stable or unstable according to the value of R. Thus the motion is not thoroughly stable. Without discussing how far these assumptions express the behaviour of actual fluids and boundaries, we may see how they affect the energy method.

We shall take the case of laminar flow between fixed planes, for which the previous calculations are available.

The stream function ψ satisfies equation (11), and the boundary conditions are

$$u = 0; \qquad -p+2\mu\,\partial v/\partial y = 0.$$

From the equations (1), these are equivalent to

$$u = 0; \qquad \rho v\,d\mathrm{U}/dy-\mu\partial^2 u/\partial y^2 = 0,$$

or, in the present notation,

$$d\psi/d\alpha = 0; \qquad d^3\psi/d\alpha^3-2k\alpha\psi = 0; \qquad \alpha = \pm p. \tag{26}$$

Using the solutions ψ_0 and ψ_2, these give

$$\psi_0'(\psi_2'''-2kp\psi_2)-\psi_2'(\psi_0'''-2kp\psi_0) = 0, \tag{27}$$

where accents denote differentiation with respect to α.

From the previous work, this equation involves odd powers of k. But k is $3i\mathrm{R}/4p^3$ and we have to determine R in terms of p from (27). It follows that in this case there is no real solution of the problem of finding the critical minimum of R.

It seems probable that it is only those motions which are completely stable in the ordinary theory which lead also to a real minimum for R. The suggestion may be stated in this manner: if a fluid motion is thoroughly stable when considered by the method of small vibrations applied to equation (2) and the boundary conditions, then it also possesses a real minimum value of R found from equation (7) and the boundary conditions. It has been pointed out that the latter equation is derived directly from the former, and it may be presumed that the minimum value of R depends in some manner upon the rates of decay of elementary vibrations and so may be used as a measure of the degree of stability of the system.

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The Solution of an Integral Equation occurring in certain Problems of Viscous Fluid Motion. *By* T. H. Havelock, *F.R.S.*

1. THERE are a few well-known solutions of problems of viscous fluid motion in which a solid body starts from rest and moves through the fluid under the action of given forces: for example, the fall of a sphere under gravity when the square of the fluid velocity is neglected, or the corresponding simplified problem of the fall of a plane in which this limitation does not arise. These problems lead to integral equations which have been solved by an application of Abel's theorem ‡. In these cases the fluid was

‡ Boggio, *Rend. d. Accad. d. Lincei*, xvi. pp. 613, 730 (1907); Basset, Quart. Journ. of Math. xli. p. 369 (1910); Rayleigh, Phil. Mag. xxi. p. 697 (1911).

supposed to be of infinite extent, and it seemed to be of interest to solve similar cases of motion when the fluid has a fixed outer boundary. In the following paper consideration has been limited to the motion of a plane between fixed parallel planes and to similar problems with cylinders, the ordinary hydrodynamical equations for non-turbulent motion not involving terms of the second order in such conditions. The results are perhaps not of practical importance, but, apart from the particular problems, the method of solution may be of interest. Stating the problem as in the cases to which reference has been made, we are led to an integral equation of Poisson's type in which the nucleus is an infinite series of exponentials. This equation can be solved by following a method suggested by Whittaker*; the solving function is obtained as an infinite series of exponentials, the exponents being the roots of a certain equation. It seems that examples of this method have not been given hitherto, though equations of this type should arise naturally in various physical problems. The particular cases worked out in detail are the fall of a thin material plane in a liquid bounded by two fixed parallel walls, and the motion of a cylindrical shell filled with liquid and acted on by a constant couple. The same method gives the solution when the force is an assigned function of the time, for instance an alternating force which is suddenly applied. Motion in an infinite fluid may be included in the scheme by replacing the infinite series of exponentials by corresponding infinite integrals. The case of systems with a natural period of oscillation will be considered in a subsequent paper.

It will be clear, from the examples, that the method of solution could be formulated in general rules for obtaining the solving function. This has not been attempted here, as an examination of convergence would be necessary to establish any general theorem. A knowledge of the differential equations and the boundary and initial conditions enables us to verify the results which are given ; in these circumstances, of course, they can be obtained by other methods without difficulty. However, there are probably other physical problems, in which the conditions are not so completely known, whose statement leads to an integral equation of the same type, and its solution can be obtained in the same manner.

2. Consider laminar fluid motion between two fixed planes $x = \pm h$, the fluid velocity being parallel to Oy. Let the

* E. T. Whittaker, Proc. Royal Socy. A, vol. 94 (1918), p. 367.

plane of yz be a thin rigid barrier which is made to move parallel to Oy with a velocity $V(t)$.

Since the equation of fluid motion is

$$\frac{\partial v}{\partial t}=\nu\frac{\partial^2 v}{\partial x^2}, \quad \ldots \ldots \quad (1)$$

with the boundary conditions $v=0$ for $x=h$ and $v=V(t)$ for $x=0$, we may write down the solution as in a similar problem in the conduction of heat; we have, for $x>0$ and $t>0$,

$$v=\sum_{n=1}^{\infty}\frac{2n\pi\nu}{h^2}e^{-n^2\pi^2\nu t/h^2}\sin\frac{n\pi x}{h}\int_0^t V(\tau)e^{n^2\pi^2\nu\tau/h^2}d\tau, \quad . \quad (2)$$

The frictional force, per unit area, on the plane of yz is the value of $2\mu(\partial v/\partial x)$ for $x=0$, counting both sides of the plane: if we suppose the plane to start from rest, so that $V(0)=0$, this gives, after integrating by parts,

$$(2\mu/h)\int_0^t V'(\tau)\{1+2\sum_1^{\infty}e^{-n^2\pi^2\nu(t-\tau)/h^2}\}d\tau. \quad . \quad . \quad (3)$$

In the class of problems we are considering, $V(t)$ is the function to be determined and it is the forces on the plane which are given. As a first example, consider motion under gravity. Suppose that the plane of yz is vertical and that it has a mass σ per unit area; we require the motion of the plane as it falls under gravity, starting from rest and having fixed parallel walls at a distance h on either side. Using (3), the equation of motion of the plane can be put at once into the form

$$V'(t)+(2\mu/\sigma h)\int_0^t V'(\tau)\{1+2\sum_1^{\infty}e^{-n^2\pi^2\nu(t-\tau)/h^2}\}d\tau=g. \quad (4)$$

This is an integral equation of Poisson's type, which can be solved for $V'(t)$ in the following manner.

3. In the paper already quoted, Whittaker considers an equation

$$\phi(x)+\int_0^x\phi(s)\kappa(x-s)ds=f(x), \quad . \quad . \quad . \quad (5)$$

in which the nucleus is the sum of μ exponentials, or

$$\kappa(x)=Pe^{px}+Qe^{qx}+\ldots\ldots+Ve^{vx}. \quad . \quad . \quad . \quad (6)$$

The solution is obtained as

$$\phi(x)=f(x)-\int_0^x f(s)K(x-s)ds, \quad . \quad . \quad . \quad (7)$$

where the solving function is also a sum of μ exponentials, or

$$K(x)=Ae^{\alpha x}+Be^{\beta x}+\ldots\ldots+Ne^{\nu x}. \quad . \quad . \quad . \quad (8)$$

It is shown that $\phi(x)=\mathrm{K}(x)$ and $f(x)=\kappa(x)$ satisfy (5): hence by substituting and equating coefficients of similar exponentials, it is found that $\alpha, \beta, \gamma, \ldots \nu$ are the roots of the algebraic equation

$$\frac{\mathrm{P}}{x-p}+\frac{\mathrm{Q}}{x-q}+\ldots\ldots+\frac{\mathrm{V}}{x-v}+1=0, \quad . \quad . \quad (9)$$

while the coefficients in $\mathrm{K}(x)$ satisfy the equations

$$\left.\begin{array}{l}\dfrac{\mathrm{A}}{\alpha-p}+\dfrac{\mathrm{B}}{\beta-p}+\ldots\ldots+\dfrac{\mathrm{N}}{\nu-p}+1=0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\mathrm{A}}{\alpha-v}+\dfrac{\mathrm{B}}{\beta-v}+\ldots\ldots+\dfrac{\mathrm{N}}{\nu-v}+1=0\end{array}\right\} \quad . \quad . \quad (10)$$

The solution of (10) leads to

$$\mathrm{K}(x)=-\frac{(\alpha-p)(\alpha-q)\ldots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\ldots\ldots$$
$$-\frac{(\nu-p)(\nu-q)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}. \quad (11)$$

Before proceeding, we may note alternative forms of these results which are of use later. If we write

$$\mathrm{F}(x)\equiv(x-p)^{\mathrm{P}}(x-q)^{\mathrm{Q}}\ldots(x-v)^{\mathrm{V}}, \quad . \quad . \quad (12)$$

the equation for the new exponents $\alpha, \beta, \gamma \ldots$ is

$$\mathrm{F}'(x)+\mathrm{F}(x)=0. \quad . \quad . \quad . \quad . \quad (13)$$

Further, if we put

$$f(x)=(x-\alpha)(x-\beta)\ldots(x-\nu)$$

and

$$\phi(x)=(x-p)(x-q)\ldots(x-v),$$

the coefficients in (11) are $-\phi(\alpha)/f'(\alpha)$, where α is a root of (13).

Whittaker remarks that if the number of exponential terms in (6) is supposed to increase indefinitely, a theorem appears to be indicated, namely, that in the solution of a Poisson's integral equation whose nucleus is expressible as a Dirichlet series, the solving function is also expressible as a Dirichlet series, but with a different set of exponents for the exponentials.

4. Returning now to equation (4), we see that it is an

example of such a theorem; without attempting any discussion of the general theorem, we proceed to solve (4) directly on the lines indicated in equations (9)–(13). In the notation of these equations, we have from (4)

$$p=0, \quad q=-\pi^2\nu/h^2, \quad r=-2^2\pi^2\nu/h^2, \quad \ldots \; ;$$
$$P=2\mu/\sigma h, \quad Q=R=\ldots\ldots=4\mu/\sigma h.$$

Equation (9) becomes a transcendental equation, namely

$$\frac{2\mu}{\sigma\nu^{\frac{1}{2}}x^{\frac{1}{2}}}\coth\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}+1=0. \quad . \quad . \quad . \quad . \quad (14)$$

The roots of this equation are negative, and it is convenient to write $x=-\nu\lambda^2/h^2$, then the values of λ are the positive roots of the equation

$$\lambda\tan\lambda=2\rho h/\sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Using A_1, A_2, ... for the coefficients of the solving function, equations (10) become

$$\frac{A_1}{\lambda_1{}^2-n^2\pi^2}+\frac{A_2}{\lambda_2{}^2-n^2\pi^2}+\frac{A_3}{\lambda_3{}^2-n^2\pi^2}+\ldots\ldots-\frac{\nu}{h^2}=0, \quad (16)$$

where $n=0, 1, 2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ are the positive roots of (15).

Assuming that a function $f(x)$ can be expanded, in the range $-1<x<1$, in a series

$$f(x)=\Sigma C\cos\lambda x,$$

we have

$$C=\frac{\lambda}{\lambda+\sin\lambda\cos\lambda}\int_{-1}^{+1}f(x)\cos\lambda x\,dx, \quad . \quad . \quad (17)$$

where λ is a root of (15). Taking $f(x)=\cos n\pi x$, we obtain the set of expansions

$$1=2\Sigma\frac{\lambda^2\sin\lambda\cos\lambda}{(\lambda+\sin\lambda\cos\lambda)(\lambda^2-n^2\pi^2)}. \quad . \quad . \quad (18)$$

Hence the solution of the set of equations in (16) is

$$A_r=\frac{2\nu\lambda_r{}^2\sin\lambda_r\cos\lambda_r}{h^2(\lambda_r+\sin\lambda_r\cos\lambda_r)}=\frac{4\mu}{\sigma h}\frac{\lambda_r{}^2}{\lambda_r{}^2+k(1+k)}, \quad (19)$$

where $k=2\rho h/\sigma$. These results can also be derived directly by extending the forms (12) and (13) to include infinite products.

Substituting in (7), we have

$$\frac{dV}{dt} = g - \frac{4\mu g}{\sigma h}\int_0^t \Sigma \frac{\lambda^2}{\lambda^2 + k(1+k)} e^{-\nu\lambda^2(t-\tau)/h^2}\,d\tau$$

$$= \frac{4g\rho h}{\sigma} \Sigma \frac{e^{-\nu\lambda^2 t/h^2}}{\lambda^2 + k(1+k)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

the terms independent of t cancelling out on summation.

The velocity at any time is given by

$$V = \frac{4g\rho h^3}{\sigma\nu} \Sigma \frac{1 - e^{-\nu\lambda^2 t/h^2}}{\lambda^2\{\lambda^2 + k(1+k)\}}. \quad , \quad . \quad . \quad (21)$$

It can be verified by summation that the limiting steady velocity has the value $g\sigma h/2\mu$. The fluid velocity at any point can be obtained by substituting V from (21) in (2) and reducing the expressions, but it is, of course, simpler to insert suitable functions of x directly in (21); we obtain

$$v = \frac{g\sigma h}{2\mu}\left(1 - \frac{x}{h}\right) - \frac{4g\rho h^3}{\sigma\nu} \Sigma \frac{\sin\{\lambda(1 - x/h)\}\, e^{-\nu\lambda^2 t/h^2}}{\lambda^2\{\lambda^2 + k(1+k)\}\sin\lambda}. \quad (22)$$

In this particular problem the result can also be obtained from the differential equation together with the boundary and initial conditions, by assuming the existence of a limiting steady state. In the preceding analysis the existence of a final steady state is associated with the occurrence of zero as one of the exponents in the nucleus of the integral equation (4).

5. It is interesting to deduce the motion in an infinite fluid from these results. In solving this case directly, Rayleigh obtains the equation of motion as

$$\frac{dV}{dt} + \frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi^{\frac{1}{2}}}\int_0^t \frac{V'(\tau)\,d\tau}{\sqrt{(t-\tau)}} = g. \quad . \quad . \quad . \quad . \quad (23)$$

Applying Abel's theorem, this is reduced to an ordinary differential equation whose solution is given as

$$4\pi^{\frac{1}{2}}\mu\rho V/g\sigma = 4\rho\nu^{\frac{1}{2}}t^{\frac{1}{2}} - \pi^{\frac{1}{2}}\sigma + 2\sigma e^{4\rho^2\nu t/\sigma^2}\int_{2\rho\nu^{\frac{1}{2}}t^{\frac{1}{2}}/\sigma}^{\infty} e^{-u^2}\,du. \quad (24)$$

We obtain (23) from (4) by giving the nucleus its limiting value, since

$$\lim_{h\to\infty} (2\mu/\sigma h) \sum_{-\infty}^{\infty} e^{-n^2\pi^2\nu(t-\tau)/h^2} = (2\rho\nu^{\frac{1}{2}}/\sigma\pi)\int_{-\infty}^{\infty} e^{-\alpha^2(t-\tau)}\,d\alpha.$$

In the same way, the solving function has a limiting form which follows directly from (15) and (19), namely

$$\frac{2\rho\nu^{\frac{1}{2}}}{\sigma\pi}\int_{-\infty}^{\infty}\frac{\alpha^2 e^{-\alpha^2(t-\tau)}}{\alpha^2+4\rho^2\nu_1/\sigma^2}\,d\alpha.$$

Using this value as before, we obtain the same result (24).

6. It is clear that the same procedure is sufficient when the applied force is any assigned function of the time. For example, if the accelerative force is $a\cos pt$ and the motion starts from rest, we have

$$\frac{dV}{dt}=a\cos pt-\int_0^t a\cos p\tau\,.\,\Sigma A e^{-\nu\lambda^2(t-\tau)/h^2}d\tau, \quad . \quad (25)$$

where the summation extends over the roots of the same equation (15), and the coefficients are given by (19) The solution follows on completing the integrations; it consists of a periodic motion in different phase from the applied force, together with the disturbance due to taking into account the initial conditions.

7. A final example may be taken from cylindrical motion when there is no limiting steady velocity. Suppose the motion to be symmetrical round an axis; then if r is distance from the axis and v is the fluid velocity, supposed perpendicular to the radius vector, we have

$$\frac{\partial v}{\partial t}=\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right). \quad . \quad . \quad . \quad (26)$$

Consider the motion of a hollow cylinder, of radius a, filled with the liquid. Suppose the motion to start from rest and let the velocity of the cylinder be $\Omega(t)$. Then it may be shown that the angular velocity of the fluid at any time is given by

$$\omega=\int_0^t \Omega'(\tau)\left\{1+2\Sigma\frac{aJ_1(pr/a)}{pr\,J_1'(p)}e^{-\nu p^2(t-\tau)/a^2}\right\}d\tau, \quad (27)$$

where the summation extends over the positive roots of $J_1(p)=0$.

Let the cylindrical shell start from rest under the action of a constant couple N, and let I be its moment of inertia, both quantities being for unit length along the axis. The retarding couple due to fluid friction is the value of $2\pi\mu r^3\partial\omega/\partial r$ when $r=a$. Hence the equation of motion of the cylinder is

$$\Omega'(t)+(4\pi\mu a^2/I)\int_0^t \Omega'(\tau)\Sigma e^{-\nu p^2(t-\tau)/a^2}d\tau=N/I, \quad (28)$$

where the summation extends over the positive roots of

$$\frac{1}{p}J_3(p)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The equation for the exponents of the solving function is

$$\frac{4\pi\mu a^4}{I}\left(\frac{1}{a^2x+\nu p_1^2}+\frac{1}{a^2x+\nu p_2^2}+\ldots\right)+1=0. \quad (30)$$

Writing $x=-\nu\lambda^2/a^2$, equation (30) reduces to

$$kJ_2(\lambda)+\lambda J_1(\lambda)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $k=2\pi\rho a^4/I$. The equation can be deduced from (29), by logarithmic differentiation, as indicated in (12) and (13).

The equations for the coefficients of the solving function become

$$\left.\begin{array}{l}
\dfrac{A_1}{\lambda_1^2-p_1^2}+\dfrac{A_2}{\lambda_2^2-p_1^2}+\dfrac{A_3}{\lambda_3^2-p_1^2}+\ldots-\dfrac{\nu}{a^2}=0 \\
\dfrac{A_1}{\lambda_1^2-p_2^2}+\dfrac{A_2}{\lambda_2^2-p_2^2}+\dfrac{A_3}{\lambda_3^2-p_2^2}+\ldots-\dfrac{\nu}{a^2}=0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\}, \quad (32)$$

where $p_1, p_2, \ldots$ are the roots of (29), and $\lambda_1, \lambda_2, \ldots$ the roots of (31).

To solve these equations, we may adopt the same plan as before. Assuming that a function $f(r)$ can be expanded, in the range $0<r<1$, in the series

$$f(r)=\Sigma BJ_2(\lambda r),$$

the summation extending over the positive roots of (31) we have

$$B=\frac{2\lambda^2}{\{\lambda^2+k(k+4)\}J_2^2(\lambda)}\int_0^1 f(r)J_2(\lambda r)r\,dr. \quad . \quad (33)$$

Now take $f(r)=J_2(pr)$, where p is a positive root of (29); after obtaining the expansion and putting $r=1$, we arrive at the result

$$1=\sum_\lambda\frac{2k\lambda^2}{\{\lambda^2+k(k+4)\}(\lambda^2-p^2)}, \quad . \quad . \quad . \quad (34)$$

p being any one root of (29) and the summation being with respect to the roots of (31).

Comparing with (32) it follows that

$$A_s = 2k\nu\lambda_s^2/a^2\{\lambda_s^2 + k(k+4)\}. \quad . \quad . \quad . \quad (35)$$

With this expression for the solving function, (28) gives

$$\frac{d\Omega}{dt} = \frac{N}{I} - \frac{2k\nu N}{a^2 I}\int_0^t \Sigma \frac{\lambda^2 e^{-\nu\lambda^2(t-\tau)/a^2}}{\lambda^2 + k(k+4)}\, d\tau. \quad . \quad . \quad (36)$$

By expanding r^2 by (33) and putting $r=1$, it can be shown that

$$1 = \Sigma \frac{2(k+4)}{\lambda^2 + k(k+4)}. \quad . \quad . \quad . \quad . \quad (37)$$

Carrying out the integration in (36) and using (37), we find

$$\frac{d\Omega}{dt} = \frac{N}{I + \frac{1}{2}\pi\rho a^4} + \frac{2kN}{I}\Sigma \frac{e^{-\nu\lambda^2 t/a^2}}{\lambda^2 + k(k+4)}. \quad . \quad . \quad (38)$$

The angular acceleration has a finite limiting value in this case, the same as if the cylinder and enclosed liquid were rotating like a rigid body. We notice that in this case zero is excluded from the roots of the equation (29) for the exponents of the nucleus.

Integrating (38) we obtain the angular velocity of the cylinder at any time; then, using the differential equation (26), we may complete the solution by writing down the angular velocity of the liquid. It is found to be given by

$$\omega = \frac{N}{I + \frac{1}{2}\pi\rho a^4}\left\{t + \frac{r^2}{8\nu} - \frac{k+6}{12(k+4)}\right\}$$
$$- \frac{2kN}{I}\Sigma \frac{a^3 J_1(\lambda r/a)e^{-\nu\lambda^2 t/a^2}}{\nu r\lambda^2\{\lambda^2 + k(k+4)\}J_1(\lambda)}.$$

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On the Decay of Oscillation of a Solid Body in a Viscous Fluid. *By* T. H. HAVELOCK, *F.R.S.*

1. THE decay of rotational oscillation of a cylinder or a sphere in a viscous liquid is a well-known problem in Hydrodynamics; among more recent researches, reference may be made to the work of Verschaffelt †, Coster ‡, and others. In those papers it is remarked that

† G. E. Verschaffelt, Amsterdam Proc. xviii. p. 840 (1916); also Comm. Leiden, cli. (1917).
‡ D. Coster, Phil. Mag. xxxvii. p. 587 (1919).

the ordinary solution of a damped harmonic vibration requires modification when the initial conditions are taken into account, but no explicit solution of this nature seems to have been given; in certain experimental refinements, the disturbance may be of some importance. In the following notes, I have worked out in detail first the simpler case of a plane oscillating between two fixed planes. The problem can be solved by various methods: by normal functions, or, more readily, by operational methods. I have chosen to use it as an example of a type of integral equation, for which reference may be made to a previous paper*. In this case the equation of motion is an integro-differential equation of Volterra's type, and it can be solved by a repeated application of Whittaker's method which was used in the simpler cases; the solution may be of interest apart from the particular problem. The results are then verified by using Bromwich's method of complex integration. Finally, the solution is indicated for a sphere oscillating within a fixed outer sphere, and the results are discussed in connexion with the experiments to which reference has already been made.

2. Suppose that a viscous liquid can move in laminar motion between two fixed planes $x=\pm h$. Let the plane of yz be a thin rigid barrier of mass σ per unit area, and let it be acted on by an elastic force parallel to Oy such that, if the liquid were absent, the plane would vibrate with a natural period $2\pi/p$. Further, suppose the motion starts from rest with the plane displaced a distance a from its equilibrium position. The equation of motion of the plane is

$$\sigma\frac{d^2y}{dt^2}-2\mu\left(\frac{\partial v}{\partial x}\right)_0+\sigma p^2y=0, \quad . \quad . \quad . \quad . \quad (1)$$

where v is the fluid velocity.

Now if the plane of yz has a velocity $\mathrm{V}(t)$, the fluid velocity may be written in the form

$$v=\frac{2}{h}\sum_1^\infty\frac{n\pi\nu}{h}e^{-n^2\pi^2\nu t/h^2}\sin\frac{n\pi x}{h}\int_0^t \mathrm{V}(\tau)e^{n^2\pi^2\nu\tau/h^2}d\tau. \quad . \quad (2)$$

Taking the value of $\partial v/\partial x$ for $x=0$, integrating by parts and noting that in this problem $\mathrm{V}(0)=0$, equation (1) gives

$$\frac{d^2y}{dt^2}+\frac{2\mu}{\sigma h}\int_0^t\frac{d^2y}{d\tau^2}\left\{1+2\sum_1^\infty e^{-n^2\pi^2\nu(t-\tau)/h^2}\right\}d\tau+p^2y=0. \quad (3)$$

* *Suprà*, p. 620.

Consider the integral equation

$$\phi(t)+\int_0^t \phi(\tau)\kappa(t-\tau)\,d\tau=f(t),$$

where the nucleus is the sum of n exponentials

$$\kappa(t)=\sum_1^n \mathrm{P}_r e^{p_r t}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Whittaker's solution * is given as

$$\phi(t)=f(t)-\int_0^t f(\tau)\,\mathrm{K}(t-\tau)\,d\tau, \quad . \quad . \quad . \qquad (6)$$

the solving function being also the sum of n exponentials

$$\mathrm{K}(t)=\sum_1^n \mathrm{A}_r e^{\alpha_r t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

The indices α are the roots of the equation

$$\frac{\mathrm{P}_1}{x-p_1}+\frac{\mathrm{P}_2}{x-p_2}+\ldots+\frac{\mathrm{P}_n}{x-p_n}+1=0. \quad . \quad . \qquad (8)$$

Further, if we form the functions

$$\left.\begin{aligned}\psi(x)&\equiv\left(1-\frac{x}{p_1}\right)\left(1-\frac{x}{p_2}\right)\ldots\left(1-\frac{x}{p_n}\right),\\ \theta(x)&\equiv\left(1-\frac{x}{\alpha_1}\right)\left(1-\frac{x}{\alpha_2}\right)\ldots\left(1-\frac{x}{\alpha_n}\right),\end{aligned}\right\} \quad . \quad . \qquad (9)$$

it may be shown that the coefficients of the solving function are given by

$$\mathrm{A}=\left(\frac{\mathrm{P}_1}{p_1}+\ldots+\frac{\mathrm{P}_n}{p_n}-1\right)^{-1}\psi(\alpha)/\theta'(\alpha), \quad . \qquad (10)$$

where α is a root of (8). It should be noted that if p_1 is zero, and we write $\psi(x)=x(1-x/p_2)\ldots(1-x/p_n)$, then

$$\mathrm{A}=-\psi(\alpha)/\mathrm{P}_1\theta'(\alpha). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

We shall assume that these results hold in the limit when the number of exponential terms becomes infinite. Equation (3) then comes under this form, except that it is an integro-differential equation. Equation (8) for the exponents of the solving function gives, on summation,

$$\frac{2\mu}{\sigma\nu^{\frac{1}{2}}x}\coth\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}+1=0. \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Also we have

$$\psi(x)=(\nu^{\frac{1}{2}}x^{\frac{1}{2}}/h)\sinh\,(hx^{\frac{1}{2}}/\nu^{\frac{1}{2}}),$$

$$\theta(x)=\cosh\,(hx^{\frac{1}{2}}/\nu^{\frac{1}{2}})+(\sigma\nu^{\frac{1}{2}}x^{\frac{1}{2}}/2\mu)\sinh\,(hx^{\frac{1}{2}}/\nu^{\frac{1}{2}})$$

* E. T. Whittaker, Proc. Roy. Soc. A, xciv. p. 367 (1918).

The method of formation of $\theta(x)$ is clear from the equations for a finite number of terms; multiply the left-hand side of (12) by $\psi(x)$ and a factor to make the value unity for x zero. From (11) and (12), we have

$$A = -\alpha \Big/ \frac{2\mu}{\sigma h}\left(\frac{h^2}{2\nu} + \frac{\sigma h}{4\mu} - \frac{\sigma^2 h^2 \alpha}{8\mu^2}\right). \quad . \quad . \quad (13)$$

Writing the roots of (12) as $\alpha = -\nu\lambda^2/h^2$ and collecting the results from (6), (12), and (13), the first step in the solution of (3) gives

$$-\frac{d^2y}{dt^2} = p^2 y - \frac{4\mu p^2}{\sigma h}\int_0^t y(\tau)\Sigma\frac{\lambda^2 e^{-\nu\lambda^2(t-\tau)/h^2}}{\lambda^2 + k(1+k)}d\tau, \quad . \quad (14)$$

where the summation extends over the positive roots of

$$\lambda \tan\lambda = 2\rho h/\sigma = k. \quad . \quad . \quad . \quad . \quad (15)$$

Following the method of reduction for this type of equation*, integrate (14) with respect to t from 0 to θ, using Dirichlet's formula to transform the order of integration of the last term. Since the initial value of dy/dt is zero, this leads to

$$-\frac{dy(\theta)}{d\theta} = \frac{4\mu p^2 h}{\sigma\nu}\int_0^\theta y(t)\Sigma\frac{e^{-\nu\lambda^2(\theta-t)/h^2}}{\lambda^2 + k(1+k)}dt. \quad . \quad (16)$$

Integrate (16), in the same manner, with respect to θ from 0 to T; finally, for convenience, replace T by t and t by τ, respectively, in the result. Then we obtain

$$y(t) + \int_0^t y(\tau)\left[\frac{\sigma p^2 h}{2\mu} - \frac{4\mu p^2 h^3}{\sigma\nu^2}\Sigma\frac{e^{-\nu\lambda^2(t-\tau)/h^2}}{\lambda^2\{\lambda^2 + k(1+k)\}}\right]d\tau = a. \quad (17)$$

The solution of (17) can be completed by means of (6), (8), and (11). The new exponents are given by

$$\frac{\sigma p^2 h}{x} + \frac{4\mu p^2 h^3}{\sigma\nu^2}\Sigma\frac{1}{\lambda^2\{\lambda^2 + k(1+k)\}(x + \nu\lambda^2/h^2)} + 1 = 0. \quad (18)$$

Resolving the summation into one of simple partial fractions and using the properties of the roots of (15), this equation can be reduced to

$$x^2 + \frac{2\rho\nu^{\frac{1}{2}}x^{\frac{3}{2}}}{\sigma}\coth\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} + p^2 = 0. \quad . \quad . \quad . \quad (19)$$

* Volterra, 'Leçons sur les Équations Intégrales,' p. 140.

In the previous notation, we have

$$\left.\begin{aligned}\psi(x)&=x\left(\cosh\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}+\frac{\sigma\nu^{\frac{1}{2}}x^{\frac{1}{2}}}{2\mu}\sinh\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\right),\\ \theta(x)&=\left(\frac{\nu^{\frac{1}{2}}}{hx^{\frac{1}{2}}}+\frac{\nu^{\frac{1}{2}}x^{\frac{3}{2}}}{p^2h}\right)\sinh\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}+\frac{2\mu x}{\sigma p^2h}\cosh\frac{hx^{\frac{1}{2}}}{\nu^{\frac{1}{2}}},\end{aligned}\right\}. \quad (20)$$

the formation of the latter being clearly indicated in the reduction from (18) to (19). The coefficients of the solving function can now be formed by (11). Finally, substituting in (6) and carrying out the integration, we arrive at the result

$$y=\frac{4\mu p^2a}{\sigma h}\Sigma\frac{\alpha e^{\alpha t}}{\alpha^4-(2\mu/\sigma h)(1+2ph/\sigma)\alpha^3+2p^2\alpha^2+6\mu p^2\alpha/\sigma h+p^4}, \quad . \quad . \quad . \quad (21)$$

the summation extending over the roots of (19).

3. We may verify the result by other methods which are available in this case. We choose Bromwich's method of complex integration*, referring to his paper for the general principles, and writing down the results briefly for the present problem.

Suppose the fluid velocity and the displacement of the plane to be given by

$$v=\frac{1}{2\pi i}\int ue^{\alpha t}\,d\alpha\,; \quad y=\frac{1}{2\pi i}\int \eta e^{\alpha t}\,d\alpha\,; \quad . \quad . \quad (22)$$

where u and η are functions of α, and the paths of integration are in the plane of a complex variable α and enclose all the poles of these functions. The differential equation of fluid motion, $\partial v/\partial t=\nu\partial^2v/\partial x^2$, with the conditions $u=0$ for $x=h$ and $u=d\eta/dt$ for $x=0$, gives the solution, for x positive,

$$u=\frac{d\eta}{dt}\frac{\sinh\{\alpha^{\frac{1}{2}}(h-x)/\nu^{\frac{1}{2}}\}}{\sinh(\alpha^{\frac{1}{2}}h/\nu^{\frac{1}{2}})}. \quad . \quad . \quad . \quad (23)$$

From the boundary condition (1), after introducing terms due to the initial conditions $y=a$ and $dy/dt=0$ for $t=0$, and using (23), we obtain

$$\sigma\alpha^2\eta+\frac{2\mu\alpha^{\frac{3}{2}}}{\nu^{\frac{1}{2}}}\eta\coth\frac{\alpha^{\frac{1}{2}}h}{\nu^{\frac{1}{2}}}+\sigma p^2\eta=\left(\sigma\alpha+\frac{2\mu\alpha^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\coth\frac{\alpha^{\frac{1}{2}}h}{\nu^{\frac{1}{2}}}\right)a. \quad (24)$$

Hence we have

$$y=\frac{1}{2\pi i}\int\frac{a\{\alpha+(2\mu\alpha^{\frac{1}{2}}/\sigma\nu^{\frac{1}{2}})\coth(h\alpha^{\frac{1}{2}}/\nu^{\frac{1}{2}})\}e^{\alpha t}\,d\alpha}{\alpha^2+(2\mu\alpha^{\frac{3}{2}}/\sigma\nu^{\frac{1}{2}})\coth(h\alpha^{\frac{1}{2}}/\nu^{\frac{1}{2}})+p^2}. \quad . \quad . \quad (25)$$

Forming the residues of the integrand at the zeros of the

* T. J. I'A. Bromwich, Proc. Lond. Math. Soc. xv. p. 401 (1916).

denominator, we obtain the same solution (21). The comparison brings out the connexion between the method of solution of the particular form of integral equation and the use of normal functions in dynamical problems. The latter methods would not be available if we had not a complete knowledge of the differential equations of the problem: for instance, if it were stated directly as an integro-differential equation like (3) in some problem of 'heredity.'

4. The nature of the roots of (19) may be studied most easily by graphical methods, or by using the form (18) or equivalent expansions. It appears that, leaving aside the possibility of multiple roots, there is an infinite series of real negative roots and, in addition, a pair of roots which may be complex, or real and negative. In the latter case the motion is aperiodic; in the former, the two complex roots give the damped harmonic vibration while the remaining roots complete the solution according to (21) for the given initial conditions. In the theory of determinations of viscosity by oscillating cylinders or spheres it is usual to assume a damped harmonic vibration, neglecting all the other terms.

Verschaffelt remarks that for a motion that is not purely damped harmonic, the proportionality of the resistance to the velocity no longer exists, and that it would then probably be impossible to establish a general differential equation for the motion. We have seen, however, that it may be expressed by an integro-differential equation as in (3). It seems that in experiments under usual conditions, the final state of a damped harmonic motion is practically reached after a comparatively short time (a few minutes).

With numerical values of the usual order, it is easy to see that the lowest real negative root of (19) is much larger numerically than the real (negative) part of the complex roots. The matter would require closer examination if the motion were entirely aperiodic,* as in some experiments. In the case of a sphere making oscillations of finite amplitude, Verschaffelt has studied small damping effects due to approximations involving the quadratic terms in the hydrodynamical equations; this introduces damping coefficients of three or five times the first approximation, and it may be that in such cases the purely aperiodic terms in the solution should also be taken into account.

5. It may be of interest to record the complete solution, neglecting quadratic terms, for a sphere oscillating in a liquid enclosed within a fixed concentric shell.

Let ω be the angular velocity in the liquid, θ the angular

displacement of the sphere, a its radius and I its moment of inertia ; and let b be the radius of the fixed outer sphere. Then the equation of motion of the rotating sphere is

$$\mathrm{I}\frac{d^2\theta}{dt^2}-\frac{8}{3}\mu\pi a^4\left(\frac{\partial\omega}{dr}\right)_a+p^2\mathrm{I}\theta=0, \quad . \quad . \quad . \quad (26)$$

with $\theta=\theta_0$ and $d\theta/dt=0$ for $t=0$.

In the fluid we have

$$\frac{\partial\omega}{\partial t}=\nu\left(\frac{\partial^2\omega}{\partial r^2}+\frac{4}{r}\frac{\partial\omega}{\partial r}\right), \quad . \quad . \quad . \quad . \quad (27)$$

with $\omega=0$ $r=b$, and $\omega=d\theta/dt$ for $r=a$.

Using the method of § 3, we write

$$\omega=\frac{1}{2\pi i}\int ue^{\alpha t}\,d\alpha\ ; \quad \theta=\frac{1}{2\pi i}\int\eta e^{\alpha t}\,d\alpha. \quad . \quad . \quad (28)$$

Then equation (27) gives the solution

$$u=\frac{a^3}{r^3}\frac{d\theta}{dt}\frac{k(b-r)\cosh\{k(b-r)\}+(k^2br-1)\sinh\{k(b-r)\}}{k(b-a)\cosh\{k(b-a)\}+(k^2ba-1)\sinh\{k(b-a)\}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where $k=\alpha^{\frac{1}{2}}/\nu^{\frac{1}{2}}$.

Modifying (26) so as to take account of the initial conditions, we have for η the equation

$$\eta f(\alpha)=\mathrm{F}(\alpha), \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where

$$f(\alpha)=\mathrm{I}\alpha^2+8\mu\pi a^3\alpha+\mathrm{I}p^2+\tfrac{8}{3}\pi\rho a^5\alpha^2 \times\frac{bk\cosh k(b-a)-\sinh k(b-a)}{k(b-a)\cosh k(b-a)+(k^2ab-1)\sinh k(b-a)},$$

$$\mathrm{F}(\alpha)=\mathrm{I}\alpha+8\mu\pi a^3+\tfrac{8}{3}\pi\rho a^5\alpha \times\frac{bk\cosh k(b-a)-\sinh k(b-a)}{k(b-a)\cosh k(b-a)+(k^2ab-1)\sinh k(b-a)}.$$

The angular displacement of the sphere is then

$$\theta=\frac{\theta_0}{2\pi i}\int\frac{\mathrm{F}(\alpha)}{f(\alpha)}e^{\alpha t}\,d\alpha=\theta_0\Sigma\frac{\mathrm{F}(\alpha)}{f'(\alpha)}e^{\alpha t}, \quad . \quad . \quad . \quad (31)$$

where the summation extends over the roots of $f(\alpha)=0$, and it is assumed that these are all simple roots.

In practice we may usually separate the roots into two classes : first a pair of roots which may be either complex or real and negative, then a series of real negative roots in the neighbourhood of $-\pi^2\nu/(b-a)^2$, $-4\pi^2\nu/(b-a)^2$ and so on. In deducing the form of (31) for a sphere in an infinite liquid the sum of the terms from the latter series of roots must be replaced by a corresponding infinite integral.

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The Effect of Shallow Water on Wave Resistance.

By T. H. HAVELOCK, F.R.S.

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1. The general character of experimental results dealing with the effect of shallow water on ship resistance may be stated briefly as follows:—At low velocities the resistance in shallow water is greater than in deep water, the speed at which the excess is first appreciable varying with the type of vessel. As the speed increases, the excess resistance increases up to a maximum at a certain critical velocity, and then diminishes. With still further increase of speed, the resistance in shallow water ultimately becomes, and remains, less than that in deep water at the same speed. The maximum effect is the more pronounced the shallower the water. For further details and references one may refer to standard treatises, but one quotation may be made in regard to the critical velocity: "This maximum appears to be at about a speed such that a trochoidal wave travelling at this speed in water of the same depth is about $1\frac{1}{4}$ times as long as the vessel. . . . It was at one time supposed that the speed for maximum increase in resistance was that of the wave of translation. This, however, holds only for water whose depth is less than 0·2 times the length of the vessel. For greater depths the speed of the wave of translation rapidly becomes greater than the speed of maximum increase of resistance."* In a recent analysis of the data, H. M. Weitbrecht† expresses a similar conclusion by stating that for each depth of water there is a critical velocity, but that the critical velocity does not vary as the square root of the corresponding depth.

It should be noted that experimental results are for the total resistance. If we assume that this can be separated into three terms, which are simply additive, namely, eddy, frictional, and wave-making resistance, it must be admitted that probably all are affected by limited depth of water. However, the main differences are due to the altered wave-making, and the general explanation is to be found in the fact that there is a limiting velocity, $\sqrt{(gh)}$ for simple straight-crested waves on water of depth h.

Leaving aside the difficult problem of a solid body towed or driven through the water, we may study the allied problem of a given distribution of surface pressure and the associated wave resistance. Previous calculations of wave resistance have been limited to a line distribution of pressure, involving

* D. W. Taylor, 'Speed and Power of Ships,' vol. 1, p. 114; also G. S. Baker, 'Ship Form, Resistance and Screw Propulsion,' p. 134.

† H. M. Weitbrecht, 'Jahrbuch d. Schiffbautech. Gesell.,' vol. 22, p. 122 (1921).

therefore, only straight-crested parallel waves and so emphasising the connection between the critical velocity and that of the wave of translation. In the present paper I obtain an expression for the wave resistance of a surface pressure symmetrical about a point, and moving over water of finite depth. The result is in the form of a definite integral, which has been evaluated by numerical and graphical methods so as to give graphs of the variation of wave resistance with speed for different values of the ratio of the depth of water to the length associated with the pressure distribution. The graphs are of special interest in the cases intermediate between the two extremes of deep water and shallow water. They show the double effect of limited depth, in lowering the normal wave-making speed of the ship and in increasing the magnitude of the effect as the speed approaches that of the wave of translation. The results are discussed in their bearing upon the experimental results which have just been described.

2. In a previous paper* I worked out the case of a symmetrical surface pressure moving over deep water. The present analysis is on exactly similar lines, except for suitable changes in the expressions; it may be sufficient, therefore, to set forth the calculation briefly, referring to the previous paper for further detail in the argument.

Take axes Ox, Oy in the undisturbed horizontal surface of water of depth h and Oz vertically upwards. For an initial impulse symmetrical about the origin, that is if the initial data are

$$\rho\phi_0 = \mathrm{F}(\varpi), \qquad \zeta = 0, \tag{1}$$

where $\varpi^2 = x^2+y^2$, the velocity potential and surface elevation in the subsequent fluid motion are given by

$$\rho\phi = \int_0^\infty f(\kappa) \cosh\kappa(z+h) \operatorname{sech}\kappa h \, \mathrm{J}_0(\kappa\varpi) \cos(\kappa \mathrm{V}t)\, \kappa\, d\kappa,$$

$$g\rho\zeta = -\int_0^\infty f(\kappa)\, \mathrm{J}_0(\kappa\varpi) \sin(\kappa\mathrm{V}t)\, \kappa^2 \mathrm{V}\, d\kappa, \tag{2}$$

where $$\mathrm{V}^2 = (g/\kappa)\tanh\kappa h,$$

$$f(\kappa) = \int_0^\infty \mathrm{F}(\alpha)\, \mathrm{J}_0(\kappa\alpha)\, \alpha\, d\alpha. \tag{3}$$

We obtain the effect of a travelling pressure system by integrating with respect to the time. We shall suppose that the system has been moving for a long time with uniform velocity, c, in the direction of Ox. Transferring to a moving origin at the centre of the system, we replace x in (2) by $x+ct$, and we find for the surface elevation

$$g\rho\zeta = -\int_0^\infty e^{-1/2\mu t}\, dt \int_0^\infty f(\kappa)\, \mathrm{J}_0[\kappa\{(x+ct)^2+y^2\}^{1/2}] \sin(\kappa\mathrm{V}t)\, \kappa^2\mathrm{V}\, d\kappa, \tag{4}$$

* 'Roy. Soc. Proc.,' A, vol. 95, p. 354 (1919).

where $f(\kappa)$ is found from the assigned pressure distribution, $p = \mathrm{F}(\varpi)$, by means of (3). The factor exp. $(-\frac{1}{2}\mu t)$ serves to keep the integrals determinate, so that they give a solution which corresponds to the main part of the surface waves trailing aft from the moving disturbance. It is to be noted that ultimately μ is made zero in the final results, and it is only retained in the intermediate analysis to a degree sufficient to attain its chief purpose. It should be stated also that all the analysis is subject to the usual limitation that the slope of the surface is supposed to be always small.

We take the wave resistance to be the resolved part of the pressure system in the direction of motion, or

$$\mathrm{R} = \int \mathrm{F}(\varpi) \frac{\partial \zeta}{\partial x} d\mathrm{S}, \tag{5}$$

taken over the whole surface.

The disturbance (4) may be analysed into plane waves ranged at all possible angles to Ox. Substituting

$$\pi \mathrm{J}_0 [\kappa \{ (x+ct)^2 + y^2 \}^{1/2}] = \int_0^{\pi} e^{i\kappa (x+ct)\cos\phi} \cos(\kappa y \sin\phi)\, d\phi, \tag{6}$$

we can integrate with respect to t, and obtain, after rejecting superfluous terms in μ,

$$\int_0^{\infty} e^{-1/2\mu t} e^{i\kappa ct \cos\phi} \sin(\kappa \mathrm{V} t)\, dt = -\frac{\mathrm{V}\sec^2\phi}{\kappa c^2 - g\sec^2\phi \tanh\kappa h + i\mu c \sec\phi}. \tag{7}$$

Using this in (4), the surface elevation can be expressed in the form

$$2\pi\rho\zeta = \int_{-\pi/2}^{\pi/2} \sec^2\phi\, d\phi \int_0^{\infty} \kappa f(\kappa) \tanh\kappa h \left\{ \frac{e^{i\kappa(x\cos\phi + y\sin\phi)}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h + i\mu c\sec\phi} + \frac{e^{-i\kappa(x\cos\phi + y\sin\phi)}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h - i\mu c\sec\phi} \right\} d\kappa \tag{8}$$

3. We simplify the calculations by specifying the surface distribution of pressure as

$$p = \mathrm{F}(\varpi) = \mathrm{A}l/(l^2+\varpi^2)^{3/2}, \tag{9}$$

where A and l are constants. It follows from (3) that $f(\kappa) = \mathrm{A}e^{-\kappa l}$. Now in (8) consider an element making an angle ϕ with the axis Ox. Change to axes Ox', Oy', given by $x' = x\cos\phi + y\sin\phi$, $y' = y\cos\phi - x\sin\phi$. Then the integral with respect to κ becomes

$$\int_0^{\infty} \kappa e^{-\kappa l} \tanh\kappa h \left\{ \frac{e^{i\kappa x'}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h + i\mu c\sec\phi} + \frac{e^{-i\kappa x'}}{\kappa c^2 - g\sec^2\phi\tanh\kappa h - i\mu c\sec\phi} \right\} d\kappa. \tag{10}$$

As in similar plane wave problems, this integral can be modified by integrating round a suitable contour in the plane of a complex variable; the expressions then divide into two types according as the integrand has or has not a pole within the contour. The surface disturbance corresponding to (10) is seen then to consist, in general, of a surface elevation symmetrical with respect to the line $x\cos\phi+y\sin\phi=0$, together with a regular train of waves in the rear of this line; but the latter part only occurs if $c^2\cos^2\phi<gh$. In evaluating the wave resistance by (5) for the symmetrical distribution (9), we see that we need only consider the regular train of waves. By calculating the residue of the integrand in (10), collecting the results and finally making μ zero, we find that the regular waves, when they occur, are given by

$$\frac{4\pi Ac^2\kappa^2 e^{-\kappa l}\sin(\kappa x')}{g\sec^2\phi\,(c^2-gh\sec^2\phi)+\kappa^2c^4h}, \tag{11}$$

where κ is the root of

$$\kappa c^2-g\sec^2\phi\tanh\kappa h=0\,; \qquad gh\sec^2\phi>c^2. \tag{12}$$

From (5) and (11), the contribution of this element to the wave resistance is

$$\frac{4\pi Ac^2\kappa^3 e^{-\kappa l}\cos\phi}{g\sec^2\phi\,(c^2-gh\sec^2\phi)+\kappa^2c^4h}\int_{-\infty}^{\infty}dy'\int_{-\infty}^{0}\frac{Al\cos(\kappa x')}{(x'^2+y'^2+l^2)^{3/2}}\,dx'$$
$$=\frac{4\pi^2A^2c^2\kappa^3 e^{-2\kappa l}\cos\phi}{g\sec^2\phi\,(c^2-gh\sec^2\phi)+\kappa^2c^4h}. \tag{13}$$

Summing for the different elements, from (13) and (7), we have finally for the wave resistance

$$R=\frac{4\pi A^2c^2}{\rho}\int_{\phi_0}^{\pi/2}\frac{\kappa^3e^{-2\kappa l}\sec\phi\,d\phi}{g\sec^2\phi\,(c^2-gh\sec^2\phi)+\kappa^2c^4h}, \tag{14}$$

where κ satisfies $\kappa c^2=g\sec^2\phi\tanh\kappa h$, and the lower limit ϕ_0 is given by

$$\phi_0=0,\ \text{for } c^2<gh\,; \qquad \phi_0=\text{arc}\cos(gh/c^2)^{1/2},\ \text{for } c^2>gh. \tag{15}$$

4. We may notice, in the first place, that (14) reduces to the expression given previously for deep water; making $h\to\infty$, we find

$$R=(4\pi g^2A^2/\rho c^6)\int_0^{\pi/2}\sec^5\phi\,e^{-2(gl/c^2)\sec^2\phi}\,d\phi$$
$$=\frac{\pi^2A^2x^3e^{-x}}{2g\rho l^3}\left\{iH_0^{(1)}(ix)-\frac{1+2x}{2x}H_1^{(1)}(ix)\right\}, \tag{16}$$

where $x=gl/c^2$, and the result is expressed in terms of Bessel functions of which Tables are available. For finite values of the ratio h/l, the value of R for given values of c can only be obtained from (14) by numerical and graphical methods. After some preliminary trial, the following plan was

adopted. With $p = h/l$, and $\alpha = \kappa h$, using the relation between κ and ϕ, (14) can be put in the form

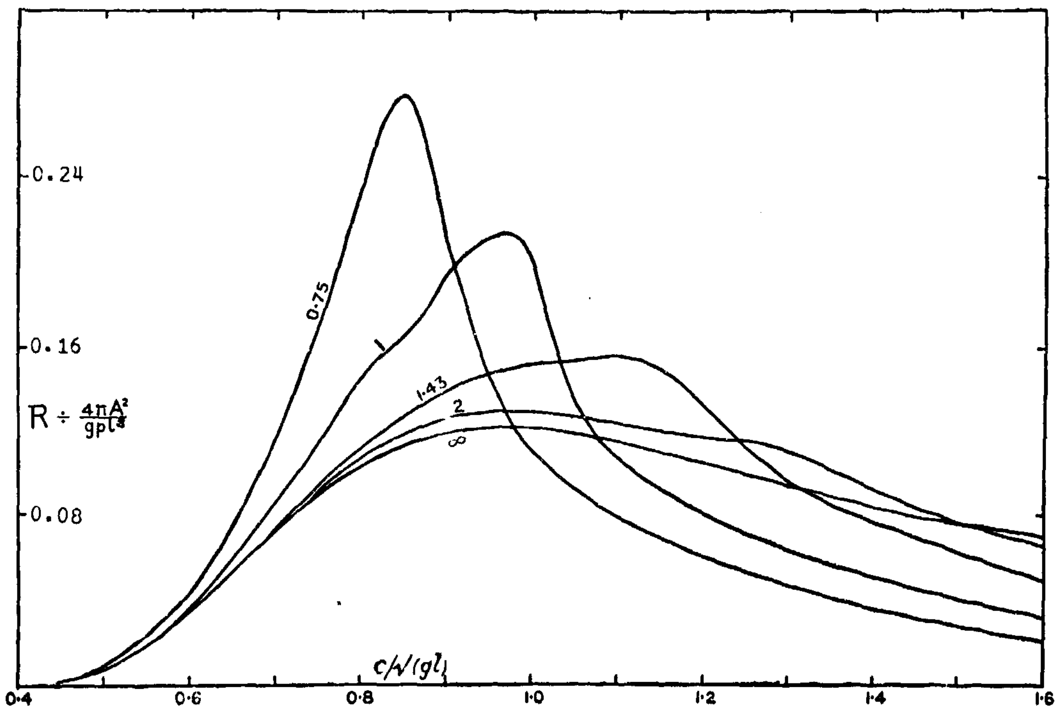
$$\mathrm{R} = \frac{4\pi \mathrm{A}^2 x^{1/2}}{g\rho l^3 p^{5/2}} \int_{\phi_0}^{\pi/2} \frac{\alpha^{7/2} e^{-2\alpha/p} \coth^{1/2} \alpha}{\alpha^2 + \alpha \coth \alpha - \alpha^2 \coth^2 \alpha} \, d\phi, \tag{17}$$

with

$$\alpha \coth \alpha = px \sec^2 \phi. \tag{18}$$

For a given value of p, the integrand of (17), which we may denote by $f(\alpha)$, was calculated for values of α ranging from zero to 3 at intervals of 0·2, and in certain cases also at unit intervals up to the value 10. Taking next an assigned value of x, the value of ϕ corresponding to each value of α was found from (18). The integrand $f(\alpha)$ was then graphed on a base of ϕ, giving a curve for each value of x; the area of the curve was taken by an Amsler radial planimeter, and then the value of (17) was obtained. The calculations are rather lengthy and it is unnecessary to repeat them here.

The process was carried out for $p = 2$, 1·43, 1, 0·75, with about a dozen values of x in each case; some estimates were also made for $p = 0{\cdot}5$, to confirm the general deductions. Further, the values for $p = \infty$ were calculated from (16). The results are shown in the figure, where the unit for R is $4\pi\mathrm{A}^2/g\rho l^3$, and for c is $\sqrt{(gl)}$.



5. The curve for deep water, $p = \infty$, has a single maximum at a velocity slightly less than $\sqrt{(gl)}$. At this velocity the corresponding length of

simple transverse waves is about $2\pi l$; this may be called the principal wave-making length of the disturbance, to use a term from the theory of ship resistance. Taking next the curve for $p = 2$, we can see indications of two maxima. The first occurs at about the point 0·97 on the velocity scale; it clearly corresponds to the deep water maximum, and comes lower down the scale, because waves of given length occur at a lower velocity as the depth diminishes. There is also a second maximum at a velocity of about 1·25; this is due to the other factor in the resistance, namely, the increased effect as the velocity approaches the velocity $\sqrt{(gh)}$ of the so-called wave of translation, which in this case is at the point 1·41 on the velocity scale.

From the next curves, $p = 1{\cdot}43$ and $p = 1$, we see the increasing importance of the latter effect as the depth becomes less. For the curve $p = 1{\cdot}43$, there is a maximum near the velocity 1·1, the corresponding value of $\sqrt{(gh)}$ on the scale being 1·2. There is no other actual maximum, but there is an enhanced resistance at about 0·92, followed by a flattening of the curve between that point and the point 1·05; we may take the increased effect at 0·92 to correspond to the deep water maximum in the lower curves. Similarly for the curve $p = 1$, the corresponding values are: increased effect at about 0·81, diminished slope of curve between 0·82 and 0·9, maximum at 0·97, velocity of wave of translation 1·0. The last curve, $p = 0{\cdot}75$, shows that, as the depth becomes small, the second effect becomes the predominant feature; the excess resistance increases rapidly in magnitude, and occurs practically at the velocity $\sqrt{(gh)}$. This effect is still more pronounced for $p = 0{\cdot}5$, but the results are not shown in the figure. It is obvious that, as the ratio of h/l diminishes, the disturbance becomes more like that due to a line disturbance; in simple calculations on the latter assumption, the resistance increases indefinitely at the velocity $\sqrt{(gh)}$, and falls suddenly to zero above that velocity. It will be seen from the figure that in all cases the resistance falls after the velocity $\sqrt{(gh)}$, as, in fact, may be deduced directly from the expression (17).

In a comparison between these results and the experimental curves of ship resistance described in § 1, it is advisable to consider in each case the difference between the resistance in water of a given depth and that in deep water; in this sense the results agree in character. Thus the first effect of finite depth may be regarded as due to the lowering of the chief wave-making velocity; it is only when the depth of water becomes of the same order as the beam of the ship that the critical velocity is practically that of the wave of translation.

In describing the experimental curves, it was stated that the excess resistance has a maximum value at a certain critical velocity. But there is

one exceptional set of curves, obtained at the United States Model Basin,* which shows two maxima: a phenomenon which has not received explanation. It is conceivable that this may be a case in which the two maxima indicated in the intermediate curves of the present paper have become prominent through some unusual features of the model. In this connection, it must be remembered that the present calculations are based upon a surface pressure of specially simple type, one symmetrical round a point; one could extend the calculations by integration, as in the previous results for deep water, so as to apply to a pressure distribution, giving a better analogy with ship form. It may be anticipated that the results would be of the same character in general, though no doubt better agreement could be obtained in certain details.

* D. W. Taylor, *loc. cit.*, p. 115.

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Studies in Wave Resistance: Influence of the Form of the Water-plane Section of the Ship.

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Introduction.

1. The problem which is investigated in some detail in the following paper is the wave resistance of a vertical post in a uniform stream. The horizontal section of the post is of ship-shape form and the lines are varied in a certain manner while keeping the area of the section constant.

A direct study of ship waves as a three-dimensional problem for a ship of finite dimensions has not yet been accomplished. From one point of view the problem has been attacked by the method of an equivalent distribution of pressure on the surface of the water. Some advance has also been made in the case of submerged bodies; I have shown previously how to calculate the wave resistance of a body whose form is derived by combining the stream-lines of a uniform current with certain distributions of sources and sinks, under the limitation that the dimensions of the body are small compared with its depth. On the other hand Michell,‡ in an extremely

‡ J. H. Michell, 'Phil. Mag.,' vol. 45, p. 106 (1898).

interesting paper, gave a general expression for wave resistance; but it suffers from a serious limitation, in that the surface of the ship must be everywhere inclined at only a small angle to its vertical median plane.

In § 2 a short synopsis of Michell's theory is given.

In § 3 this is applied to the case of a submerged body and the result compared with the work to which reference has been made; the two methods are quite different and have different limitations, but it appears that the results agree when these conditions overlap and are common to both.

The main problem is treated by an application of Michell's analysis in circumstances in which its limitations are not of serious importance, namely, when the body is a vertical post of infinite depth and of small beam compared with its length. We may regard this as a ship in which the effect of the vertical sides will be exaggerated, and we may study the changes produced in the resistance curves by varying the form of the level lines. The practical problems which have been kept in view in devising special cases are such as the effect of straight or hollow lines at the bow, the effect of finer entrance and increased beam while displacement remains constant, and similar questions.

In § 4 a set of parabolic curves for the level lines is specified so as to illustrate these points, and the corresponding value of the wave resistance obtained in general form as a function of the velocity. Certain new types of integral which occur in the analysis are examined in § 5; they can be expressed in terms of the second Bessel functions Y_0 and Y_1 together with the integral of Y_0, and are evaluated numerically by means of recent tables of Struve's functions.

In §§ 6–10, four types of model are examined, and the wave resistance calculated for various velocities in each case. The chief results are shown in the resistance curves of fig. 2 For comparison with experimental curves from ship models, the base is the quantity $V/\sqrt{L}$, where V is the speed in knots and L the length in feet. The models with finer entrance, or with hollow lines, have smaller resistance up to $V/\sqrt{L} = 1{\cdot}1$ or $1{\cdot}2$; but above this speed the models with fuller ends have the less resistance. These, and other results of some interest agree with deductions from the corresponding practical study of ship resistance; in § 11 a summary of these deductions is given and a comparison is made with the results of the present calculations.

General Analysis.

2. Take Ox, Oy in the undisturbed surface of the water and Oz vertically downwards; and suppose the ship to be symmetrical with respect to the plane $y = 0$. Assuming the ship to be at rest, and the water at a great

distance to have a uniform velocity c in the negative direction of Ox, the velocity potential is taken as $cx+\phi$; the squares of the velocity due to the disturbance ϕ are to be neglected. At the surface $z=0$, the kinematical condition is

$$\frac{\partial\phi}{\partial z}=c\frac{\partial\zeta}{\partial x}, \tag{1}$$

where ζ is the surface depression.
The condition for constant pressure at $z=0$ gives $c\partial\phi/\partial x-g\zeta=0$, or

$$\frac{c^2}{g}\frac{\partial^2\phi}{\partial x^2}=\frac{\partial\phi}{\partial z}. \tag{2}$$

At the bottom of the water $\partial\phi/\partial z=0$; in what follows we shall assume the water to be of infinite depth. The remaining boundary condition is that $\partial\phi/\partial y=0$ when $y=0$, except over the surface of the ship; in the latter case, with ν as the normal,

$$\frac{\partial}{\partial\nu}(cx+\phi)=0.$$

If the inclination of the ship's surface to the plane $y=0$ is everywhere small, the latter condition reduces to

$$\frac{\partial\phi}{\partial y}=c\frac{\partial\eta}{\partial x}=cf'(x,z), \tag{3}$$

where $\eta=f(x,z)$ is the equation to the ship's surface; to the same order the condition (3) may be taken to hold at $y=0$ over the median plane of the ship.

A potential function to satisfy these conditions may be built up by a summation of simple harmonic terms in the co-ordinates; it is sufficient here to state Michell's expression, namely,

$$\begin{aligned}\phi = &-\frac{2c}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f'(\xi,\zeta)\frac{\cos(nz-\epsilon)\cos(n\zeta-\epsilon)}{(m^2+n^2)^{1/2}}\\ &\qquad\qquad\cos\{m(\xi-x)\}e^{-y(m^2+n^2)^{1/2}}\,d\xi\,d\zeta\,dm\,dn\\ &+\frac{2c^3}{\pi g}\int_{g/c^2}^\infty\int_0^\infty\int_{-\infty}^\infty f'(\xi,\zeta)\frac{me^{-m^2c^2(z+\zeta)/g}}{(m^2c^4/g^2-1)^{1/2}}\\ &\qquad\qquad\sin\{m(x-\xi)+my(m^2c^4/g^2-1)^{1/2}\}\,d\xi\,d\zeta\,dm\\ &-\frac{2c^3}{\pi g}\int_0^{g/c^2}\int_0^\infty\int_{-\infty}^\infty f'(\xi,\zeta)\frac{me^{-m^2c^2(z+\zeta)/g}}{(1-m^2c^4/g^2)^{1/2}}\\ &\qquad\qquad\cos\{m(\xi-x)\}e^{-my(1-m^2c^4/g^2)^{1/2}}\,d\xi\,d\zeta\,dm,\end{aligned} \tag{4}$$

where $\tan\epsilon=-c^2m^2/gn$.

It may be verified directly that each term in (4) is a potential function and

satisfies all the above conditions except (3); and further that (3) is satisfied by the complete expression on account of the expansion

$$f'(x,z) = \frac{2}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f'(\xi,\zeta)$$

$$\cos(nz-\epsilon)\cos(n\zeta-\epsilon)\cos m(\xi-x)\,d\xi\,d\zeta\,dm\,dn$$

$$+\frac{2c^2}{\pi g}\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f'(\xi,\zeta)\,m^2 e^{-c^2m^2(z+\zeta)/g}\cos m(\xi-x)\,d\xi\,d\zeta\,dm. \qquad (5)$$

An expansion which may be verified without difficulty, ϵ having the value given in (4); it is assumed that the function $f'(x,z)$ is such that the various integrals are convergent.

The expression (4) holds for y positive. The first and third integrals represent local symmetrical disturbances, while the second integral represents the waves which follow the ship if we imagine it to be advancing into still water.

If δp is the increase of fluid pressure due to the disturbance ϕ, the wave resistance is given by

$$R = -2\iint \delta p\,.\,\frac{\partial\eta}{\partial x}\,dx\,dz = 2\rho c\iint \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}\,dx\,dz, \qquad (6)$$

the integration extending over the vertical median plane of the ship.

The first and third terms in (4) contribute nothing to R, and we have

$$R = \frac{4\rho c^4}{\pi g}\int_{g/c^2}^\infty\int_0^\infty\int_{-\infty}^\infty\int_0^\infty\int_{-\infty}^\infty f'(x,z)\,f'(\xi,\zeta)\,\frac{m^2e^{-m^2c^2(z+\zeta)/g}}{(m^2c^4/g^2-1)^{1/2}}$$

$$\cos m(x-\xi)\,dx\,dz\,d\xi\,d\zeta\,dm$$

$$= \frac{4\rho c^4}{\pi g}\int_{g/c^2}^\infty (I^2+J^2)\,\frac{m^2dm}{(m^2c^4/g^2-1)^{1/2}},$$

where

$$I = \int_0^\infty\int_{-\infty}^\infty f'(x,z)\,e^{-m^2c^2z/g}\cos mx\,dx\,dz$$

$$J = \int_0^\infty\int_{-\infty}^\infty f'(x,z)\,e^{-m^2c^2z/g}\sin mx\,dx\,dz. \qquad (7)$$

This is Michell's expression for the wave resistance. We shall take the origin at the midship section and assume the ship to be symmetrical fore and aft; in these circumstances, $I = 0$.

Submerged Spheroid.

3. The application of (7) is limited by the assumption involved in (3), that the inclination of the surface of the ship to the median plane $y = 0$ is always small. To illustrate this limitation we may consider a particular case in

which we can compare the result by another method. In a previous paper* I have shown how to find the wave resistance of submerged bodies of various forms. Apart from the usual simplification of neglecting the square of the fluid velocity in the wave disturbance, the specific limitation in that analysis was that the dimensions of the submerged body should be small compared with the depth at which it moves; but on the other hand, the kinematical condition at the surface of the body was taken in its exact form. In particular, if the body is a prolate spheroid of semi-axis a, eccentricity ϵ, moving with velocity c in the direction of its axis and at a depth f, the wave resistance was found to be

$$R = 128\pi^2 g\rho a^3\epsilon^3 A^2 \int_0^{\pi/2} \sec^2\phi\, e^{-2\kappa_0 f\sec^2\phi} \{J_{3/2}(\kappa_0 a\epsilon\sec\phi)\}^2 d\phi, \tag{8}$$

where $\kappa_0 = g/c^2$ and $A = [4\epsilon/(1-\epsilon^2) - 2\log\{(1+\epsilon)/(1-\epsilon)\}]^{-1}$.

The limitation in (8) is that a is small compared with f, but there is no direct limitation on the form, for example the expression includes the case of the sphere with $\epsilon = 0$. Now, if we apply Michell's formula (7) to this case, we shall obtain a result in which there is no limitation of the ratio of a to f; but on the other hand the inclination of the surface must be small, so the expression will only hold in the limit as ϵ approaches unity.

The equation of the spheroid being

$$\frac{x^2}{a^2} + \frac{\eta^2}{b^2} + \frac{(z-f)^2}{b^2} = 1,$$

we have

$$\partial\eta/\partial x = f'(x, z) = -b^2 x/a\,\{b^2(a^2-x^2) - a^2(z-f)^2\}^{1/2}. \tag{9}$$

Thus from (7)

$$J = -\frac{b^2}{a} e^{-m^2c^2f/g} \iint \frac{x e^{m^2c^2\zeta/g}}{\{b^2(a^2-x^2) - a^2\zeta^2\}^{1/2}} \sin mx\, dx\, d\zeta, \tag{10}$$

where we have put $\zeta = z - f$, and the integration extends over the ellipse $x^2/a^2 + \zeta^2/b^2 = 1$.

Integrating with respect to x first, we have

$$\int_{-p}^{p} \frac{x\sin mx\, dx}{\{a^2(b^2-\zeta^2) - b^2x^2\}^{1/2}} = \frac{\pi a}{b^2}(b^2-\zeta^2)^{1/2} J_1\left(\frac{ma}{b}\sqrt{b^2-\zeta^2}\right),$$

where p has been used for $a(1-\zeta^2/b^2)^{1/2}$.

Hence we obtain

$$\begin{aligned} J &= -\pi b^2 e^{-m^2c^2f/g} \int_0^{\pi} e^{(m^2c^2b/g)\cos\theta} J_1(ma\sin\theta)\sin^2\theta\, d\theta \\ &= -2\pi b^2 e^{-m^2c^2f/g} \sum_0^{\infty} \frac{2^{n-1/2}}{(ma)^{n+1/2}} \frac{\Gamma(n+\frac{1}{2})}{(2n)!} \left(\frac{m^2c^2b}{g}\right)^{2n} J_{n+3/2}(ma). \end{aligned} \tag{11}$$

* 'Roy. Soc. Proc.,' A, vol. 95, p. 354 (1919).

If we use only the first term in this expansion in powers of b, we have

$$J = -(2\pi^3/ma)^{1/2}\, b^2 e^{-m^2c^2f/g}\, J_{3/2}(ma). \tag{12}$$

With this value (7) gives

$$R = \frac{8\pi^2\rho c^4 b^4}{ga}\int_{g/c^2}^{\infty}\{J_{3/2}(ma)\}^2 e^{-2m^2c^2f/g}\frac{m\,dm}{(m^2c^4/g^2-1)^{1/2}}$$

$$= 8\pi^2 g\rho a^3(1-\epsilon^2)^2\int_0^{\pi/2}\sec^2\phi\, e^{-2\kappa_0 f\sec^2\phi}\{J_{3/2}(\kappa_0 a\sec\phi)\}^2\, d\phi. \tag{13}$$

With ϵ nearly equal to unity, it is easily verified that (13) agrees with (8). On the one hand, the result (8) includes the sphere ($\epsilon = 0$), under the restriction that f is large; on the other hand (7) and (11) give a formal solution for any depth f, but only serve for ϵ nearly unity. The two methods are very different, and it is of interest that the results agree under conditions in which the two approximations overlap.

Formulæ for General Type of Model.

4. The limitations of Michell's formula do not admit of its application to actual ship forms; for although the sides of a ship may be at small angles to the median vertical plane, the bottom of the ship does not fulfil this condition. It is proposed to use the method here in such conditions that this objection does not hold, by supposing the ship to be of infinite draught. In other words, we consider the wave resistance of a post extending vertically downwards through the water from the surface, its section by a horizontal plane being the same at all depths and having its breadth small compared with its length. This enables us to elucidate certain points of interest in ship resistance.

We suppose the ship to be symmetrical fore and aft, and we take the origin at the mid-ship section. Then since in (7), $f'(x, z)$ is independent of z, we have

$$R = \frac{4g\rho}{\pi}\int_{g/c^2}^{\infty}\frac{J^2\, dm}{m^2(m^2c^4/g^2-1)^{1/2}}, \tag{14}$$

where

$$J = \int f'(x)\sin mx\, dx, \tag{15}$$

the integration covering the length of the ship, and the equation to the half-section being $y = f(x)$.

We wish to study the effect of altering the form of the section while keeping the length and the total displacement unaltered, the beam varying slightly according to the curvature of the lines. These conditions can be satisfied by taking the form of the water-plane section, for y positive and x ranging between $\pm l$, to be

$$y = \frac{b}{1-\frac{1}{5}l^2/d^2}\left(1-\frac{x^2}{l^2}\right)\left(1-\frac{l^2+x^2}{6d^2}\right). \tag{16}$$

Here $2l$ is the constant length, and $\frac{8}{3}bl$ the constant area of horizontal section of the ship; the beam is

$$2b\,(1-l^2/6d^2)/(1-l^2/5d^2). \tag{17}$$

The points of inflection in the curve are at $x = \pm d$. For $d = \infty$, we have the ordinary parabolic form with beam $2b$. With $d = l$ the bow and stern lines are still straight, but the ship has a finer entrance and a slightly larger beam. With $d < l$, the lines at bow and stern are hollow, that is, the sides are concave outwards. We shall study in detail later four values of l/d, namely, 0, 1, 1·25 and 1·5.

From (15) and (16) we have

$$J = -\frac{4b}{l^2(1-l^2/5d^2)}\int_0^l (x-\tfrac{1}{3}x^3/d^2)\sin mx\,dx.$$

Evaluating and putting in (14) we obtain

$$R = 64\pi^{-1}g\rho b^2 l^{-4}(1-l^2/5d^2)^{-2} \times$$

$$\int_{g/c^2}^{\infty}\left\{\left(\frac{l}{m}-\frac{l^3}{3d^2m}+\frac{2l}{d^2m^3}\right)\cos ml-\left(\frac{1}{m^2}-\frac{l^2}{d^2m^2}+\frac{2}{d^2m^4}\right)\sin ml\right\}^2 \frac{dm}{m^2\,(m^2c^4/g^2-1)^{1/2}}. \tag{18}$$

We shall use the notation

$$\delta = l/d; \qquad L = 2l; \qquad p = gL/c^2.$$

Altering the variable in (18) and expanding the terms, it can be expressed in the form

$$R = \frac{512g\rho b^2 l}{\pi(1-\frac{1}{5}\delta^2)^2p^3}\int_0^{\pi/2}\Big[\tfrac{1}{2}\cos^3\phi\,\Big\{(1-\tfrac{1}{3}\delta^2)^2+\frac{4}{p^2}(1+2\delta^2-\tfrac{1}{3}\delta^4)\cos^2\phi$$

$$+\frac{64}{p^4}\delta^2\cos^4\phi+\frac{256}{p^6}\delta^4\cos^6\phi\Big\}+\tfrac{1}{2}\cos^3\phi\Big\{(1-\tfrac{1}{3}\delta^2)^2$$

$$-\frac{4}{p^2}(1-6\delta^2+\tfrac{7}{3}\delta^4)\cos^2\phi-\frac{64}{p^4}\delta^2(1-2\delta^2)\cos^4\phi$$

$$-\frac{256}{p^6}\delta^4\cos^6\phi\Big\}\cos(p\sec\phi)-\cos^2\phi\,\Big\{\frac{2}{p}(1-\tfrac{4}{3}\delta^2+\tfrac{1}{3}\delta^4)\cos\phi$$

$$+\frac{32}{p^3}\delta^2(1-\tfrac{2}{3}\delta^2)\cos^3\phi+\frac{128}{p^5}\delta^4\cos^5\phi\Big\}\sin(p\sec\phi)\Big]\,d\phi. \qquad (19,$$

5. The integrals in (19) which do not seem to have been studied explicitly are of the following forms

$$P_{2n}(p) = (-1)^n\int_0^{\pi/2}\cos^{2n}\phi\,\sin(p\sec\phi)\,d\phi, \tag{20}$$

$$P_{2n+1}(p) = (-1)^{n+1}\int_0^{\pi/2}\cos^{2n+1}\phi\,\cos(p\sec\phi)\,d\phi, \tag{21}$$

with n a positive integer. The cases which occur in (19) could be tabulated directly by means of convergent series and asymptotic series. They can, however, be derived by repeated integration of the second Bessel function, and can be expressed in terms of functions of which Tables are now available.

The functions satisfy the relations

$$P_n = P'_{n+1},$$
$$nP_n = p(P_{n-1}+P_{n-3})-(n-1)P_{n-2}. \tag{22}$$

Further, in reducing a function of positive order n by this relation, (22) holds as far as

$$2P_2 = p(P_1+P_{-1})-P_0,$$
$$P_1 = p(P_0+P_{-2}). \tag{23}$$

Now we have

$$P_0 = \int_0^{\pi/2} \sin(p\sec\phi)\,d\phi = -\frac{\pi}{2}\int_0^p Y_0(p)\,dp, \tag{24}$$

where Y_0 is the second Bessel function defined by

$$Y_0 = -\frac{2}{\pi}\int_0^{\pi/2} \sec\phi\cos(p\sec\phi)\,d\phi. \tag{25}$$

We shall use the notation

$$Y_0^{-1} = \int_0^p Y_0(p)\,dp. \tag{26}$$

Since we have

$$P_{-1} = -\frac{\pi}{2}Y_0; \qquad P_{-2} = -\frac{\pi}{2}Y_0' = \frac{\pi}{2}Y_1, \tag{27}$$

it follows that by using (22) and (23), we may express the unknown integrals in (19) in terms of Y_0^{-1}, Y_0 and Y_1.

Some numerical values of Y_0^{-1} have been published recently by G. N. Watson; these are not sufficient for our purpose, but Watson also gives Tables of Struve's functions H_0 and H_1 ranging from 0 to 16. In terms of these functions

$$Y_0^{-1} = pY_0-\frac{\pi}{2}p(Y_0H_1-Y_1H_0). \tag{28}^*$$

Watson's Tables of Struve's functions and of Y_0 and Y_1 have been used in the calculations that follow.

6. Returning to (19) we evaluate the simple integrals and reduce the others in the manner indicated in the previous section; omitting the algebraic reductions, the final result is

* G. N. Watson, 'Treatise on Bessel Functions,' p. 752 (1923).

$$R = \frac{512 g\rho b^2 l}{\pi\left(1-\frac{1}{5}\delta^2\right)^2 p^3}\left[\frac{1}{3}\left(1-\frac{1}{3}\delta^2\right)^2 + \frac{16}{15}\left(1+2\delta^2-\frac{1}{3}\delta^4\right)\frac{1}{p^2} + \frac{512}{35}\frac{\delta^2}{p^4} + \frac{16384}{315}\frac{\delta^4}{p^6}\right.$$

$$-\frac{\pi}{2}\left\{\left(\frac{1}{60}-\frac{2}{315}\delta^2+\frac{1}{1620}\delta^4\right)p^3+\frac{1}{6}\left(\frac{1}{2}-\frac{1}{5}\delta^2+\frac{1}{42}\delta^4\right)p\right.$$

$$\left.+\frac{1}{2}\delta^2\left(1-\frac{1}{5}\delta^2\right)\frac{1}{p}\right\}Y_0^{-1}$$

$$-\frac{\pi}{2}\left\{\left(\frac{1}{60}-\frac{2}{315}\delta^2+\frac{1}{1620}\delta^4\right)p^2+\frac{8}{15}-\frac{73}{210}\delta^2+\frac{109}{1890}\delta^4\right.$$

$$\left.+\frac{256}{35}\delta^2\left(1-\frac{4}{9}\delta^2\right)\frac{1}{p^2}+\frac{8192}{315}\frac{\delta^4}{p^4}\right\}Y_0$$

$$+\frac{\pi}{2}\left\{\left(\frac{1}{60}-\frac{2}{315}\delta^2+\frac{1}{1620}\delta^4\right)p^3+\frac{1}{15}\left(1-\frac{17}{42}\delta^2+\frac{19}{378}\delta^4\right)p\right.$$

$$\left.\left.+\left(\frac{16}{15}-\frac{32}{21}\delta^2+\frac{16}{35}\delta^4\right)\frac{1}{p}+\frac{512}{35}\delta^2\left(1-\frac{8}{9}\delta^2\right)\frac{1}{p^3}+\frac{16384}{315}\frac{\delta^4}{p^5}\right\}Y_1\right]. \quad (29)$$

This, with $p = gL/c^2$, gives the wave resistance as a function of the velocity.

For large values of p it is simpler to calculate directly from an asymptotic expansion. This may be obtained directly from the integral expression for R, or by substituting in (29) the asymptotic expansions of Y_0, Y_1 and Y_0^{-1}. The latter method gives a check for the coefficients in (29), since the positive powers of p must disappear from the expansion; in this way the first few terms of the expansion are found to be

$$R \sim \frac{512 g\rho b^2 l}{\pi\left(1-\frac{1}{5}\delta^2\right)^2 p^3}\left[\frac{1}{3}\left(1-\frac{1}{3}\delta^2\right)^2+\frac{16}{15}\left(1+2\delta^2-\frac{1}{3}\delta^4,\ \frac{1}{p^2}\right.\right.$$

$$-\left(\frac{\pi}{2p}\right)^{1/2}\left\{\left(\frac{1}{2}-\frac{1}{3}\delta^2+\frac{1}{18}\delta^4\right)\sin\left(p-\frac{\pi}{4}\right)\right.$$

$$\left.\left.+\left(\frac{15}{16}-\frac{47}{24}\delta^2+\frac{79}{144}\delta^4\right)\frac{1}{p}\cos\left(p-\frac{\pi}{4}\right)\right\}\right]. \quad (30)$$

We shall consider now four cases numerically.

Calculations for Four Models.

7. In model A we take $\delta = 0$; so that the level lines of the ship are the parabolic curves

$$y = b\left(1-x^2/l^2\right).$$

The expression for the wave resistance reduces to

$$R = \frac{512 g\rho b^2 l}{\pi p^3}\left\{\frac{1}{3}+\frac{16}{15}\frac{1}{p^2}-\frac{\pi}{2}\left(\frac{1}{60}p^3+\frac{1}{12}p\right)Y_0^{-1}\right.$$

$$\left.-\frac{\pi}{2}\left(\frac{1}{60}p^2+\frac{8}{15}\right)Y_0+\frac{\pi}{2}\left(\frac{1}{60}p^3+\frac{1}{15}p+\frac{16}{15}\frac{1}{p}\right)Y_1\right\}, \quad (31)$$

with the asymptotic expansion

$$R \sim \frac{512 g\rho b^2 l}{\pi p^3}\left[\frac{1}{3}+\frac{16}{15}\frac{1}{p^2}-\left(\frac{\pi}{2p}\right)^{1/2}\left\{\frac{1}{2}\sin\left(p-\frac{\pi}{4}\right)+\frac{15}{16}\frac{1}{p}\cos\left(p-\frac{\pi}{4}\right)\right\}\right]. \quad (32)$$

For small values of p, (31) is not satisfactory as the quantity within large brackets is the small difference between large numbers; it is better then to use an ascending series in p which can be found by substituting expansions of Y_0^{-1}, Y_0 and Y_1. The first few terms are

$$R = \frac{512}{\pi} g\rho b^2 l\left\{\left(\frac{1}{144}p-\frac{1}{5760}p^3+\ldots\right)\log\left(\frac{2}{p}-\gamma\right) + \frac{7}{576}p-\frac{73}{230400}p^3+\ldots\right\}, \quad (33)$$

where γ is Euler's constant, 0·57722.

From these expressions the values in Table I have been calculated.

Table I.—Resistance of Model A.

p.	$c/\sqrt{(gL)}$.	$10^3 R/g\rho b^2 l$.	p.	$c/\sqrt{(gL)}$.	$10^3 R/g\rho b^2 l$.
19·64	0·226	6·9	7·8	0·358	56
18·07	0·235	13·4	7·086	0·376	131
16·5	0·246	12·9	6	0·408	406
14·92	0·259	8·7	5	0·447	897
14	0·267	13	3·96	0·503	1502
13·361	0·273	21·4	2	0·707	2392
11·78	0·292	52	1	1·0	2050
10·22	0·313	58·2	0·89	1·058	1930
9·24	0·329	41·7	0·5	1·414	1434
8·64	0·340	35·8	0·25	2·0	904

A certain portion of the range will be studied in detail later; the Table gives a general view of the variation of the resistance with the velocity. At low velocities the resistance is small and oscillates in value; then at a speed of about $0·4\sqrt{(gL)}$ it begins to rise rapidly and reaches a maximum at about $\sqrt{(\frac{1}{2}gL)}$, after which the resistance decreases continually and converges to zero for infinitely large velocities. Doubtless the conditions under which the expressions were obtained would be violated at very high velocities, but it is of interest to trace the variation in value over the whole range. Absolute values could be obtained from Table I for a plank of given dimensions and of the specified form; these would be comparable with experimental results for a plank of finite depth if the velocity were such that the effect of the surface waves could be neglected at the depth of the lower edge of the plank.

8. For Model B we take $\delta = 1$. The formulæ are now

$$R = \frac{800 g\rho b^2 l}{\pi p^3}\left[\frac{4}{27}+\frac{128}{45}\frac{1}{p^2}+\frac{512}{35}\frac{1}{p^4}+\frac{16384}{315}\frac{1}{p^6}-\frac{\pi}{2}\left(\frac{31}{2835}p^3\right.\right.$$
$$\left.+\frac{17}{315}p+\frac{2}{5}\frac{1}{p}\right)Y_0^{-1}-\frac{\pi}{2}\left(\frac{31}{2835}p^2+\frac{46}{189}+\frac{256}{63}\frac{1}{p^2}+\frac{8192}{315}\frac{1}{p^4}\right)Y_0$$
$$\left.+\frac{\pi}{2}\left(\frac{31}{2835}p^3+\frac{122}{2835}p+\frac{512}{315}\frac{1}{p^3}+\frac{16384}{315}\frac{1}{p^5}\right)Y_1\right]. \quad (34)$$

The corresponding asymptotic expansion is

$$R \sim \frac{800 g\rho b^2 l}{\pi p^3}\left[\frac{4}{27}+\frac{128}{45}\frac{1}{p^2}-\left(\frac{\pi}{2p}\right)^{1/2}\left\{\frac{2}{9}\sin\left(p-\frac{\pi}{4}\right)-\frac{17}{36}\frac{1}{p}\cos\left(p-\frac{\pi}{4}\right)\right\}\right]. \quad (35)$$

The general course of R is similar to that for Model A except that the values are less for velocities below about $0{\cdot}36\sqrt{(gL)}$ and higher for velocities above that value; for example, at $p = 6$ the resistance of B is 482 units, and at $p = 1$ it is 2546 units, while the corresponding values for A are 406 and 2050 respectively.

9. The third case is Model C with $\delta = 1{\cdot}25$. We have then

$$R = \frac{131072}{121\pi}\frac{g\rho b^2 l}{p^3}\left\{\frac{529}{6912}+\frac{2543}{720}\frac{1}{p^2}+\frac{160}{7}\frac{1}{p^4}+\frac{8000}{63}\frac{1}{p^6}\right.$$
$$-\frac{\pi}{2}\left(\frac{23959}{2903040}p^3+\frac{2641}{64512}p+\frac{275}{512}\frac{1}{p}\right)Y_0^{-1}-\frac{\pi}{2}\left(\frac{23959}{2903040}p^2+\frac{63373}{483840}\right.$$
$$\left.+\frac{220}{63}\frac{1}{p^2}+\frac{4000}{63}\frac{1}{p^4}\right)Y_0+\frac{\pi}{2}\left(\frac{23959}{2903040}p^3+\frac{47443}{1451520}p-\frac{111}{560}\frac{1}{p}\right.$$
$$\left.\left.-\frac{560}{63}\frac{1}{p^3}+\frac{8000}{63}\frac{1}{p^5}\right)Y_1\right\}, \quad (36)$$

and for large values of p,

$$R \sim \frac{131072}{121\pi}\frac{g\rho b^2 l}{p^3}\left[\frac{529}{6912}+\frac{2543}{720}\frac{1}{p^2}-\left(\frac{\pi}{2p}\right)^{1/2}\left\{\frac{529}{4608}\sin\left(p-\frac{\pi}{4}\right)\right.\right.$$
$$\left.\left.-\frac{28865}{36864}\frac{1}{p}\cos\left(p-\frac{\pi}{4}\right)\right\}\right]. \quad (37)$$

The remaining example, Model D, is a more pronounced variation from the standard form A. With $\delta = 1{\cdot}5$, the forward point of inflection in the water-plane curve is at one-sixth of the length of the ship from the bow.

We have in this case

$$R = \frac{204800}{121\pi}\frac{g\rho b^2 l}{p^3}\left[\frac{1}{48}+\frac{61}{15}\frac{1}{p^2}+\frac{1152}{35}\frac{1}{p^4}+\frac{9216}{35}\frac{1}{p^6}-\frac{\pi}{2}\left(\frac{37}{6720}p^3\right.\right.$$
$$\left.+\frac{191}{6720}p+\frac{99}{160}\frac{1}{p}\right)Y_0^{-1}-\frac{\pi}{2}\left(\frac{37}{6720}p^2+\frac{29}{672}+\frac{4608}{35}\frac{1}{p^4}\right)Y_0$$
$$\left.+\frac{\pi}{2}\left(\frac{37}{6720}p^3+\frac{77}{3360}p-\frac{1}{21}\frac{1}{p}-\frac{1152}{35}\frac{1}{p^3}+\frac{9216}{35}\frac{1}{p^5}\right)Y_1\right]. \quad (38)$$

For small velocities the appropriate expansion is

$$R \sim \frac{204800}{121\pi}\frac{g\rho b^2 l}{p^3}\left[\frac{1}{48}+\frac{61}{15}\frac{1}{p^2}-\left(\frac{\pi}{2p}\right)^{1/2}\left\{\frac{1}{32}\sin\left(p-\frac{\pi}{4}\right)-\frac{177}{256}\frac{1}{p}\cos\left(p-\frac{\pi}{4}\right)\right\}\right]. \quad (39)$$

A study of the numerical coefficients in these various formulæ gives some indication of the manner in which the resistance varies with the form, and this is confirmed by actual calculations which have been made in each case as in Table I for Model A. The general variation of the resistance is the same, but the differences noted between Models A and B become more pronounced for C and D; the resistances are less at low velocities and greater at high velocities as ve progress from A to D. The results may now be collected and examined graphically.

10. Fig. 1 shows the lines of models A and D, the curves being one-quarter of the water-plane section in each case. In the comparison we have in view with ship models the ratio of beam to length is of the order of 1 to 10. In order to make the diagram show the difference on a small scale, the ratio of beam to length in fig. 1 is 1 to 5. Further, only the extreme models A and D are shown; the lines for B and C would fall between those of A and D.

FIG. 1

The variations in form are summarised in Table II.

Table II.—Models of Constant Length and Displacement.

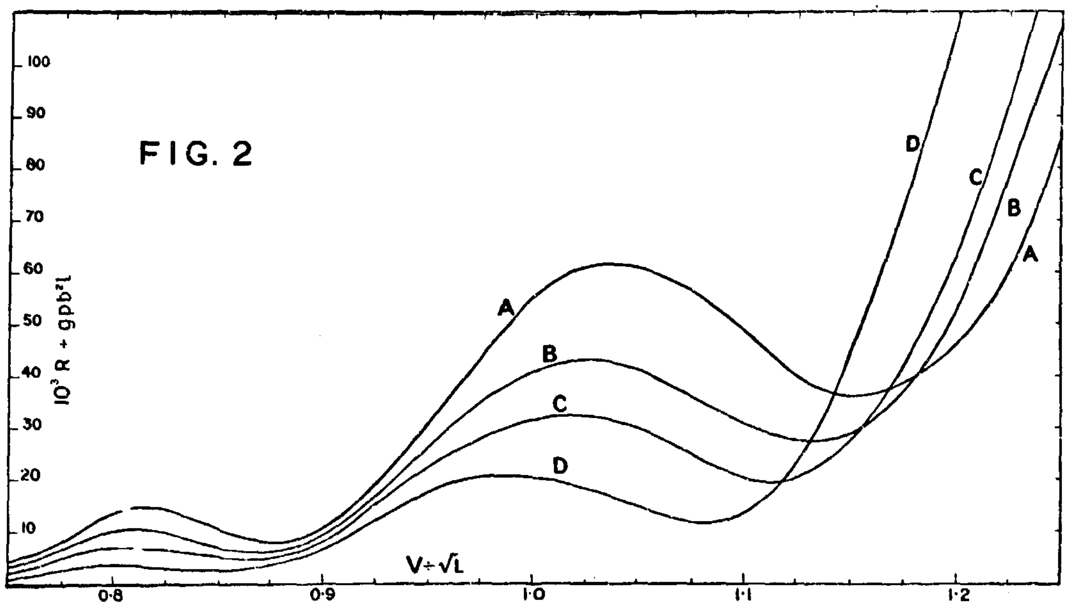
Model.	Beam.	Water-plane coefficient.	Bow and stern lines.
A	1·0	0·667	Straight.
B	1·042	0·64	Straight.
C	1·076	0·62	Hollow.
D	1·136	0·587	Hollow.

For comparison with ship resistance, it is convenient to use the same co-ordinates as are used in experimental results. In graphing the resistance

we use as a base the quantity $V/\sqrt{L}$, where V is the speed in knots and L is the length in feet; thus we have, in the previous notation,

$$p = \frac{gL}{c^2} = 11{\cdot}594 \frac{L}{V^2}, \text{ approx.} \tag{40}$$

The range for $V/\sqrt{L}$, which is of special interest, is from about 0·75 to 1·25. Fig. 2 shows the curves of wave resistance for the four models, obtained by calculating R from the expressions (31) to (39).



Comparison with Ship Resistance and General Conclusions.

11. In studying these curves in relation to experimental data from ship models, one cannot make a direct numerical comparison of absolute values. In the present calculations one has the advantage of isolating some single feature and of seeing how its variation affects the results, for instance, the form of the level lines. On the other hand, in experimental curves from models there is no simple separation. In practice, the form of the ship is expressed roughly by certain coefficients of fineness: the water-plane coefficient being the ratio of the area of the water-plane section to a rectangle enclosing the section, the mid-ship area coefficient similarly defined, the prismatic coefficient, the curve of sectional areas, and so forth. In experiments these coefficients may be varied in a systematic manner, but in their effect on the ship's form they are not in any mathematical sense independent variables; this leads to some difficulty of interpretation from a theoretical point of view.

The first point to be noticed is the prominent hump on the resistance curves in the neighbourhood of $V/\sqrt{L} = 1$. This is a well-known feature of ship resistance; it has been stated as an empirical rule that this hump occurs at $V = 1{\cdot}05\sqrt{L}$, or again that it occurs at $V = 1{\cdot}34\sqrt{(PL)}$ where P is the prismatic coefficient. In fig. 2 the values of $V/\sqrt{L}$ for the more normal models A and B are 1·04 and 1·03 respectively, while for the more extreme forms C and D with hollow lines they are about 1·02 and 0·98; the model D has obviously lines which are unusually fine at the bow.

In the figure the humps and hollows are, in general, more pronounced than in experimental curves. The familiar pattern of ship waves is usually described as made up of transverse waves and diverging waves, the former being the chief factor in the wave resistance; there is also a tendency to associate the transverse waves with the stream-lines which travel along the bottom of the ship and the diverging waves with the action of the vertical sides of the ship at the bow, but this is misleading. In the present calculations we have models in which none of the stream-lines can go underneath the ship; they are all forced sideways from the bow. It appears that the effect of the flat bottom of the ship, and of its finite draught, may be rather to smooth out the oscillations in the resistance curve. A general feature of the curves which is in agreement with experiment is that the oscillations become progressively less prominent as we take the models in the order A, B, C, D; this is especially noticeable in models C and D, which have hollow lines.

The most interesting and important characteristic of the set of curves is their intersection in pairs at values of $V/\sqrt{L}$ ranging from 1·12 to 1·18. Compare, for instance, models C and A. At low speeds C, with its finer entrance, has a decided advantage; at 1·18 the resistances are equal, while above this speed the advantage remains with model A, with blunter ends but with less beam. It has been remarked that one cannot make exact comparison with experimental results from ship models, but a general survey of the data bears out these calculations. Without going into detail, it may suffice to give a few references to standard treatises on ship resistance where the results are summarised.

G. S. Baker remarks: "In the section dealing with the relative merits of hollow *versus* straight lines, and elsewhere, it has been shown that for vessels of fine form intended to work at speeds in the neighbourhood of $V = \sqrt{L}$ there is a decided gain in working the level lines with some hollow in them. It has also been known that for such fine forms at very high speeds the hollow should be reduced to get the best effect."*

* G. S. Baker, 'Ship Form Resistance and Screw Propulsion,' p. 87, 2nd edn. 1920.

D. W. Taylor,* referring to a series of experiments with models of the same displacement and of varying, midship section coefficients, states that the models with full midship-section coefficients drive a little easier up to $V \div \sqrt{L} = 1{\cdot}1$ to $1{\cdot}2$, and the models with fine coefficients have a shade the best of it at higher speeds. Again, the same author analyses the results of another set of experiments thus: "Fig. 67 shows curves of residuary resistance for five pairs of 400-foot ships, each pair having the same displacement and derived from the same parent lines, but differing in midship section area or longitudinal coefficient. It is seen that at 21 knots No. 10 with 0·64 longitudinal coefficient has 2·3 times the residuary resistance of its mate No. 9 with 0·56 longitudinal coefficient. But at 24½ knots they have the same resistance. Again, No. 4 of 0·64 coefficient at 21 knots has nearly twice the residuary resistance of No. 3 of 0·56 coefficient. At 25½ knots they have the same residuary resistance, and at higher speeds No. 4 has the best of it, having but 0·9 of the residuary resistance of No. 3 at 35 knots.

"These results, which are thoroughly typical, are susceptible of a very simple qualitative explanation. A small longitudinal coefficient means large area of midship-section and fine ends. A large longitudinal coefficient means small area of midship-section and full ends."

It will be noticed that the experimental curves referred to in this extract intersect in the neighbourhood of the point $V/\sqrt{L} = 1{\cdot}2$. The curves of fig. 2 also intersect near this point. The lines of the models A, B and C were chosen to be of suitable form, limited by the necessity for a simple mathematical expression which led to integrals that could be evaluated. It may be claimed that the curves so obtained agree with experimental data, and, further, that they repay detailed study, in that the variations in resistance are connected definitely with a precise variation in the form of the model.

* D. W. Taylor, 'Speed and Power of Ships,' pp. 96 and 97.

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Studies in Wave Resistance: the Effect of Parallel Middle Body.

By T. H. HAVELOCK, F.R.S.

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Introduction and Summary.

1. If a ship is altered by inserting different lengths of parallel middle body between the same bow and stern, the main features of the variation in the wave resistance may be inferred from the principle of wave interference, and may be expressed in terms of a certain length, sometimes called the wave-making length of the ship. The problem proposed for examination is the alteration in this length with varying length of parallel middle body at the same speed, and, further, its variation for a given ship at different speeds. Recent discussions have attracted renewed attention to this problem. It may be said that there are two approximations based on experimental results of various kinds obtained from ship models. On the one hand the wave-making length is supposed to be approximately independent of speed for a given ship, and to increase directly with the increase of parallel middle body; on the other hand, an empirical formula which agrees with experimental results over a certain range makes the length increase with velocity, the increase being one-quarter of the increase in the wave-length of regular transverse waves.

The following contribution to the solution of this problem is mathematical, and necessarily deals with a simplified form of ship. It is true that one cannot compare absolute values of the wave resistance with those of actual ship models; but it has been shown in former studies of the dependence of wave resistance on ship form that one obtains a rather remarkable agreement, at least in the character of the results and in the positions at which changes occur. Leaving detailed discussion of the present extension till later, it may be stated that as regards the two approximate formulæ mentioned above the results are intermediate; after an initial decrease the wave-making length increases with velocity, but not so rapidly as in the quarter wave-length formula.

In §2 an expression is developed for the wave motion due to any distribution of doublets in a vertical plane in a uniform stream, and in §3 this is associated with the form of the ship's surface. Applying the formulæ to a ship of infinite draught, with parabolic curves for the entrance and run and with parallel middle body, we obtain a general expression for the wave resistance (§4). After computation of the functions involved (§5), a detailed numerical study is made for

a ship with entrance and run each of 80 feet and with parallel middle body increased from zero up to 340 feet, as in W. Froude's well-known experiments (§6). Fig. 2 shows the curves of wave resistance for five different velocities, the base being the length of parallel middle body. A short account of model results and of recent discussions is given in §7, and the present calculations are reviewed in the remaining sections. The information from the curves of fig. 2 is extended by an equation whose roots give the complete series of maxima and minima (§8). The roots are found numerically for three series. With both the length of the ship and the speed varying, we obtain the roots for the maxima for which on a simple theory the wave-making length is equal to one and a half wave-lengths ; Table III shows the actual variation of this length. Then two series of roots are found for a ship of constant length at varying velocities, one for a ship of 160 feet without parallel middle body, and the other when 240 feet of parallel middle body have been inserted ; these are given with other quantities in Tables IV and V, and the results are discussed in relation to experimental data.

Expressions for Wave Resistance.

2. A uniform stream of deep water moves with velocity c in the negative direction of Ox, the axes Ox, Oy being in the undisturbed surface, and the axis Oz vertically upwards. Suppose there is a doublet of moment M in the liquid at the point $(h, 0, -f)$ with its axis parallel to Ox. With the usual limitation of assuming the additional fluid velocity at the surface to be small compared with c, one can write down complete expressions for the velocity potential, and so deduce the wave disturbance and the corresponding wave resistance. It is convenient to begin here by quoting from a previous paper* the wave resistance, altered to the present notation, as

$$R = 16\pi g^4 \rho M^2 c^{-8} \int_0^{\frac{\pi}{2}} \sec^5 \phi e^{-(2gf/c^2)\sec^2\phi} \, d\phi. \tag{1}$$

In the same paper it was also shown how to generalize this expression, first, for any two doublets at given points in the plane $y=0$ and then for any continuous distribution in the same plane. Equation (37) of that paper gives the result for a continuous line distribution of doublets along the line $y=0$, $z=-f$; an obvious extension gives now

$$R = 16\pi g^4 \rho c^{-8} \int_0^\infty df \int_0^\infty df' \int_{-\infty}^\infty dh \int_{-\infty}^\infty dh' \int_0^{\frac{\pi}{2}} \psi(h, f)\psi(h', f')$$
$$\times \sec^5 \phi e^{-\{g(f+f')/c^2\}\sec^2\phi} \cos[\{g(h-h')/c^2\}\sec\phi] \, d\phi, \tag{2}$$

* 'Roy. Soc. Proc.,' A, vol. 95, p. 358 (1919).

for a distribution of doublets in the plane $y = 0$, the moment per unit area being $\psi(h, f)$ and the integrations extending over the whole distribution. The function ψ must be such that the integrals are convergent, as well as the corresponding expressions for the velocity potential and the surface disturbance. We now integrate (2) by parts with respect to h and h', and as we shall deal with distributions which are of finite extent in the x co-ordinate, we obtain

$$R = 16\pi g^2 \rho c^{-4} \int_0^\infty df \int_0^\infty df' \int_{-\infty}^\infty dh \int_{-\infty}^\infty dh' \int_0^{\frac{\pi}{2}} \partial\psi/\partial h \,.\, \partial\psi'/\partial h' \,.\, \sec^3\phi$$
$$\times\, e^{-\{g(f+f')/c^2\}\sec^2\phi} \cos\left[\{g(h-h')/c^2\}\sec\phi\right] \partial\phi. \qquad (3)$$

The fluid motion is symmetrical with respect to the plane $y = 0$; we may therefore confine our attention to the fluid on one side of this plane and we may interpret (3) in terms of the distribution of normal fluid velocity over the plane $y = 0$. For, from the definition of ψ, the normal fluid velocity at the point $(h, 0, f)$ is $2\pi\partial\psi/\partial h$. Substituting in (3) we should then have the wave resistance for a given distribution of normal fluid velocity over the plane $y = 0$. From the latter point of view the solution can also be obtained by methods of harmonic analysis; the expression for the wave resistance, used in a former paper,* agrees with (3) found by the method of sources and sinks.

3. In the application to ship waves the same assumptions are made as in the paper just quoted. The plane $y = 0$ is the fore and aft median plane of the ship, and the inclination of the ship's surface to this plane is supposed small. The ship is then replaced by an equivalent distribution of normal fluid velocity over its section by the plane $y = 0$, namely the component of the stream velocity c over the actual surface of the ship; thus if

$$y = F(x, z) \qquad (4)$$

is the equation of the ship's surface, we use in (3)

$$2\pi \frac{\partial\psi}{\partial h} = c \frac{\partial F}{\partial x}. \qquad (5)$$

A difficulty which may arise in the general solution should be mentioned, but need not be considered further in the present applications. A mathematical infinity may occur in some of the expressions; this may be removed by introducing a suitable factor to ensure convergency, but in any case it only occurs in those parts of the velocity potential and surface disturbance which represent the local symmetrical disturbance. The integrals for the wave disturbance, and consequently expression (3) for the wave resistance, remain finite.

* 'Roy. Soc. Proc.,' A, vol. 103, p. 574 (1923).

4. Suppose the ship to be symmetrical fore and aft, and take the origin at the midship section. To simplify the calculations we assume, as in previous studies, the ship to be of infinite draught and to be of constant horizontal section, as shown in fig. 1.

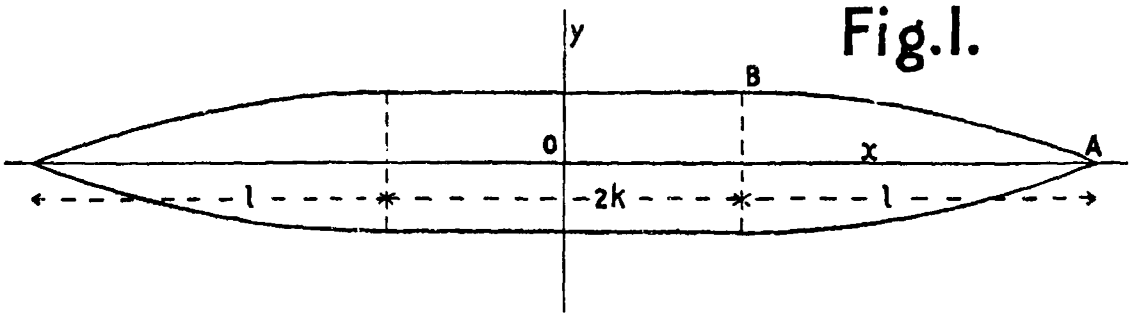


Fig. I.

The length of parallel middle body is $2k$, and l is the length of entrance or run; the curved surface is of parabolic form, the equation of AB being

$$y = b\{1-(x-k)^2/l^2\}. \tag{6}$$

Substitute from (5) and (6) in (3). Since the normal velocity is zero over the parallel middle body, and since the ship is symmetrical, the integrations in h and h' simplify considerably; also, we may carry out the integrations in f and f' and so obtain

$$\mathrm{R} = 4\rho c^2\pi^{-1}\int_0^{\frac{\pi}{2}} \mathrm{J}^2 \cos\phi\, d\phi,$$

where

$$\mathrm{J} = \frac{4b}{l^2}\int_0^l h \sin\left\{\frac{g}{c^2}(h+k)\sec\phi\right\} dh. \tag{7}$$

Evaluating J, we find after some reduction

$$\begin{aligned}
\mathrm{R} = \frac{64g\rho b^2 l}{\pi}\left(\frac{c^2}{gl}\right)^3 \int_0^{\frac{\pi}{2}} \Big[& \tfrac{1}{2}\cos^3\phi + \frac{c^4}{g^2l^2}\cos^5\phi - \frac{c^2}{gl}\cos^4\phi \sin\left(\frac{gl}{c^2}\sec\phi\right) \\
& - \frac{c^4}{g^2l^2}\cos^5\phi\cos\left(\frac{gl}{c^2}\sec\phi\right) + \tfrac{1}{2}\left(\cos^3\phi - \frac{c^4}{g^2l^2}\cos^5\phi\right)\cos\left\{\frac{2g(k+l)}{c^2}\sec\phi\right\} \\
& - \frac{c^2}{gl}\cos^4\phi\sin\left\{\frac{2g(k+l)}{c^2}\sec\phi\right\} + \frac{c^2}{gl}\cos^4\phi\sin\left\{\frac{g(2k+l)}{c^2}\sec\phi\right\} \\
& + \frac{c^4}{g^2l^2}\cos^5\phi\cos\left\{\frac{g(2k+l)}{c^2}\sec\phi\right\} - \tfrac{1}{2}\frac{c^4}{g^2l^2}\cos^5\phi\cos\left(\frac{2gk}{c^2}\sec\phi\right)\Big] d\phi.
\end{aligned} \tag{8}$$

Using the notation

$$P_{2n}(p) = (-1)^n \int_0^{\frac{\pi}{2}} \cos^{2n}\phi \sin(p \sec\phi)\, d\phi,$$

$$P_{2n+1}(p) = (-1)^{n+1} \int_0^{\frac{\pi}{2}} \cos^{2n+1}\phi \cos(p \sec\phi)\, d\phi, \tag{9}$$

we have

$$\begin{aligned} R = \frac{64g\rho b^2 l}{\pi}\left(\frac{c^2}{gl}\right)^3 \Big[& \tfrac{1}{3} + \tfrac{8}{15}\frac{c^4}{g^2l^2} - \frac{c^2}{gl} P_4\left(\frac{gl}{c^2}\right) + \frac{c^4}{g^2l^2} P_5\left(\frac{gl}{c^2}\right) \\ & + \tfrac{1}{2}P_3(p_1) - \frac{c^2}{gl} P_4(p_1) + \tfrac{1}{2}\frac{c^4}{g^2l^2} P_5(p_1) + \frac{c^2}{gl} P_4(p_2) \\ & - \frac{c^4}{g^2l^2} P_5(p_2) + \tfrac{1}{2}\frac{c^4}{g^2l^2} P_5(p_3) \Big], \end{aligned} \tag{10}$$

where

$$p_1 = g(2k+2l)/c^2, \quad p_2 = g(2k+l)/c^2, \quad p_3 = 2gk/c^2.$$

Tabulation of Functions.

5. In order to obtain the curves we require, we have to evaluate (10) for a large number of values of k, and in each case for several values of c; it was necessary to prepare tables and graphs of the P functions. Y_0 and Y_1 being Bessel functions of the second kind and Y_0^{-1} being defined by

$$Y_0^{-1} = \int_0^p Y_0(p)\, dp, \tag{11}$$

we have, from sequence relations given previously,*

$$P_3 = -\frac{\pi}{12}\{(p^3-3p)Y_0^{-1} + p^2 Y_0 - (p^3-4p)Y_1\}$$

$$P_4 = -\frac{\pi}{48}\{(p^4-6p^2+9)Y_0^{-1} + (p^3-9p)Y_0 - (p^4-7p^2)Y_1\}$$

$$P_5 = -\frac{\pi}{240}\{(p^5-10p^3+45p)Y_0^{-1} + (p^4-13p^2)Y_0 - (p^5-11p^3+64p)Y_1\}. \tag{12}$$

The values of Y_0 and Y_1 and of Struve's Functions H_0 and H_1, given in G. N. Watson's "Treatise on Bessel Functions," were used to calculate values of Y_0^{-1} from the formula

$$Y_0^{-1} = pY_0 - \frac{\pi}{2}p(Y_0H_1 - Y_1H_0). \tag{13}$$

* 'Roy. Soc. Proc.,' A, vol. 103, p. 578 (1923).

and then values of the P functions were found from (12). With increasing values of p, the multipliers in (12) become large and this method loses accuracy unless the Bessel functions are known to a large number of places. It is then preferable, and sufficiently accurate for the present purpose, to calculate from a few terms of asymptotic expansions. These can be found independently, or derived from those of the Bessel Functions; they are

$$P_3 \sim \sqrt{\frac{\pi}{2p}}\left\{-\left(1-\frac{1065}{128}\frac{1}{p^2}+\frac{269\cdot 7}{p^4}\right)\sin\left(p-\frac{\pi}{4}\right)\right.$$
$$\left.+\left(\frac{17}{8}\frac{1}{p}-\frac{42\cdot 964}{p^3}+\frac{1980}{p^5}\right)\cos\left(p-\frac{\pi}{4}\right)\right\},$$
$$P_4 \sim \sqrt{\frac{\pi}{2p}}\left\{\left(1-\frac{1569}{128}\frac{1}{p^2}+\frac{527\cdot 3}{p^4}\right)\cos\left(p-\frac{\pi}{4}\right)\right.$$
$$\left.+\left(\frac{21}{8}\frac{1}{p}-\frac{73\cdot 608}{p^3}+\frac{4353}{p^5}\right)\sin\left(p-\frac{\pi}{4}\right)\right\},$$
$$P_5 \sim \sqrt{\frac{\pi}{2p}}\left\{\left(1-\frac{2169}{128}\frac{1}{p^2}+\frac{933\cdot 2}{p^4}\right)\sin\left(p-\frac{\pi}{4}\right)\right.$$
$$\left.-\left(\frac{25}{8}\frac{1}{p}-\frac{115\cdot 97}{p^3}+\frac{8554}{p^5}\right)\cos\left(p-\frac{\pi}{4}\right)\right\} \qquad (14)$$

Although systematic computation of these functions has not been attempted to any high degree of accuracy, it was found necessary to calculate a large number of values from p zero up to p equal to 40. Some of these are recorded in Table I.

Table I.

p.	P_3.	P_4.	P_5.	p.	P_3.	P_4.	P_5.
0	+0·6666	0	−0·5333	3·6	−0·3515	−0·3606	+0·3457
0·4	+0·5880	+0·2563	−0·4795	4·0	−0·1517	−0·4624	+0·1784
0·8	+0·3876	+0·4551	−0·3361	4·4	+0·0597	−0·4828	+0·0150
1·0	+0·2569	+0·5198	−0·2381	4·8	+0·2580	−0·4220	−0·1800
1·2	+0·1171	+0·5573	−0·1300	5·9	+0·3317	−0·3570	−0·2721
1·6	−0·1590	+0·5480	+0·0903	6	+0·4478	+0·0741	−0·4230
2·0	−0·3867	+0·4366	+0·2940	7	−0·1380	+0·3932	−0·1623
2·4	−0·5106	+0·2509	+0·4405	8	0·2659	+0·3195	+0·2291
2·8	−0·5651	+0·0281	+0·4902	9	−0·3884	0·0406	+0·3799
3·0	−0·5136	−0·0828	+0·4849	10	−0·1475	−0·3348	+0·1669
3·2	−0·4998	−0·1880	+0·4579				

Many intermediate values of the functions were required, and the only practicable plan was to construct graphs from which these could be taken with an accuracy of three figures. This was obtained by drawing graphs

of the three functions over the range from 0 to 40, the scale for p being 1 inch for unity and the scale for the ordinates being 10 inches for unity; these gave the required accuracy, supplemented at critical points by numerical calculation. The graphs are not reproduced here, as they lose their practical value unless on a very large scale; they are, of course, similar in character to graphs of the Bessel functions—oscillating curves diminishing in absolute value with increasing argument.

Resistance Curves.

6. We can now make a numerical study of the wave resistance given by (10). We might adopt dimensionless variables, such as gl/c^2 and k/l, but the calculations were begun with the intention of comparing the results with W. Froude's curves; we take therefore

$$\left.\begin{aligned} l &= \text{length of entrance} = \text{length of run} = 80 \text{ feet} \\ 2\,k &= \text{length of parallel middle body}, \end{aligned}\right\} \tag{15}$$

with $2\,k$ increased from zero up to 340 feet. For an assigned velocity c, the values of R were found for every 20 feet of parallel middle body; as a rule, intermediate values were also calculated so as to define the maxima and minima with sufficient accuracy.

Two examples of the work may suffice. With

$$g/c^2 = 0{\cdot}045; \quad c = 26{\cdot}75 \text{ ft./sec.}; \quad \mathrm{V} = 15{\cdot}83 \text{ knots}; \tag{16}$$

we have, from (10),

$$\mathrm{R} = \frac{g\rho b^2 l}{8\pi} \times 10{\cdot}974 \times \Big[0{\cdot}5026 + \left(\tfrac{1}{2}\mathrm{P}_3 - \frac{1}{3{\cdot}6}\mathrm{P}_4 + \frac{1}{25{\cdot}92}\mathrm{P}_5\right)\{0{\cdot}09k+7{\cdot}2\}$$

$$+ \left(\frac{1}{3{\cdot}6}\mathrm{P}_4 - \frac{1}{12{\cdot}96}\mathrm{P}_5\right)\{0{\cdot}09k+3{\cdot}6\} + \frac{1}{25{\cdot}92}\mathrm{P}_5\,\{0{\cdot}09k\}\Big]. \tag{17}$$

The notation { } denotes the argument of the P functions in the preceding bracket.

For increments of 10 in the value of k, the P functions were required at intervals of $0{\cdot}9$ from zero up to $22{\cdot}5$. Again, with

$$g/c^2 = 0{\cdot}02; \quad c = 40{\cdot}13 \text{ ft./sec.}; \quad \mathrm{V} = 23{\cdot}76 \text{ knots}, \tag{18}$$

we have

$$\mathrm{R} = \frac{g\rho b^2 l}{8\pi} \times 125 \times \Big[0{\cdot}2344 + \left(\tfrac{1}{2}\mathrm{P}_3 - \frac{1}{1{\cdot}6}\mathrm{P}_4 + \frac{1}{5{\cdot}12}\mathrm{P}_5\right)\{0{\cdot}04k+3{\cdot}2\}$$

$$+ \left(\frac{1}{1{\cdot}6}\mathrm{P}_4 - \frac{1}{2{\cdot}56}\mathrm{P}_5\right)\{0{\cdot}04k+1{\cdot}6\} + \frac{1}{5{\cdot}12}\mathrm{P}_5\,\{0{\cdot}04k\}\Big]. \tag{19}$$

In this case the arguments of the P functions increase by intervals of 0·8 from zero up to 10.

It may be noticed from (17) and (19) how the relative importance of the oscillating terms alters with the velocity.

This process was carried out for nine different velocities, namely :—

$$g/c^2 = 0\cdot 1,\quad 0\cdot 0625,\quad 0\cdot 05,\quad 0\cdot 045,\quad 0\cdot 04125,\quad 0\cdot 0375,\quad 0\cdot 03125,\quad 0\cdot 02,\quad 0\cdot 0075. \tag{20}$$

Five of the curves are shown in fig. 2, which gives the quantity $8\pi R/g\rho b^2 l$ on a base $2k$ representing the length of parallel middle body ; the curves for higher speeds are not reproduced, as the scale would obscure the effects, but the data are used in the discussion.

Approximate Formulæ.

7. It is convenient to summarise now the experimental data and empirical formulæ derived from them.

The investigation of W. Froude* was the first direct study of interference of bow and stern waves made by testing models with the same bow and stern, but with increasing lengths of parallel middle body. We associate with this work the subsequent paper by R. E. Froude,† who applied the principle of interference to the resistance of a given model at different speeds. Founded on this work, the approximate theory has been developed : the bow produces a wave system beginning, so far as regular transverse waves are concerned, with a crest slightly aft of the bow, while the stern originates a system beginning with a trough a little aft of the after-body shoulder. Assume that this wave-making length, say Z, is approximately independent of speed, and further assume that the wave resistance is chiefly due to the transverse waves. If, then, λ is the wave-length of regular transverse waves for velocity c, the so-called humps and hollows on the resistance curve occur at speeds for which Z is an odd or even multiple of $\frac{1}{2}\lambda$. Or, if we assume an approximate formula

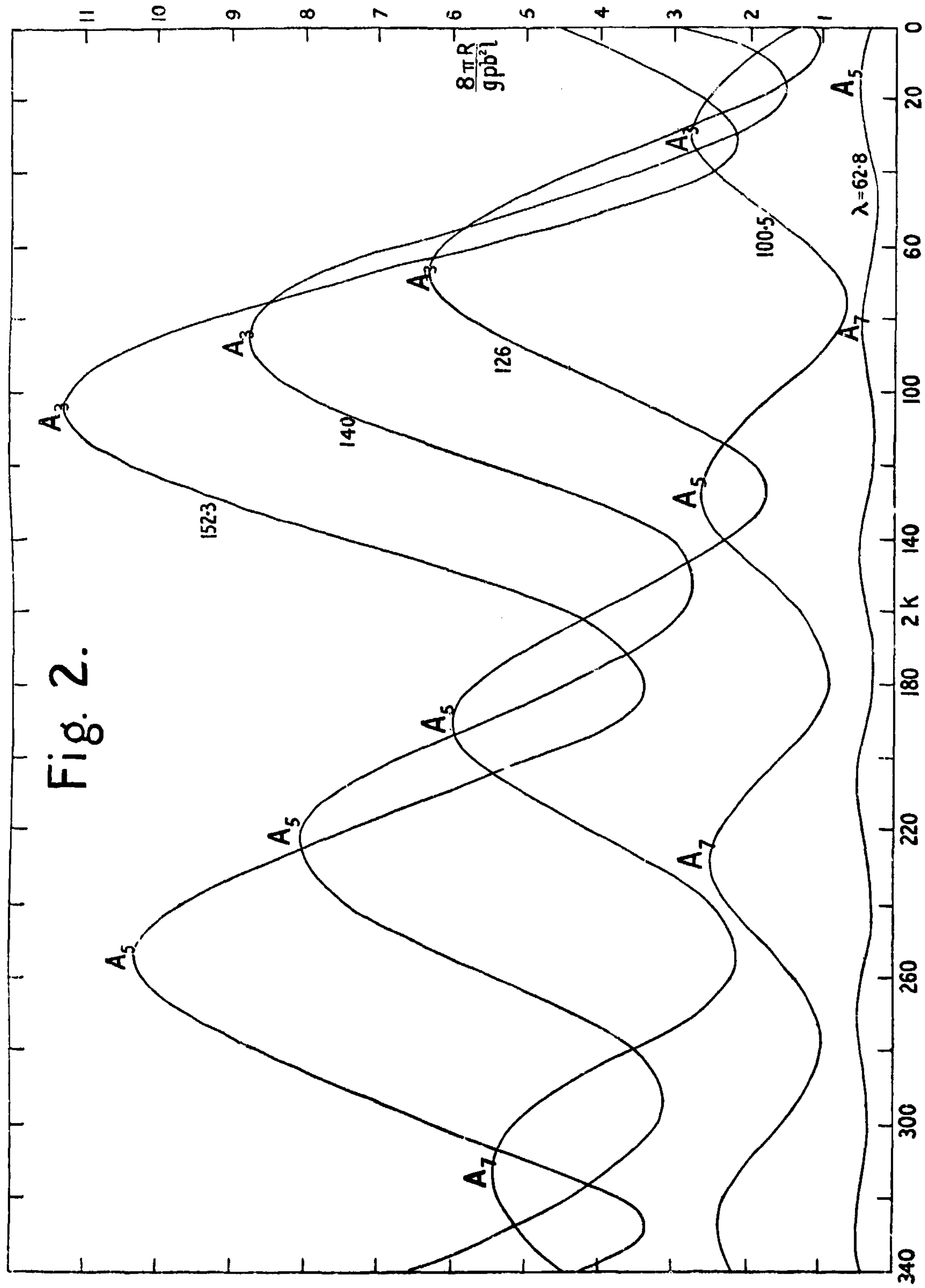
$$R = A - B\cos(gZ/c^2), \tag{21}$$

where A and B are undetermined functions of velocity, the humps and hollows correspond to the maxima and minima of the cosine factor ; hence we have the sequence

$$1,\quad \frac{1}{\sqrt{2}},\quad \frac{1}{\sqrt{3}},\quad \frac{1}{\sqrt{4}},\quad \ldots \tag{22}$$

* W. Froude, 'Trans. Nav. Arch.,' vol. 18, p. 77 (1877).

† R. E. Froude, 'Trans. Nav. Arch.,' vol. 22, p. 220 (1881).



$\frac{8\pi R}{g\rho b^2 l}$
2 k
$\lambda = 62 \cdot 8$
100·5
126
140
152·3
A_3
A_5
A_7

Fig. 2.

for the ratios of the velocities at which these occur, beginning with the final hump on the curve of R plotted on a velocity base.

This was the sequence verified experimentally by R. E. Froude. It should be noted, however, that these points are not actual maxima or minima on the resistance-velocity curve ; although their approximate position is fairly obvious from inspection, they cannot be defined accurately without a knowledge of the mean resistance curve.

Turning to W. Froude's work, it is obvious we should have similar phenomena if the effect of introducing parallel middle body is simply an addition to the wave-making length. This is the case if we consider any curve of R on a base of parallel middle body for a given speed. Here we are dealing with actual maxima and minima ; and Froude's curves show that, within experimental error, the separation between consecutive maxima is approximately equal to the wave-length λ. On this theory the quantity Z derived from each curve should be the same for all velocities, but Froude did not examine that point. The second approximate theory, which we shall consider now, asserts in fact that Z is not constant in these curves.

From a study of various model results, a formula connecting Z with ship form was given by G. S. Baker and J. L. Kent* ; the formula was later associated with direct observation of wave profiles in certain cases. For a recent critical account of this formula, reference should also be made to two papers by J. Tutin† and to the discussions published in connection with them.

The formula is equivalent to defining the wave-making length Z by the equation

$$Z = \mathrm{PL}+\tfrac{1}{4}\lambda = \mathrm{PL}+\pi c^2/2g, \tag{23}$$

where L is the total length of the ship, and P is the prismatic coefficient of form. Since P is the ratio of the volume of the ship to the volume of a prism of the same length and with section equal to the midship section of the ship, we have in the present notation

$$\mathrm{PL} = 2k+2\mathrm{P}_e l, \tag{24}$$

where P_e is the coefficient for the entrance or run ; and at any given speed there is a similar relation between R and $2k$ as on the previous theory. The chief interest of (23) lies in the second term, which makes Z increase with the

* G. S. Baker and J. L. Kent, 'Trans. Nav. Arch.,' vol. 55, Pt. II, p. 37 (1913); *also* J. L. Kent, 'Trans. Nav. Arch.,' vol. 57, p. 154 (1915).

† J. Tutin, 'Trans. Nav. Arch.,' vol. 66, p. 240 (1924); *also* 'Trans. N. E. Coast Inst. Eng. and Ship.,' vol. 41 (1925).

velocity, in contrast to R. E. Froude's results. The expression for R corresponding to (21) is now

$$R = A + B \sin (gPL/c^2), \tag{25}$$

and instead of the sequence (22) we have

$$1, \quad \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{5}}, \quad \frac{1}{\sqrt{9}}, \ \dots \tag{26}$$

for the ratios of the significant velocities.

The authors of this formula based it upon observations over a certain intermediate range of velocities. If we omit the first two or three terms in the sequences (22) and (26), there is a range in which the ratios do not differ very much; further, if we are considering a resistance-velocity curve, the points in question are not defined with precision. However, these remarks do not apply to the final hump on such a curve, and in that case the available evidence seems to favour the first sequence (22).

It is different when we turn to resistance curves, such as those given by W. Froude and by Baker and Kent, in which the base line is length of parallel middle body. In these curves we may follow the position of a certain maximum as the velocity is increased. If the wave-making length Z is constant, it follows from (21) that if λ is the wave-length and $2k$ the length of parallel middle body at which the maximum occurs, we should have

$$\frac{2n+1}{2}\lambda - 2k = \text{constant}; \tag{27}$$

while, on the other hand, from (23) and (25) we should have

$$\frac{4n+1}{4}\lambda - 2k = \text{constant}, \tag{28}$$

where n is zero or an assigned positive integer.

It is certainly the case that over the range which has been examined the second relation (28) fits the data very well. For comparison with present calculations we may take one example from the results of Baker and Kent. The figures are given by Kent, in a recent discussion already quoted, for the case $n = 2$. They relate to models ranging in length from 11·2 feet to 20·5 feet by the insertion of parallel middle body; and the velocities vary from 290 to 370 feet per minute. We transform the results to apply to ships with entrance and run equal to 160 feet by multiplying all lengths, including wave-lengths, by the factor 160/11·2. In the present notation we obtain thus Table II.

Table II.

λ.	$2k$.	$\frac{5}{2}\lambda-2k$.	$\frac{9}{4}\lambda-2k$.
65	43	120	103
73	60	122	104
79	74	124	104
90	96	129	106
96	110	130	106
106	133	132	105

It should also be stated that Kent has observed the wave profile for a certain model at two speeds, and his analysis of the waves agrees with the view that in that case the distance between the first regular bow crest and the first stern trough had increased by one-quarter of the increase of wave-length.

Discussion of Results.

8. We return now to the curves of fig. 2 obtained by the present calculations. Absolute values are not under consideration, and we notice one or two other respects in which the curves differ from experimental results. The interference effects are very greatly increased, and this is no doubt largely due to the infinite draught of the ship; further, as might also have been anticipated, the oscillations in any curve do not fall off so rapidly with increasing length of the ship as in practice.

Consider now the positions of the maxima and minima. Take, for instance, the curve for $g/c^2 = 0{\cdot}0625$, that is, for $\lambda = 100{\cdot}5$ feet. Successive maxima occur at $2k = 28$, 128, 228; the differences are equal to the wave-length, to the order of approximation. This rule holds for any of the curves for moderate velocities. Again, considering the actual positions, the maximum at $2k = 128$ for the same curve evidently corresponds to $n = 2$ in the formulæ (27) and (28), the wave-length being $100{\cdot}5$. In Table II we had $2k = 133$ for a wave-length of 106. Thus, to a first approximation the actual positions are in very fair agreement; more could not be expected, for the experimental results vary slightly according to the lines of the model, and no attempt has been made here to fit closely the form of any particular model.

We have now to group corresponding maxima at different speeds. It is easily seen that the crests A_3 on fig. 2 must correspond to $n = 1$, A_5 to $n = 2$, and so on; the troughs are given by the intermediate values $n = \frac{1}{2}, \frac{3}{2}, \ldots$. We have to follow out any one of these series and find the relation between λ and $2k$; before doing so, we extend the calculations beyond the curves shown in fig. 2.

9. It is not necessary to graph a large range of resistance curves at each speed to find the positions of the maxima or minima. Turning to the general expression (10) for the wave resistance, we require the roots of the equation

$$dR/dk = 0. \tag{29}$$

Since $P'_{n+1} = P_n$, we find that this reduces to

$$\left(\tfrac{1}{2}P_2 - \frac{c^2}{gl}P_3 + \tfrac{1}{2}\frac{c^4}{g^2l^2}P_4\right)\left\{\frac{g(2k+2l)}{c^2}\right\} + \left(\frac{c^2}{gl}P_3 - \frac{c^4}{g^2l^2}P_4\right)\left\{\frac{g(2k+l)}{c^2}\right\} + \tfrac{1}{2}\frac{c^4}{g^2l^2}P_4\left\{\frac{2gk}{c^2}\right\} = 0. \tag{30}$$

But we have

$$pP_2(p) = 4P_3(p) - pP_4(p) + 5P_5(p), \tag{31}$$

and if we write

$$x = gl/c^2 = 2\pi l/\lambda; \quad y = 2gk/c^2 = 4\pi k/\lambda = 2kx/l, \tag{32}$$

the equation (30) becomes, in terms of functions which have been tabulated here,

$$\left(-\frac{y}{xy+2x^2}P_3 - \frac{x^2-1}{2x^2}P_4 + \frac{5}{2y+4x}P_5\right)\{y+2x\} + \left(\frac{1}{x}P_3 - \frac{1}{x^2}P_4\right)\{y+x\} + \frac{1}{2x^2}P_4\{y\} = 0. \tag{33}$$

The problem is the determination of pairs of positive values of x and y satisfying this equation. The approximate formulæ (27) and (28) are equivalent to arranging these in series in linear relationships. For a numerical study of the roots of (33), we have to use the tables and graphs of the P-functions to which reference was made in §5. Starting with some value of x, we find the corresponding value of y from (33), and it is not difficult when we take another value of x to decide which is the corresponding root in y; the preliminary survey of the curves in fig. 2 enables us to follow out any required sequence. We choose here the series corresponding to $n = 1$—that is, the series of crests which includes those marked A_3 in fig. 2. It was found that with the large-scale graphs of the P-functions, the value of the left-hand side of (33) could be calculated with sufficient accuracy for a graphical method to give the required root; except that for high velocities—that is, low values of x—the graphs had to be supplemented by direct calculations.

Omitting the details of the work, the following pairs of roots were obtained :—

x	5·97	5	4	3·6	3	2·5	2·3	2	1·6
y	0	1·75	3·3	3·83	4·7	5·36	5·66	6·1	6·56

On the scale used for fig. 2, we have $l = 80$ ft.; from these values of x and y we get from (32) the values of λ and $2k$ and so the results collected in Table III.

Table III.

$2k$.	λ.	$\frac{3}{2}\lambda-2k$.	$\frac{5}{4}\lambda-2k$.	$2k$.	λ.	$\frac{3}{2}\lambda-2k$.	$\frac{5}{4}\lambda-2k$.
0	84	126	105	171·5	201	131	81
28	100·5	123	98	196	218·6	132	77
66	126	123	92	244	251·3	133	70
85	140	125	90	328	314	143	65
125	167·5	126	84	—	—	—	—

From the third column we see that the wave-making length Z of the approximate theory is not constant. There is first a small decrease, which we should find emphasised if we examined a higher order of crest, say, for $n = 2$; then for a short range it is practically constant, after which it increases steadily with the velocity. However, we see from the fourth column that the rate of increase is not so large as in the alternative approximate formula.

If we had taken any other series of corresponding crests or troughs, we should have found similar results at moderate velocities, but with greater increase at high speeds where the distance between successive maxima differs somewhat from the wave-length.

9. Consider now the resistance-velocity curve for any given length of ship. It has already been stated that the points of maximum excess or defect on such a curve cannot be found precisely; however, they will be in the neighbourhood of the velocities for which $d\mathrm{R}/dk$ is zero for the given length of ship, this being, in fact, the assumption involved in the usual comparison of experimental data of the two kinds.

We shall work out two examples. First, for a ship with no parallel middle body, equation (33) reduces to

$$\left(\frac{1-x^2}{2x^2}\mathrm{P}_4+\frac{5}{4x}\mathrm{P}_5\right)\{2x\}+\left(\frac{1}{x}\mathrm{P}_3-\frac{1}{x^2}\mathrm{P}_4\right)\{x\}=0. \tag{34}$$

The first seven roots and the corresponding results are shown in Table IV.

Table IV.

x	1·93	4·14	5·97	7·41	8·79	10·55	12·14
λ	260	121·4	83·9	67·8	57·2	47·65	41·4
Z	130	121·4	126	135·6	142	143	145
$V/\sqrt{L}$	1·7	1·17	0·97	0·87	0·8	0·73	0·68

The roots correspond to the series of humps and hollows on the resistance curve. The third row shows the wave-making length Z, and in the last row are the values of $V/\sqrt{L}$, where V is in knots and L is the length of the ship in feet. The velocities which would be assigned from an inspection of the actual resistance curve would naturally be a little higher than those found from (34), especially where the mean resistance curve rises rapidly. We have already considered the increase of Z from moderate to higher velocities; we notice here that it is not sufficient to affect appreciably the value of $V/\sqrt{L}$ for the position of the final hump. Table IV. brings out a new point, namely, the increase of Z with decreasing velocity. It is easy to see how this arises. We may express it in this way: the particular model has straight lines at bow and stern, including a finite angle, and as the velocity decreases there is an increase in the relative importance of the wave-making properties of the ends compared with the parts where the change of curvature is gradual; or, analytically from (34), when x is large we can use the asymptotic values of the P functions, and the roots approximate to those of $P_4\ (2x) = 0$ and succeed each other at intervals of $\pi/2$. It has not been found possible to analyse experimental curves to see if this effect occurs; the interference at low velocities is small and unimportant in practice, and the curves are not sufficiently accurate for the purpose. One reference may, however, be given where this effect seems to have been observed.

In a contribution to a recent discussion quoted in § 7, G. Kempf describes some experiments made at the Hamburg Experimental Tank. The model was of cylindrical form with a hemispherical entrance and a run formed by the rotation of a sine curve; it is stated that Z was not constant at all speeds, but that the value of $\sqrt{Z}$ increased 10 per cent. with decreasing speed from V_3 to V_7. It may be noted, as a coincidence, that in Table IV., Z increases from 126 at V_3 to 145 at V_7, and this is an increase of 7 per cent. in $\sqrt{Z}$.

To show the effect of parallel middle body, we consider finally a ship of 400 ft., with the same entrance and run as before, but with 240 ft. of parallel middle body.

Since $y = 3x$, equation (33) becomes in this case

$$\left(-\frac{3}{5x}P_3+\frac{1-x^2}{2x^2}P_4+\frac{1}{2x}P_5\right)\{5x\}+\left(\frac{1}{x}P_3-\frac{1}{x^2}P_4\right)\{4x\}+\frac{1}{2x^2}P_4\{3x\}=0. \tag{35}$$

Table V gives the roots and the similar quantities deduced from them.

Table V.

x	0·625	1·27	2·02	2·73	3·45	4·14	4·83
λ	804	396	249	184	146	121	104
Z	402	396	373	368	365	364	364
$V/\sqrt{L}$	1·9	1·33	1·06	0·91	0·81	0·74	0·68

In this Table the value of Z is given for the whole ship, without deducting the length of parallel middle body; it is seen that in this case Z increases with increasing velocity over the range examined. It is difficult to determine the first root accurately by graphical methods, but it seems probable that Z is then approximately equal to the length of the ship. The increase in the higher values of $V/\sqrt{L}$ as compared with Table IV. is of interest.

The particular dimensions of this case have been chosen because they are the same as the model for which R. E. Froude obtained the resistance curve and made the deductions described in § 7; from inspection of the curve, Froude gave the following values of $V/\sqrt{L}$ for the series of humps and hollows, namely, 1·8, 1·28, 1·045, 0·905, 0·81, 0·73, 0·68. These may be compared with the last row in Table V.

10. The comparison which has been made between the present calculations and experimental data has provided various points of interest. The general agreement is rather striking when one remembers not only the general limitations of the hydrodynamical theory, but the fact that the lines of the ship have been given a simple form, and further that it has been assumed to be of infinite draught. Except for the labour involved in the calculations, it would not be difficult to improve the investigation in both the latter respects. For instance, in former studies other forms for the ship's lines were used in cases where there was no parallel middle body; these could be used for the present problem, and without writing down explicit equations now it may be stated that the result is to vary the coefficients of the P functions in equation (33), and also to introduce functions of higher orders. This will no doubt affect to some extent the rate of change of the wave-making length; but one cannot say in advance to what extent, and the point is one which must be left for consideration in any future extension of the calculations.

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Wave Resistance: the Effect of Varying Draught.

By T. H. HAVELOCK, F.R.S.

(Received June 5, 1925.)

1. In previous studies in the theory of wave resistance, while the water-plane section of the model was of a reasonably ship-like form, the draught was assumed to be infinite. In the following paper the model has the same simple lines and has vertical sides, but the draught is finite. The investigation shows how the resistance at different speeds depends on the draught, but it was undertaken specially for other reasons. In view of certain applications, it was important to find how the interference effects due to bow and stern waves are affected by varying draught. It is shown now that these become less prominent with diminishing draught, but the maxima and minima occur at practically the same positions. Further, when the ratio of draught to length is of the order of the values in actual ship models, one is in a position to attempt a comparison between the absolute values of theoretical and experimental results.

Curves are shown in fig. 2 (p. 590) for the variation of resistance with velocity in three cases when the draught is infinite, and when it is one-tenth and one-twentieth of the length of the model. The latter values cover approximately the usual ratios in practice. On the same diagram are reproduced experimental curves for three models of different types, the data being reduced to the same non-dimensional co-ordinates. Making allowance for the differences of form between these models and for the simplified form for which the calculations have been made, the results show that the calculated values are of the right order of magnitude over a considerable range of velocity. Differences in the two sets of curves, such as the greater prominence of interference effects in the theoretical curves, are discussed.

The first sections of the paper deal with the mathematical expressions for the resistance, and their transformation into forms suitable for calculation; graphs of certain integrals are given in fig. 1 (p. 586).

2. Take axes Ox, Oy in the undisturbed surface of a stream flowing with uniform velocity c in the negative direction of Ox, and take Oz vertically upwards. If there is a distribution of doublets in the liquid in the plane $y = 0$, with axes parallel to Ox, and of moment $\psi(h, 0, f)$ per unit area, the corresponding wave resistance is given by*

$$R = 16\pi g^2 \rho c^{-4} \int_0^\infty df \int_0^\infty df' \int_{-\infty}^\infty dh \int_{-\infty}^\infty dh' \int_0^{\pi/2} \partial\psi/\partial h \,.\, \partial\psi'/\partial h' \,.\, \sec^3\phi$$
$$\times e^{-\{g(f+f')/c^2\}\sec^2\phi} \cos\left[\{g(h-h')/c^2\}\sec\phi\right] d\phi \quad (1)$$

Over the plane $y = 0$ the normal fluid velocity at the point $(h, 0, f)$ is $2\pi\partial\psi/\partial h$. Taking $y = 0$ as the fore-and-aft median plane of the ship, we assume the action of the ship to be equivalent to a distribution of normal velocity over its section by this plane, the distribution being such that if $y = F(x, z)$ is the equation of the ship's surface, we substitute in (1)

$$2\pi \frac{\partial\psi}{\partial h} = c \frac{\partial}{\partial h} F(h, f). \quad (2)$$

To simplify the calculations as far as possible, we shall assume the ship to be symmetrical fore and aft, and to have vertical sides so as to be of constant horizontal section. The water-plane section is taken to be of parabolic form, the equation for y positive being

$$y = b(1 - x^2/l^2). \quad (3)$$

The length of the ship is $2l$, its beam $2b$, and it is of constant draught d.

* 'Roy. Soc. Proc.,' A, vol. 108, p. 79 (1925).

We substitute from (2) and (3) in (1). Carrying out the integrations in f, f', h and h', we obtain, after some reductions,

$$\mathrm{R} = \frac{256\, g\rho b^2 l}{\pi p^3} \int_0^{\pi/2} (1 - e^{-\alpha \sec^2 \phi})^2 \Big\{ \cos^3 \phi + \frac{4}{p^2} \cos^5 \phi$$

$$+ \left(\cos^3 \phi - \frac{4}{p^2} \cos^5 \phi\right) \cos (p \sec \phi) - \frac{4}{p} \cos^4 \phi \sin (p \sec \phi) \Big\}\, d\phi, \quad (4)$$

where $p = 2gl/c^2$, and $\alpha = gd/c^2$.

3. In reducing this expression to a form su‘‘able for calculation, we take first the terms which are non-oscillating regarded as functions of c.

A typical integral is

$$\int_0^{\pi/2} \cos^5 \phi e^{-\beta \sec^2 \phi}\, d\phi. \quad (5)$$

Changing the variable, this becomes

$$e^{-\beta} \int_0^{\infty} (1+t^2)^{-\frac{3}{2}} e^{-\beta t^2} dt = \tfrac{1}{2}\pi^{\frac{1}{2}} \beta e^{-\frac{1}{2}\beta}\, \mathrm{W}_{-\frac{3}{2},\, -\frac{3}{2}}(\beta), \quad (6)$$

where W is a confluent hypergeometric function. We can obtain an expansion by using the contour integral for the general hypergeome ic function of this type. In this case we obtain

$$\mathrm{W}_{-\frac{3}{2},\, -\frac{3}{2}}(\beta) = \frac{\beta^{-\frac{3}{2}} e^{-\frac{1}{2}\beta}}{2\pi i} \int_{-\infty i}^{\infty i} \frac{\Gamma(s)\, \Gamma(-s+\frac{7}{2})\, \Gamma(-s+\frac{1}{2})}{\Gamma(\frac{7}{2})\, \Gamma(\frac{1}{2})} \beta^s\, ds, \quad (7)$$

the contour separating the poles of $\Gamma(s)$ from those of $\Gamma(-s+\frac{7}{2})\, \Gamma(-s+\frac{1}{2})$. We have, therefore, to evaluate the residue of the integrand at the simple poles $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, and at the series of double poles $s = \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \ldots$. The latter residues give logarithmic terms. Carrying out the calculation, we obtain the expansion

$$\mathrm{W}_{-\frac{3}{2},\, -\frac{3}{2}}(\beta) = \tfrac{8}{15}\pi^{-\frac{1}{2}} \beta^{-\frac{3}{2}} e^{-\frac{1}{2}\beta} \Big\{ -2\beta^{\frac{1}{2}} + \tfrac{1}{2}\beta^{\frac{3}{2}} - \tfrac{3}{8}\beta^{\frac{5}{2}}$$

$$- \sum_{n=3}^{\infty} \frac{\Gamma(n+1/2)\, \pi^{-\frac{1}{2}}}{\Gamma(n+1)\, \Gamma(n-2)} \beta^{n+1/2} \left(\log \tfrac{1}{4}\gamma\beta + 2 \sum_{p=1}^{n} \frac{1}{2p-1} - \sum_{1}^{n} \frac{1}{p} - \sum_{1}^{n-3} \frac{1}{p} \right) \Big\}, \quad (8)$$

with $\log \gamma = 0{\cdot}57722 \ldots$.

We obtain thus

$$\int_0^{\pi/2} \cos^5 \phi e^{-\beta \sec^2 \phi}\, d\phi = \frac{8}{15} e^{-\beta} \Big\{ 1 - \frac{1}{4}\beta + \frac{3}{16}\beta^2 + \frac{37}{192}\beta^3 + \frac{113}{3072}\beta^4$$

$$- \frac{263}{20480}\beta^5 + \ldots + \left(\frac{5}{32}\beta^3 + \frac{35}{256}\beta^4 + \frac{63}{1024}\beta^5 + \ldots \right) \log \frac{1}{4}\gamma\beta \Big\}. \quad (9)$$

A similar integral which we require in (4) can be derived by differentiation, and we have

$$\int_0^{\pi/2} \cos^3\phi e^{-\beta\sec^2\phi}\,d\phi = \frac{2}{3}e^{-\beta}\left\{1-\frac{1}{2}\beta-\frac{7}{16}\beta^2-\frac{7}{96}\beta^3+\frac{97}{3072}\beta^4 \right.$$
$$\left. +\frac{683}{30720}\beta^5+\ldots-\left(\frac{3}{8}\beta^2+\frac{5}{16}\beta^3+\frac{35}{256}\beta^4+\frac{21}{512}\beta^5+\ldots\right)\log\frac{1}{4}\gamma\beta\right\}. \quad (10)$$

For large values of β, asymptotic expansions can be found in the usual manner by transforming the integrals. They are

$$\int_0^{\pi/2} \cos^5\phi e^{-\beta\sec^2\phi}\,d\phi \sim \frac{1}{2}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}}e^{-\beta}\left(1-\frac{7}{4}\frac{1}{\beta}+\frac{189}{32}\frac{1}{\beta^2}-\frac{3465}{128}\frac{1}{\beta^3}\right.$$
$$\left.+\frac{315315}{2048}\frac{1}{\beta^4}-\ldots\right), \quad (11)$$

$$\int_0^{\pi/2} \cos^3\phi e^{-\beta\sec^2\phi}\,d\phi \sim \frac{1}{2}\pi^{\frac{1}{2}}\beta^{-\frac{1}{2}}e^{-\beta}\left(1-\frac{5}{4}\frac{1}{\beta}+\frac{105}{32}\frac{1}{\beta^2}-\frac{4725}{384}\frac{1}{\beta^3}\right.$$
$$\left.+\frac{363825}{6144}\frac{1}{\beta^4}-\ldots\right). \quad (12)$$

On applying the expression (4) to numerical cases of interest, it was found that the integrals we have just considered were required over a range intermediate between those suitable for the series given in (9)-(12). It was, therefore, necessary to calculate the values of integrals such as (5) directly by numerical methods. It was sufficiently accurate to evaluate the integrand, for each value of β, at intervals of five degrees throughout the range of integration, and then to use Simpson's rule. The series given above were used for checking and supplementing the values so obtained.

For the purposes of expression (4) the results were collected in tables and graphs of the integrals

$$I_3 = \int_0^{\pi/2}(1-e^{-\alpha\sec^2\phi})^2\cos^3\phi\,d\phi, \quad (13)$$

$$I_5 = \int_0^{\pi/2}(1-e^{-\alpha\sec^2\phi})^2\cos^5\phi\,d\phi. \quad (14)$$

The graphs are shown, on α as base, in fig. 1.

It can be seen from (4) how the mean resistance, apart from superposed interference effects, depends upon the integrals I_3 and I_5; and since α is gd/c^2, the curves in fig. 1 show how the effect of finite draught becomes appreciable when the wave-length is comparable with the draught, the ordinates falling off rapidly in value after that as α becomes smaller.

4. We have now to consider the remaining terms in (4), namely, the last three terms under the integral sign.

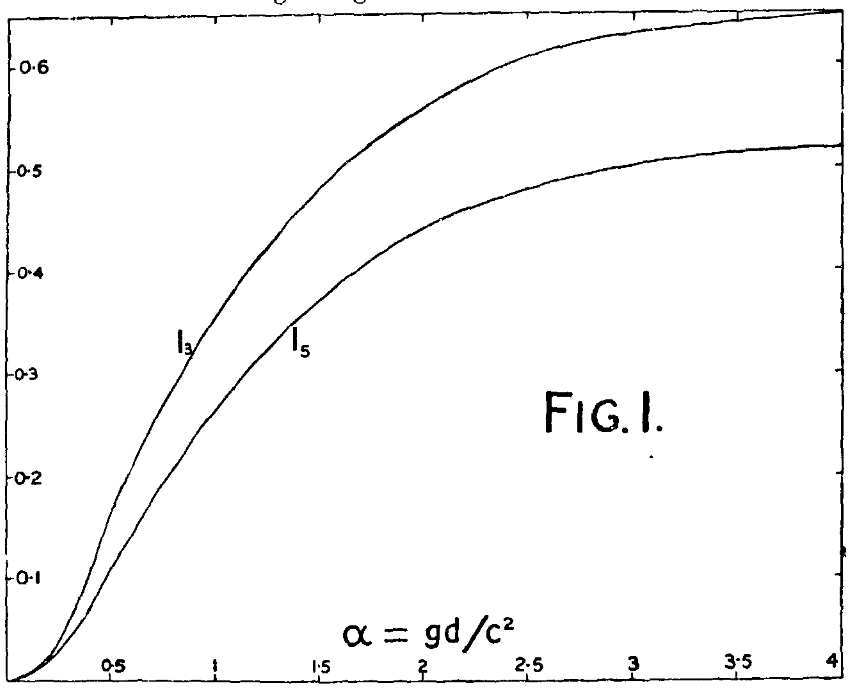


FIG. I.

A complete valuation of these over the whole range of velocity would be troublesome; but fortunately the range of most interest, and one in which there is more chance of agreement between theory and observation, corresponds to fairly large values of p, roughly between 10 and 40. We can obtain an asymptotic expansion which is suitable for this range. We shall write

$$\alpha = gd/c^2 = \beta p, \tag{15}$$

where β is the ratio of draught to length.

Then, with the understanding that real parts of complex expressions have to be taken, we have

$$\begin{aligned}
\int_0^{\pi/2} (1 - e^{-\beta p \sec^2 \phi})^2 \left\{ \left(\cos^3 \phi - \frac{4}{p^2} \cos^5 \phi \right) \cos (p \sec \phi) \right. & \\
\left. - \frac{4}{p} \cos^4 \phi \sin (p \sec \phi) \right\} d\phi & \\
= \int_0^{\pi/2} (1 - e^{-\beta p \sec^2 \phi})^2 \cos^5 \phi \left(\sec \phi + \frac{2i}{p} \right)^2 e^{ip \sec \phi} \, d\phi & \\
= \frac{e^{ip}}{\sqrt{(2p)}} \int_0^\infty \frac{e^{it} \left\{ 1 - e^{-\beta p \left(1 + \frac{t}{p}\right)^2} \right\}^2 \left(1 + \frac{2i + t}{p} \right)^2}{t^{\frac{1}{2}} \left(1 + \frac{t}{p} \right)^6 \left(1 + \frac{t}{2p} \right)^{\frac{1}{2}}} dt & \\
= (\pi/2p)^{\frac{1}{2}} e^{ip} \{ Q(0) - 2e^{-\beta p} Q(2\beta) + e^{-2\beta p} Q(4\beta) \}, &
\end{aligned} \tag{16}$$

where

$$\pi^{\frac{1}{2}}Q(\gamma)\int_0^\infty \frac{\left(1+\dfrac{2i+t}{p}\right)^2 \cdot e^{-(\gamma-i)\,t-\gamma t^2/2p}\,dt}{t^{\frac{1}{2}}\left(1+\dfrac{t}{p}\right)^6\left(1+\dfrac{t}{2p}\right)}. \tag{17}$$

The integrand in (17) is expanded in ascending powers of $1/p$ and then integrated term by term, using the formula

$$\int_0^\infty t^{r-\frac{1}{2}}e^{-(\gamma-i)t}\,dt = \Gamma\left(r+\tfrac{1}{2}\right)\delta^{\,r+\frac{1}{2}}e^{(r+\frac{1}{2})i\theta};$$

$$\delta = (1+\gamma^2)^{-\frac{1}{2}};\quad \cot\theta = \gamma. \tag{18}$$

The expression was carried out completely to include all terms of order p^{-3}, and the leading terms in p^{-4} were also determined so as to check the order of numerical approximation. Leaving out the intermediate expansions, it may suffice to record the final result ; we find

$$\begin{aligned}
Q(\gamma) \backsim\ & \delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} + \left(4i\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} - \frac{17}{8}\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta} - \frac{3}{8}\gamma\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta}\right)\frac{1}{p} \\
& - \left\{4\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} + \frac{21}{2}i\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta} - \left(\frac{1065}{128} - \frac{3}{2}\gamma i\right)\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} - \frac{255}{64}\gamma\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} - \frac{105}{128}\gamma^2\delta^{\frac{9}{2}}e^{\frac{9}{2}i\theta}\right\}\frac{1}{p^2} \\
& + \left\{\frac{25}{2}\delta^{\frac{1}{2}}e^{\frac{1}{2}i\theta} + \left(\frac{3}{2}\gamma + \frac{1569}{32}i\right)\delta^{\frac{3}{2}}e^{\frac{3}{2}i\theta}\right. \\
& \qquad + \frac{15}{8}\left(\frac{21}{2}\gamma i - \frac{2933}{128}\right)\delta^{\frac{5}{2}}e^{\frac{5}{2}i\theta} + \frac{105}{16}\left(\frac{1}{2}\gamma^2 i - \frac{355}{64}\gamma\right)\delta^{\frac{7}{2}}e^{\frac{7}{2}i\theta} \\
& \qquad \left. - \frac{16065}{1024}\gamma^2\delta^{\frac{11}{2}}e^{\frac{11}{2}i\theta} - \frac{3465}{1024}\gamma^3\delta^{\frac{13}{2}}e^{\frac{13}{2}i\theta}\right\}\frac{1}{p^3} + \ldots
\end{aligned} \tag{19}$$

Collecting these results, we have now reduced (4) to the form

$$R = \frac{256g\rho b^2 l}{\pi p^3}\Big[\,I_3(\beta p) + \frac{4}{p^2}I_5(\beta p) + \text{real part of}$$

$$(\pi/2p)^{\frac{1}{2}}e^{ip}\{Q(0) - 2e^{-\beta p}Q(2\beta) + e^{-2\beta p}Q(4\beta)\}\Big], \tag{20}$$

where the integrals I are defined in (13), (14) and graphed in fig. 1, and the asymptotic expansion of Q as a function of p is given in (19).

5. Numerical calculations from (18) and (19) are tedious, and we have chosen the parameters so that we require the numerical values of the coefficients

in (19) for four values of γ, namely, 0, 0·1, 0·2, 0·4. Omitting the details of the work, we have in these cases

$$Q(0) \backsim 0{\cdot}707\,(1+i) - \frac{1{\cdot}326\,(1-i)}{p} - \frac{1{\cdot}237\,(1+i)}{p^2} - \frac{4{\cdot}755\,(1-i)}{p^3} + \ldots,$$

$$Q(0{\cdot}1) \backsim 0{\cdot}74+0{\cdot}669i - \frac{1{\cdot}456-1{\cdot}3i}{p} - \frac{1{\cdot}605+0{\cdot}916i}{p^2} - \frac{3{\cdot}87-4{\cdot}13i}{p^3} + \ldots,$$

$$Q(0{\cdot}2) \backsim 0{\cdot}766+0{\cdot}628i - \frac{1{\cdot}473-1{\cdot}263i}{p} - \frac{1{\cdot}634+0{\cdot}47i}{p^2} + \frac{0{\cdot}571+6{\cdot}46i}{p^3} + \ldots,$$

$$Q(0{\cdot}4) \backsim 0{\cdot}798+0{\cdot}541i - \frac{1{\cdot}635-1{\cdot}315i}{p} - \frac{1{\cdot}329-0{\cdot}338i}{p^2} + \ldots. \qquad (21)$$

6. We proceed now to calculate and graph the wave resistance as a function of the velocity for three different draughts. The curves are shown in fig. 2 (p. 590) in non-dimensional co-ordinates, the ordinates being $R/g\rho b^2 l$ and the abscissæ $V/\sqrt{L}$. In the notation used, we have $2b =$ beam, $2l =$ length, $d =$ draught, $V =$ velocity in knots and $L =$ length in feet; thus $V/\sqrt{L} = \sqrt{(11{\cdot}594/p)}$, approximately.

The first case is that of infinite draught, for which $\beta = d/2l = \infty$. Here (20) reduces to

$$R = \frac{256\,g\rho b^2 l}{\pi p^3}\left\{\frac{2}{3} + \frac{32}{15}\frac{1}{p^2} + \text{Real}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{ip} Q(0)\right\}. \qquad (22)$$

This case has been calculated previously* from more complete formulæ; the use of (22) now serves to check the range of the asymptotic formulæ for Q. The second case is for the draught one-tenth of the length, or $\beta = 0{\cdot}1$, so that

$$R = \frac{256\,g\rho b^2 l}{\pi p^3}\left[I_3\left(\frac{1}{10}p\right) + \frac{4}{p^2} I_5\left(\frac{1}{10}p\right) + \text{Real}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{ip}\{Q(0) - 2e^{-\frac{1}{5}ip} Q(0{\cdot}2) + e^{-\frac{2}{5}ip} Q(0{\cdot}4)\}\right]. \qquad (23)$$

Finally, for the draught one-twentieth of the length, or $\beta = 0{\cdot}05$, we have

$$\frac{256\,g\rho b^2 l}{\pi p^3}\left[I_3\left(\frac{1}{20}p\right) + \frac{4}{p^2} I_5\left(\frac{1}{20}p\right) + \text{Real}\left(\frac{\pi}{2p}\right)^{\frac{1}{2}} e^{ip}\{Q(0) - 2e^{-\frac{1}{10}ip} Q(0{\cdot}1) + e^{-\frac{1}{5}ip} Q(0{\cdot}2)\}\right]. \qquad (24)$$

* 'Roy. Soc. Proc.,' A, vol. 103, p. 579 (1923).

With the use of the graphs in fig. 1 and the expressions in (21), calculations were made from (22), (23) and (24) for about fifteen values of p in each case. The results are shown in the continuous curves of fig. 2, the curves being marked with the corresponding value of β.

7. The curves show the increasing influence of smaller draught at the higher velocities. Although, from the differences in the expressions for the resistance, the maxima and minima due to interference of bow and stern systems probably do not occur at exactly the same positions, it is important to notice that the differences in this respect are inappreciable. This agrees with a similar phenomenon which has been observed experimentally in the resistance of a submerged model at different depths; although the magnitude of the interference effects varies with the depth, the positions of the maxima and minima are practically unaltered. Another point to note in the theoretical curves of fig. 2 is that at the smaller draughts the effect of interference is less pronounced.

But the chief purpose of the calculations was to find whether, with a draught similar to that of actual ship models, the calculated resistance was in reasonable agreement with experimental results.

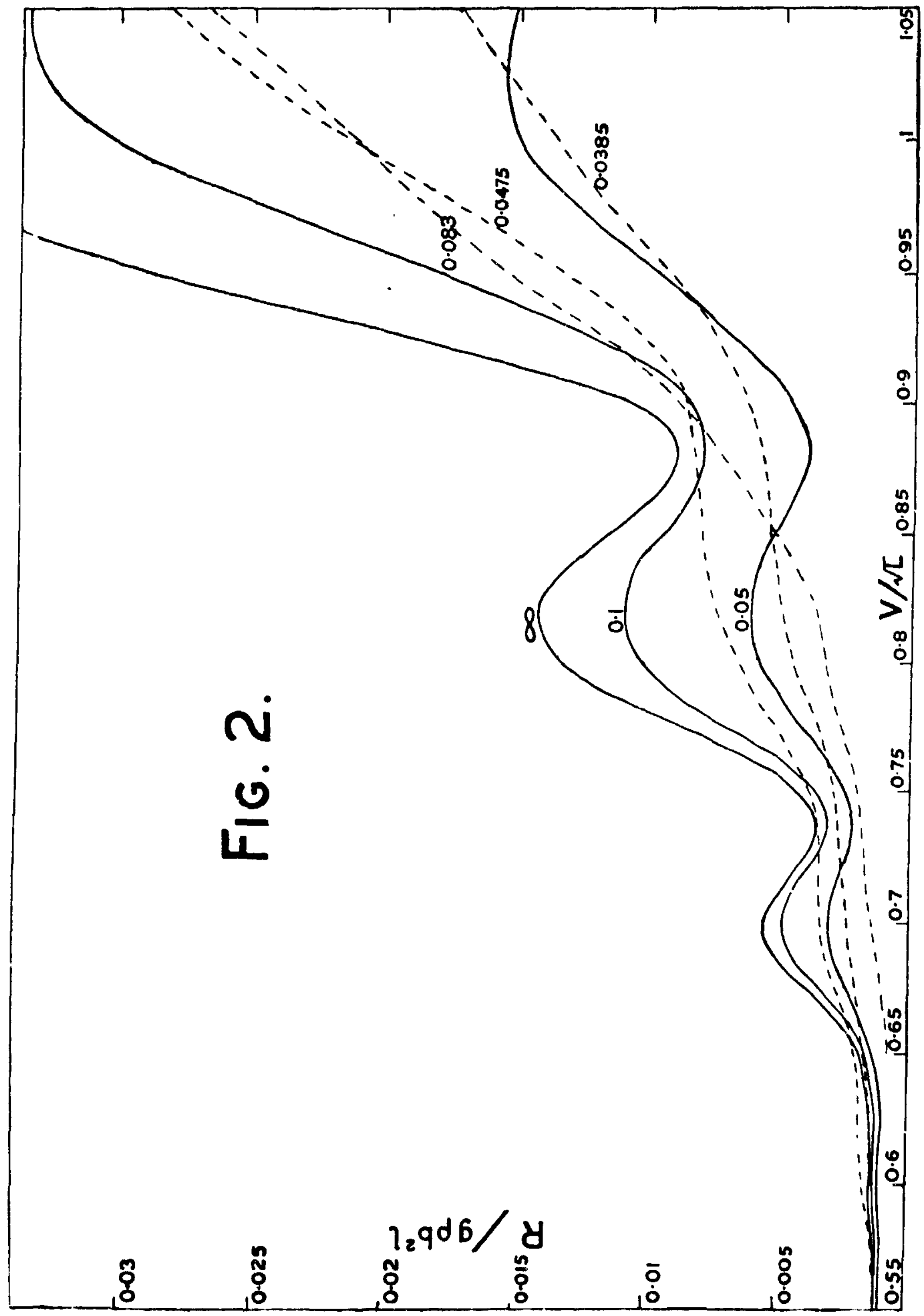
The values which have been chosen for the ratio of draught to length, namely, one-twentieth and one-tenth, cover approximately the usual range in practice. It must, of course, be remembered that the calculated results correspond to a model with vertical sides and constant horizontal cross-section; therefore one cannot expect more than agreement in order of magnitude. Three examples of experimental curves have been selected and are shown in the discontinuous curves of fig. 2.

The curves marked 0·0475 and 0·0385 have been drawn from results given by R. E. Froude* for the residuary resistance of two models of the same length and beam and having the given ratios of mean draught to length. The results were given as the resistance in tons for a ship of 400 ft. length, and have been recalculated here in the non-dimensional co-ordinates of fig. 2; the two cases are Froude's ship A with displacement 5,390 and 4,090 tons respectively. In both cases there was a certain amount of parallel middle body.

The third curve, marked 0·083 in fig. 2, has been obtained by similar reductions from experimental results given by J. L. Kent†; it refers to his model 112K, which had no parallel middle body, but had hollow lines at the bow. The curve has been filled in approximately from a smaller number of points than in the previous cases; one can, however, observe the effect of the hollow lines

* R. E. Froude, 'Trans. Inst. Nav. Arch.,' vol. 22, p. 220 (1881).

† J. L. Kent, 'Trans. Inst. Nav. Arch.,' vol. 57, p. 154 (1915).



0·03
0·025
0·02
0·015
0·01
0·005
R / gρb²L
0·55
0·6
0·65
0·7
0·75
0·8
0·85
0·9
0·95
1
1·05
V/√L
∞
0·1
0·05
0·083
0·0475
0·0385

FIG. 2.

in the general form of the curve. The effect of differences of form, other than the ratio of draught to length, is, of course, important, and the three curves reproduced in fig. 2 have been chosen so that this should not be overlooked in the comparison between the various curves.

We note first the differences between the two sets of curves in fig. 2. The theoretical curves have much more prominent humps and hollows, due to interference between bow and stern waves, especially at the lower velocities. This may be inherent in the approximations made in replacing the ship by a certain doublet distribution over the median plane. But the effect is probably due in part to the simplified form with constant horizontal section; however, this point must be left for future examination. It is hardly necessary to remark that, when one reaches the stage of comparing absolute values, the influence of viscosity and turbulence must eventually be taken into account. Further, this consideration applies not only to the theoretical curves but also to those we have called experimental; for the latter are derived from actual measurements of total resistance by deducting the frictional resistance calculated according to an empirical formula, the residuary resistance so obtained being chiefly due to wave-making. It may be that the effect of fluid friction on the wave-making could be expressed by a slight alteration of the equivalent wave-making form of the ship. The curves of fig. 2 show also small differences in the positions of the interference maxima, but this is, of course, due to the different lines of the various models.

When every allowance has been made for differences of form and other considerations, the curves of fig. 2 show over a large range ol velocity a general agreement between theory and observation, which is very interesting and suggestive. Further approach to ship-like form may enable us to remove some of the remaining differences, and should in any case be of service in the interpretation of experimental results.

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Wave Resistance: Some Cases of Unsymmetrical Forms.

By T. H. HAVELOCK, F.R.S.

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1. One of the chief features of interest in curves showing the variation of wave resistance with velocity is the occurrence of oscillations about a mean curve, which may be regarded as due to interference between the waves produced by the front and rear portions of the model. In various comparisons made between theoretical curves and such suitable experimental results as are available, the greatest divergence is perhaps in the magnitude of these oscillations, the theoretical curves showing effects many times greater than similar experimental results. There are, no doubt, many approximations in the hydro-dynamical theory which preclude too close a comparison between theoretical and experimental results in any particular case, but it seems fairly certain that the divergence in question must be largely due to neglecting the effects of fluid friction. For several reasons it is useless to attempt at present a direct introduction of viscosity into the mathematical problem, but a consideration of its general effect suggests one or two calculations which may be of interest The direct effect of viscosity upon waves already formed may be assumed to be relatively small; the important influence is one which makes the rear portion of the model less effective in generating waves than the front portion. We may imagine this as due to the skin friction decreasing the general relative velocity of model and surrounding water as we pass from the fore end to the aft end; or we may picture the so-called friction belt surrounding the model, and may consider the general effect as equivalent to a smoothing out of the curve of the rear portion of the model. Without pursuing these speculations further, they suggest calculations which can be made for models in frictionless liquid when the form of the model is unsymmetrical in this manner; and the particular point to be examined is the effect of such modification upon the magnitude of the interference phenomena.

The first sections compare, in this respect, two bodies entirely submerged in the liquid. The form in each case is a surface of revolution; one is symmetrical fore and aft and has sharp pointed ends, while in the other the rear portion is cut away so as to come to a fine point. By inspection of the expressions for the wave resistance it is seen that the oscillating terms are of a lower order of magnitude in the latter than in the former case.

The remaining sections deal with the similar problem for a model of infinite draught and constant horizontal cross-section ; the forms of the section for the two cases are shown in fig. 1. Here, with the help of tables and graphs available from previous studies, the expressions for the wave resistance have been graphed and the curves are shown in fig. 2. The result of smoothing the lines of the rear portion is very marked. It makes the curve like experimental ones in this respect at least, that the curve is a continually ascending one in the range shown ; the superposed oscillations are not large enough to make actual maxima and minima. A more complete study of the progressive effect of small changes in the rear half of the model would involve very lengthy calculations ; the examples given have been chosen for the comparatively simple form of the mathematical expressions. It is to be understood that they are not intended as a direct representation of the actual effects of fluid friction ; but they show the great difference in interference effects which are produced by an asymmetry of the general nature suggested by them.

2. The fluid motion produced by a body entirely submerged in a uniform stream may be investigated by the method of successive images. The first approximation consists of the distribution of sources and sinks which is the image of the uniform stream in the surface of the body ; the second is the image of these sources and sinks in the upper free surface of the stream, and the process could be carried on by successive images in the surface of the body and the free surface of the stream. After the second stage the expressions become very complicated, as the image of a single source in the upper free surface is a distribution of infinite extent along a horizontal line at a height above the free surface equal to the depth of the source. It would be of interest to carry the process further in some simple cases, but at present the second stage must suffice ; it can be seen that, in general, this implies that the ratio of the maximum vertical diameter of the body to its depth below the surface must be small.

For the first stage of the approximation, instead of finding the image system for a given form in a uniform stream, it is more convenient to begin with a given distribution of sources and sinks and deduce the form of the body. As we shall deal only with surfaces of revolution, we assume a line distribution of finite extent along a line parallel to the stream. Writing down Stokes' current function, the form of the body may be found by graphical methods devised by Rankine and applied to shiplike forms by D. W. Taylor and other writers.

Let the stream, of velocity c, be parallel to Ox and let there be a source

distribution of strength $f(x)$ along portion of the axis of x; then, with $\tilde{\omega}$ as distance from Ox, the velocity potential and stream function are given by

$$\phi = cx + \int \frac{f(h)\,dh}{\{(x-h)^2 + \tilde{\omega}^2\}^{\frac{1}{2}}} \tag{1}$$

$$\psi = \tfrac{1}{2}c\tilde{\omega}^2 + \int \frac{(x-h)\,f(h)\,dh}{\{(x-h)^2 + \tilde{\omega}^2\}^{\frac{1}{2}}}. \tag{2}$$

The form of the solid is obtained from the equation $\psi = 0$. The graphical method is first to graph the integral in (2) upon $\tilde{\omega}$ as a base for given values of x, obtaining a family of curves each corresponding to a constant value of x; then on the same diagram the parabola $\psi = \frac{1}{2}c\tilde{\omega}^2$ is drawn. The intersections of the parabola with the family of curves give pairs of corresponding values of x and $\tilde{\omega}$ on the zero stream line.

It is obvious that if $f(h)$ is finite, not zero, at an end of the range of sources then the body has a blunt end; and further, the length of the body is greater than the length of the range. If $f(h)$ is zero at both ends, the body has a sharp point at both ends and its length is equal to the length of the range; if, in addition, $f'(h)$ is zero at an end, the sharp point at that end is one of zero angle.

3. In considering the second approximation, namely, the image of the distribution $f(h)$ in the upper free surface of the stream, it is more convenient to use as the elementary system a doublet with its axis parallel to the stream. As we are dealing with solid bodies of finite size, we can in general replace the line of sources and sinks by an equivalent line of doublets; thus instead of (1) we have

$$\phi = cx + \int \frac{(x-h)\,\psi(h)\,dh}{\{(x-h)^2 + \tilde{\omega}^2\}^{3/2}}, \tag{3}$$

provided $\psi'(h) = f(h)$, and $\psi(h)$ is zero at both limits. Consider now a solid of revolution with its axis horizontal and at a depth f below the surface, the form being such that the image of the uniform stream in it is a line of doublets of moment $\psi(h)$. The image of this system in the free surface can be shown to be a certain distribution of doublets of infinite extent along a line at a height f above the surface. For the present purpose we shall quote the expression for the wave resistance*

$$\mathrm{R} = 16\pi g^4 \rho c^{-8} \int \psi(h)\,dh \int \psi(h')\,dh' \int_0^{\pi/2} \sec^5\phi$$
$$\times \cos\left[\{g(h-h')/c^2\}\sec\phi\right] e^{-(2gf/c^2)\sec^2\phi}\,d\phi. \tag{4}$$

* 'Roy. Soc. Proc.,' A, vol. 95, p. 363 (1919).

We shall consider two cases, one a sharp-ended form which is symmetrical fore and aft, while in the second case the aft end is curved to a fine point.

4. For the first case we take a spindle-shaped body which has been used for experimental work at the National Physical Laboratory; for this form the source distribution is

$$f(h) = a\{(h/l) - (h/l)^3\}; \quad -l < h < l. \tag{5}$$

The shape of the surface for this case has been given by Perring.* It is sufficient to state here that it is a surface of revolution symmetrical about the middle cross-section and having pointed ends with finite angle of entrance; it can be made to have any required ratio of breadth to length.

We can, in this case, carry out the integration in (2) and obtain the equation of the longitudinal section. It is found that with $2b$ as the breadth of the model, $2l$ its length, and δ the ratio of b to l, then the constant a of (5) is equal to $\frac{1}{2}\alpha bc$, where

$$\alpha = \delta/[\tfrac{1}{4}(1+\delta^2)^{\frac{1}{2}}(3+2\delta^2)+\delta^2(1+\tfrac{3}{4}\delta^2)\log(\delta/\{1+(1+\delta^2)^{\frac{1}{2}}\})]. \tag{6}$$

The equivalent distribution of doublets, given by the conditions stated in (3), is

$$\psi(h) = -\tfrac{1}{4}al\,(1 - h^2/l^2)^2. \tag{7}$$

Substituting in (4) we obtain the wave resistance

$$R = 4\pi g^4 \rho l^4 a^2 c^{-8} \int_0^{\pi/2} I^2 \sec^5 \phi\, e^{-(2gf/c^2)\sec^2\phi}\, d\phi, \tag{8}$$

where

$$I = \int_0^1 (1 - u^2)^2 \cos(glu/c^2 \cos\phi)\, du. \tag{9}$$

After evaluating (9), the expression (8) can be reduced to standard form as

$$\begin{aligned} R = \frac{256\pi g\rho b^2 l\alpha^2}{p^3} \int_0^{\pi/2} \Big[& \cos\phi + \frac{12}{p^2}\cos^3\phi + \frac{144}{p^4}\cos^5\phi \\ & - \left(\cos\phi - \frac{60}{p^2}\cos^3\phi + \frac{144}{p^4}\cos^5\phi\right)\cos(p\sec\phi) \\ & + 12\left(\frac{\cos^2\phi}{p} - \frac{12\cos^4\phi}{p^3}\right)\sin(p\sec\phi)\Big] e^{-\beta p\sec^2\phi}\, d\phi, \end{aligned} \tag{10}$$

where $\beta = f/l$, $p = 2gl/c^2$, and α is given in (6).

An asymptotic expansion suitable for large values of p could be obtained, but calculation from it is very tedious; the particular point under consideration can be made by comparison with the similar expression for the second case.

* W. G. A. Perring, 'Trans. Inst. Nav. Arch.,' vol. 67, p. 95 (1925).

5. For comparison we require a solid of revolution of which the front end is a sharp point of finite angle while the rear end is cut away to a point of zero angle; there will, of course, be a point of inflection in the curve of the rear portion.

This is obtained by taking the source distribution to be

$$f(h) = ah\,(2l - h)\,(3l + h)^2\,; \quad -3l < x < 2l. \tag{11}$$

The equivalent doublet distribution over the same range is

$$\psi(h) = -\tfrac{1}{5}\,a\,(2l - h)^2\,(3l + h)^3. \tag{12}$$

The outline of the model was found by the graphical methods described in § 2; the work is not reproduced here as it was only carried out with sufficient accuracy to verify that the curve was of the required type. A similar curve is shown later in fig. 1. The model has now a length $5l$, and it is not symmetrical fore and aft of the maximum cross-section.

From (4) we find the wave resistance

$$R = \tfrac{1}{2}\tfrac{6}{5}\pi g^4 \rho a^2 c^{-8} \int_0^{\pi/2} (I^2 + J^2) \sec^5 \phi\, e^{-(2gf/c^2)\sec^2\phi}\, d\phi, \tag{13}$$

where

$$I + iJ = \int_{-3l}^{2l} (2l - h)^2\,(3l + h)^3\, e^{igh/c^2 \cos\phi}\, dh. \tag{14}$$

Evaluating (14) and substituting in (13), the terms can be collected in the same form as in (10); if we write, with $2b$ as the maximum breadth of the model,

$$a = \alpha bc/125l^4\,, \quad p = 5gl/c^2, \quad \beta = 2f/5l, \tag{15}$$

we obtain ultimately

$$\begin{aligned} R = \frac{320\pi g\rho b^2 l\alpha^2}{p^3} \int_0^{\pi/2} \Big[& \cos\phi + \frac{18}{p^2}\cos^3\phi + \frac{432}{p^4}\cos^5\phi + \frac{7200}{p^6}\cos^7\phi \\ & - 6\left(\frac{1}{p}\cos^2\phi - \frac{128}{p^3}\cos^4\phi + \frac{1200}{p^5}\cos^6\phi\right)\sin(p\sec\phi) \\ & - 6\left(\frac{17}{p^2}\cos^3\phi - \frac{528}{p^4}\cos^5\phi + \frac{1200}{p^6}\cos^7\phi\right)\cos(p\sec\phi)\Big] e^{-\beta p\sec^2\phi}\, d\phi. \end{aligned} \tag{16}$$

We may now compare (10) and (16) as regards the matter under discussion. We imagine the resistance graphed as a function of the velocity, and we compare the relative magnitude of the oscillations superposed upon the mean curve. The terms in (10) and (16) which give rise to these oscillations are the

terms factored by sin $(p \sec \phi)$ or cos $(p \sec \phi)$. For large values of p, we have an asymptotic expansion of any of these terms in the form

$$\int_0^{\pi/2} \cos^n \phi \,.\, e^{ip \sec\phi - \beta p \sec^2\phi}\, d\phi \sim p^{-\frac{1}{2}} e^{ip} \left(a_0 + a_1 p^{-1} + a_2 p^{-2} + \ldots\right). \tag{17}$$

Moreover, in practice, the interference effects concerned are prominent for larger values of p, say, for the range 10 to 40. Now from (10) we see that the expansion of the oscillatory terms would begin with a term of order $p^{-\frac{1}{2}}$, while from (16) the lowest term is of order $p^{-\frac{3}{2}}$. It follows, therefore, that the interference effects have been largely eliminated by the alteration made in the form of the model. It may be noted that the alteration is rather extreme if considered as an illustration of practical conditions, in that the after end of the model is cut away completely to zero angle; this accounts for the complete absence of the term in $p^{-\frac{1}{2}}$ in the expression for the second case.

6. To examine the matter graphically, it is easier to consider a model of infinite draught, and of small ratio of beam to length, in the manner used in previous papers. The model is assumed to be symmetrical about a longitudinal vertical plane. Take Ox horizontally in this plane, and let Oy be also horizontal. The form of the horizontal cross-section of the model is constant; if its equation is

$$y = \mathrm{F}(x), \tag{18}$$

for positive values of y, the approximation consists in taking the doublet distribution of (4) so that

$$2\pi\, \partial\psi/\partial x = c\partial \mathrm{F}/\partial x. \tag{19}$$

Integrating (4) by parts with respect to h and h', substituting from (19), and also integrating with respect to f and f', we have

$$\mathrm{R} = \frac{4\rho c^2}{\pi} \int \frac{\partial \mathrm{F}}{\partial h}\, dh \int \frac{\partial \mathrm{F}'}{\partial h'}\, dh' \int_0^{\pi/2} \cos\left\{\frac{g\,(h-h')}{c^2} \sec\phi\right\} \cos\phi\, d\phi. \tag{20}$$

We wish to contrast two models which have the front half the same, but with the rear end smoothed off to a finer point in one case than in the other. We shall take the section of unsymmetrical form to be given by

$$y = (b/4l^3)\,(l-x)\,(2\,l+x)^2\,; \quad -2l < x < l. \tag{21}$$

For the symmetrical model we shall take the front portion to be given by (21) for x positive and by the corresponding expression for x negative. The model in one case is of length $2l$ and in the other of length $3l$. The cross-sections by a horizontal plane are shown in fig. 1.

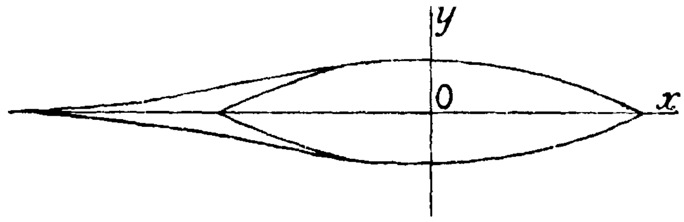

FIG. 1.

7. Taking the symmetrical case first, we obtain from (20),

$$R = 9\rho b^2 c^2 \pi^{-1} \int_0^{\pi/2} J^2 \cos\phi \, d\phi, \tag{22}$$

where

$$J = \int_0^1 (2u + u^2) \sin(glu/c^2 \cos\phi) \, du. \tag{23}$$

From these we have, after reduction, and writing p for $2gl/c^2$,

$$\begin{aligned} R = \frac{324g\rho b^2 l}{\pi p^3} \Big\{ & \frac{2}{3} + \frac{128}{135}\frac{1}{p^2} + \frac{1024}{105}\frac{1}{p^4} + P_3(p) - \frac{16}{3}\frac{1}{p} P_4(p) \\ & + \frac{112}{9}\frac{1}{p^2} P_5(p) + \frac{128}{9}\frac{1}{p^3} P_6(p) + \frac{64}{9}\frac{1}{p^4} P_7(p) - \frac{32}{3}\frac{1}{p^2} P_5(\tfrac{1}{2}p) \\ & + \frac{256}{9}\frac{1}{p^3} P_6(\tfrac{1}{2}p) - \frac{256}{9}\frac{1}{p^4} P_7(\tfrac{1}{2}p) \Big\}, \end{aligned} \tag{24}$$

with the notation

$$P_{2n}(p) = (-1)^n \int_0^{\pi/2} \cos^{2n}\phi \sin(p \sec\phi) \, d\phi,$$

$$P_{2n+1}(p) = (-1)^{n+1} \int_0^{\pi/2} \cos^{2n+1}\phi \cos(p \sec\phi) \, d\phi.$$

Using sequence relations for the P functions, we reduce (24) to a form involving only P_3, P_4 and P_5; tabulated values of these have been given previously,* and in addition large-scale graphs of the three functions were available over the range of p from zero to 40. These graphs have been used also in the present calculations; the reduced form of (24) from which these have been made is

$$\begin{aligned} R = \frac{324g\rho b^2 l}{\pi} \Big\{ & \frac{2}{3} + \frac{0\cdot9482}{p^2} + \frac{9\cdot752}{p^4} + \left(1 + \frac{2\cdot54}{p^2}\right) P_3(p) \\ & - \left(\frac{5\cdot333}{p} + \frac{11\cdot684}{p^3}\right) P_4(p) + \left(\frac{14\cdot984}{p^2} - \frac{6\cdot1}{p^4}\right) P_5(p) \\ & + \frac{2\cdot2}{p^2} P_3(\tfrac{1}{2}p) - \frac{24\cdot04}{p^3} P_4(\tfrac{1}{2}p) - \left(\frac{8\cdot466}{p^2} - \frac{24\cdot4}{p^4}\right) P_5(\tfrac{1}{2}p) \Big\}. \end{aligned} \tag{25}$$

* 'Roy. Soc. Proc.,' A, vol. 108, p. 82 (1925)

The graph is shown in curve A of fig. 2, the base being $c/\sqrt{(2gl)}$.

8. For the unsymmetrical model of fig. 1, we have

$$\mathrm{R} = (9\rho b^2 c^2/4\pi)\int_0^{\pi/2} (\mathrm{I}^2 + \mathrm{J}^2)\cos\phi\, d\phi, \tag{26}$$

where

$$\mathrm{I} + i\mathrm{J} = \int_{-2}^{1} (2u + u^2)\, e^{iglu/c^2\cos\phi}\, du. \tag{27}$$

In this case the reductions lead to,

$$\mathrm{R} = \frac{2187 g\rho b^2 l}{4\pi p^3}\left\{\frac{2}{3} + \frac{64}{15}\frac{1}{p^2} + \frac{1152}{35}\frac{1}{p^4} - \frac{4}{p}\mathrm{P}_4(p) - \frac{28}{p^2}\mathrm{P}_5(p) - \frac{72}{p^3}\mathrm{P}_6(p) - \frac{72}{p^4}\mathrm{P}_7(p)\right\}, \tag{28}$$

where p is now $3gl/c^2$.

For purposes of calculation this is put in the form

$$\mathrm{R} = \frac{2187 g\rho b^2 l}{4\pi p^3}\left\{\frac{2}{3} + \frac{64}{15}\frac{1}{p^2} + \frac{1152}{35}\frac{1}{p^4} - \frac{96}{7p^2}\mathrm{P}_3(p) - \frac{4}{p}\left(1 - \frac{102}{7p^2}\right)\mathrm{P}_4(p) - \frac{4}{7p^2}\left(73 - \frac{108}{p^2}\right)\mathrm{P}_5(p)\right\}. \tag{29}$$

The graph of (29) is shown in curve B of fig. 2.

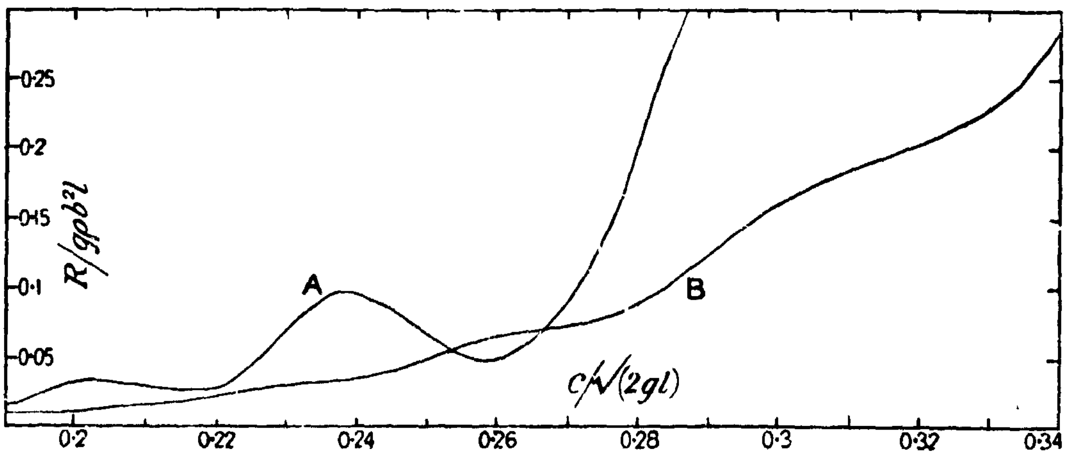

Fig. 2.

9. The curves for the two models are given with the same co-ordinates, namely, $\mathrm{R}/g\rho b^2 l$ and $c/\sqrt{(2gl)}$; since the lengths of the models are different, the maxima and minima of the superposed oscillations occur at different speeds in the two cases.

The difference in the magnitude of the interference effects is sufficiently obvious from these curves. The variation in the form of the models shown in fig. 1 is considerable, and it would have been of interest to compare forms intermediate between those shown for the rear part of the model; but equations for such curves led to expressions for the wave resistance which were too complicated for numerical calculation. However, it may be inferred that for any case in which the lines of the model are smoothed out in this manner there will be a very considerable reduction in the magnitude of the interference effects.

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SOME ASPECTS OF THE THEORY OF SHIP WAVES AND WAVE RESISTANCE

By PROF. T. H. HAVELOCK, F.R.S.

The Paper gives a general survey without detailed calculations, of attempts made during recent years to develop the mathematical theory of wave resistance. The first section is a short statement of the general problem from the theoretical point of view, while the two remaining sections describe some results which have been obtained by indirect attacks. It is shown first how calculations with travelling pressure disturbances illustrate such problems as the variation of wave resistance with speed, the interference of bow and stern waves, and the effect of shallow water. In the last section the ship is regarded as equivalent to a certain distribution of sources and sinks in the fluid; problems discussed briefly in this section are the effect of the form of the water-plane section, of the length of parallel middle body, and of varying draught. Curves are reproduced which show the results of these calculations, and some mathematical notes and further references are given in an appendix.

NEARLY forty years ago Lord Kelvin delivered to the Institution of Mechanical Engineers a lecture on ship waves which is familiar to all students of this subject. I may venture to appropriate a paragraph from that lecture and to quote it now in addressing this society: "I must premise that, when I was asked by the Council to give this lecture, I made it a condition that no practical results were to be expected from it. I explained that I could not say one word to enlighten you on practical subjects, and that I could not add one jot or tittle to what had been done by Scott Russell, by Rankine, and by the Froudes, father and son, and by practical men like the Dennys, W. H. White, and others; who have taken up the science and worked it out in practice."

My object is to discuss the wave resistance of ships as a problem in hydrodynamics. It is, of course, impossible to do so adequately without the use of mathematical analysis which would be unsuitable for a general lecture. I must therefore be content to give a mere outline sketch, aiming at giving some idea of the theoretical point of view and of the sort of contribution which mathematical theory can make to the scientific discussion of our problem. Such an outline suffers inevitably from two drawbacks: on the one hand we can only glance at the various practical problems which are suggested, and on the other we are not able to do justice to the mathematical interest of the theoretical treatment. It may, however, serve in some measure

its main purpose of being a general account which may be of value to the student of this aspect of the science and at the same time be of interest to those who have not the opportunity of studying the mathematical theory for themselves.

Some formulæ and references will be found in notes appended to this lecture, and I am indebted to the Royal Society for permission to reproduce the diagrams.

The General Problem.

We wish to know completely the fluid motion produced in the water when a ship is towed along at constant speed, and the first step is to see what information is necessary before we can attempt to find a solution. We can group this under three heads: (1) the laws of motion of the fluid, (2) the forces acting throughout, and (3) the conditions at the boundaries of the fluid. We are faced at the outset with the difficulty of saying what are the laws of motion of an actual liquid such as water. We know that water is viscous and we can write down equations taking the viscosity into account; and we can also solve the equations in simple cases if the velocities are not too large. But we also know, unfortunately, that those solutions break down completely when the motion becomes eddying or turbulent. It is not my intention to discuss here whether the difficulties arise because the solutions of the equations of viscous motion are inadequate or because the equations themselves are incomplete; in either case the inclusion of fluid friction in our problem would complicate it so much as to make progress almost impossible at present.

We are therefore compelled to assume the liquid to be frictionless. This is no doubt a serious limitation, but perhaps not so important if we confine ourselves meantime to qualitative and comparative conclusions from our results. Moreover the direct influence of viscosity upon the wave motion is comparatively small, and indirect effects might possibly be allowed for later by some adjustment of the effective form of the ship. However that may be, we can only make any advance by separating frictional resistance from wave resistance, and we therefore assume the information required under the first head to be the laws of motion of a frictionless liquid; these are equations connecting pressure, velocity, and acting forces, and their rates of change throughout the liquid. We may dispose of the second head by simply taking the acting forces to be those due to gravity. Under the third head, the conditions at the boundaries are of two kinds; at the free upper surface of the water the pressure must be the atmospheric pressure, while at the wetted surface of the ship the condition is simply that the water must remain in contact with the ship or that the component velocity of the water at right angles to the wetted surface must equal at each point the component of the ship's velocity in that direction.

Our problem is now stated in a form in which we know, from general theory, that we have all the information necessary for a complete solution; this solution would give us the velocity and pressure at every point of the water, the form of the free surface or the wave pattern, and moreover the resultant of the fluid pressures on the surface of the ship would give the wave resistance.

It is instructive to bear in mind the general problem so stated, but it must be confessed at once that the direct attack leads to calculations which have hitherto proved far too complicated for the mathematical methods available. Even if we replace the ship's surface by simple geometrical forms, the problem is extremely difficult; in fact the only direct solutions obtained so far, and they are approximate, are for spheres and other bodies of simple form entirely submerged at some distance below the surface.

It might appear that we have not gained much from our rigorous formulation of the problem, and no doubt it is not often the case that a practical problem admits of a direct and complete theoretical solution. But theory is usually built up by devising and solving simple cases; these often give in themselves valuable suggestions, and we may then endeavour to approximate more and more closely to the actual problem. The preliminary survey is necessary to guide this process along lines which are likely to prove useful.

My main task is to describe now some indirect attacks which have been made, and I shall consider these in two groups. In one case the leading idea is the pressure between the water and each element of the wetted surface of the ship, while in the other we fix our attention more upon the horizontal velocity produced in the water by the motion of the ship through it.

Travelling Pressure Disturbance.

When the ship is in steady motion there is a definite normal pressure at each element of the wetted surface. From a dynamical point of view, that is the function of the ship. We could imagine those pressures to be supplied by any means we please, for instance by jets of air properly adjusted, and the motion of the water would be exactly the same. We have now removed the ship and have applied to the surface of the water a definite distribution of pressure, definite for each velocity be it noted. The solution of this problem would give us the form taken by the surface of the water; one part of this would necessarily be a depression of the same form as the ship, while the rest would be the accompanying wave pattern. Now this is merely the general problem over again, with the complication that the pressure distribution depends upon the speed. But it suggests that we should study the wave patterns produced by simple distributions of pressure applied normally to the water surface.

Localized Pressure.—To begin with the simplest type we may picture a fine jet of air impinging on the water surface; we could call this in the extreme case a point pressure system, or more generally a distribution of surface pressure symmetrical round a vertical axis. We may imagine the jet to move horizontally with constant speed, or we may study the equivalent problem of a stationary jet of air directed down on to the surface of a uniform stream. Everyone is familiar with the simple and beautiful wave pattern produced in this way, and we are encouraged to proceed with this line of attack by the fact that the pattern is so similar in its main features to the waves produced by a ship.

The mathematical solution of this problem can be obtained completely, provided the surface waves are not too large; the wave pattern shows the well-known transverse and diverging waves contained within lines making angles of about 19° 28′ on either side of the line of motion of the system. Leaving on one side the discussion of the wave

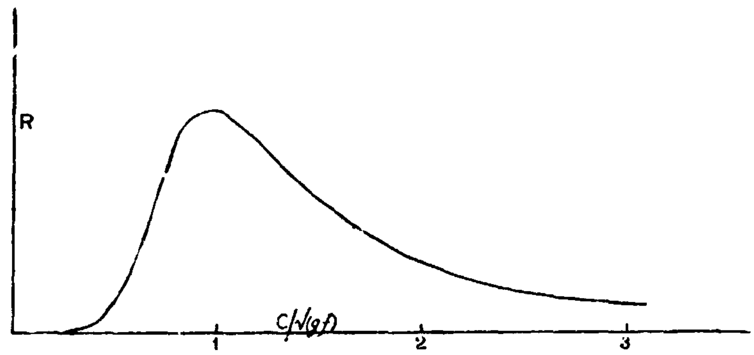

FIG. 1.

system let us consider what is perhaps less familiar, the corresponding wave resistance. We are considering a pressure system applied to the water surface and moving horizontally with constant velocity; accompanying the system there is a steady wave pattern. Suppose now that we place over the whole surface of the water a smooth rigid cover exactly fitting the surface at every point, and let this cover move horizontally with the same velocity. We could now remove the jet, or other means by which we applied the pressure system, for this function will now be performed by the rigid cover; and the fluid motion will be exactly the same as before. Moreover, at all those outlying parts where the surface pressure is the same as atmospheric pressure, the cover could obviously be cut away; and we are left with what corresponds to the ship in this problem. Let me repeat that in the actual ship problem we are given an assigned depression in the water surface, namely, the surface of the ship; we have replaced this by a problem in which the pressure distribution is assigned and the ship is, so to speak, made to fit the surface disturbance. The reason for doing this

is simply that the latter problem can be solved mathematically in certain cases.

It is clear that the wave resistance is the resultant of the surface pressures when resolved in a direction opposite to that of the motion. These calculations have been carried out (*Note* 1), but we shall only consider here the graphical form of the results

Fig. 1 shows the variation of wave resistance R with the velocity. The pressure system is of a certain localized type, symmetrical round a centre which moves over the surface with constant velocity c; the quantity f is a length which may be called the effective radius of the applied pressure system. There are various points of interest in this curve, but I shall only mention one or two which have their analogues in ship resistance. Notice that the wave resistance is very small at low speeds. Then it begins to increase rapidly and reaches a maximum

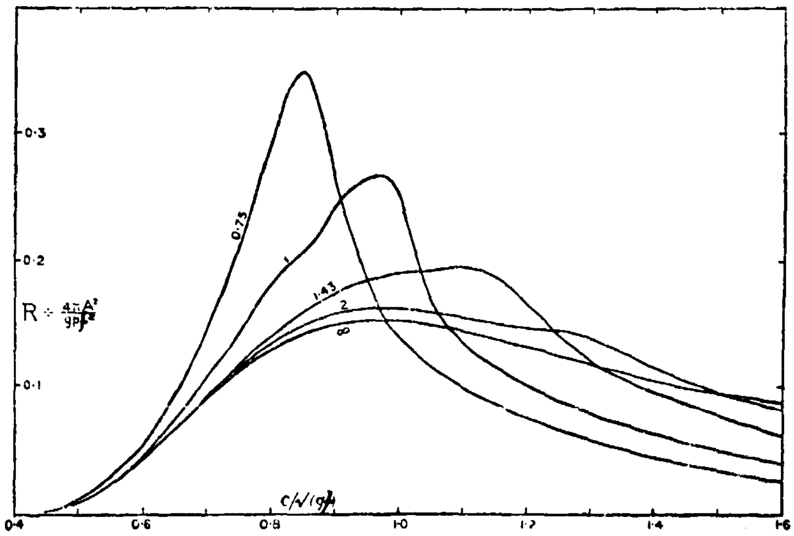

Fig. 2.

when the speed c is about equal to $\sqrt{(gf)}$; this means that the wave resistance is a maximum when the length of the transverse waves produced is of the same order as the length of the pressure system. After this stage the resistance decreases gradually to zero. A little consideration will show that this last result might have been anticipated; it may be described as a sort of planing or smoothing action of the pressure system when the velocity becomes very large.

Shallow Water.—Before we leave this elementary pressure system we may use it in another interesting problem. We have assumed so far that the water is very deep, but we can examine the effect of shallow water by adding the condition that at the bottom of the water the vertical velocity must vanish. The work becomes more difficult but formal solutions can be obtained and calculations made from them (*Note* 2). We know that on water of depth h the speed of transverse waves cannot exceed the value $\sqrt{(gh)}$, which is the speed of the so-called

wave of translation. The waves produced by our travelling pressure system agree in character with this fact. Below the speed $\surd(gh)$ the wave pattern is similar to that in deep water, the heights of the waves being increased; but at higher speeds the transverse waves have disappeared and the pattern is made up of diverging waves only.

Here are some curves, in Fig. 2, which show the corresponding changes in the wave resistance. The numbers marking the different curves are the ratios of the depth of water h to the length f which measures the linear dimensions of the applied pressure system; each curve gives the variation of wave resistance with velocity for a given depth of water. The curve marked ∞ is the curve for deep water which we have already discussed. The progressive changes in the curves as the depth is diminished should be noted; but consider in particular the curve marked 0·75. Notice the greatly increased resistance compared with deep water so long as the speed is less than a certain value, and the rapid fall after that point with the resistance ultimately becoming less than in deep water. The velocity at which the change

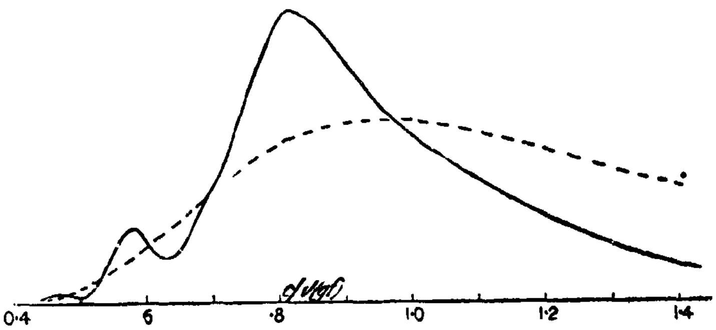

FIG. 3.

takes place in this case is, from the graph, about $0{\cdot}86\surd(gf)$; and, as the depth h is $0{\cdot}75\ \surd(gf)$, this velocity is practically equal to $\surd(gh)$, the speed of the wave of translation. This result is in general agreement with various recorded experiments on the effect of shallow water on the wave resistance of ships.

Interference Effects.—Returning to the easier case of deep water, we can illustrate the interference of bow and stern wave systems. We shall call a system in which the applied pressures exceed atmospheric pressure a positive pressure system, and one in which they are less than atmospheric pressure a negative system. Let the travelling system consist of a positive system of the kind we have been considering together with an equal negative system at a fixed distance to the rear of the positive one. The combined wave pattern is obtained simply by superposing the waves due to the two systems separately, and an expression for the wave resistance can also be obtained (*Note* 3). The resistance is not the sum of the resistances due to the two systems separately, otherwise there would be none of the so-called interference

effects; the combined effect oscillates about the mean sum according to the positions of the crests and troughs of one wave system relative to those of the other system. Fig. 3 shows a graph of the wave resistance calculated for a certain case of this combination; it shows the typical humps and hollows, and the mean curve.

It may be asked why we illustrate the wave-making action of bow and stern by positive and negative pressure systems respectively, instead of by two positive systems or two negative ones. The best answer to this question seems to be that we find that this combination gives the humps and hollows on the resistance curve in the same sort of sequence as for a ship. Another way of expressing it is this: we know from observation that the bow and stern produce wave patterns which are similar in character except that where there are crests in one pattern there are troughs in the same relative positions in the other pattern, and vice versa; the simplest combination of pressure systems which gives the same effect is obviously the one we have used, one system being positive and the other negative.

General Pressure System.—We might now attempt similar calculations for a continuous distribution of pressure such as would be associated with the motion of a ship. So far these have only been carried out in certain cases of two-dimensional fluid motion, that is when the wave motion consists only of straight-crested transverse waves; we need not consider these in detail here (*Note* 4). One point should be mentioned to avoid possible confusion. We have already remarked that the action of bow and stern is similar to that of positive and negative pressure systems. But the actual continuous distribution of pressure round a ship is different; it is symmetrical fore and aft of the midship section as far as its general character is concerned. The excess pressure begins by being positive near the bow, it then decreases rapidly to a negative value, remains more or less constant over the middle length of the ship, and then increases rapidly to a positive value again near the stern. Now a little consideration shows that the places which give the main part of the wave effect of the whole system are not the regions where the pressure is uniform, whether it is positive or negative, but those places where the pressure is changing rapidly. Here we have near the bow a rapid change from positive to negative, while at the stern the change is from negative to positive; the nett result is that in the wave patterns arising from bow and stern respectively the relative positions of crest and trough are interchanged.

One recognizes that the results which have been reviewed in this section are necessarily only illustrative of the actual ship problem. They are nevertheless interesting and suggestive, and students of the subject will be familiar with the use that has been made by various writers of the notion of pressure distribution in interpreting curves of wave resistance obtained from experiments with ship models.

Distributions of Sources and Sinks.

Let us consider now another method of treating the wave-making action of a ship. It is obvious that the bow and entrance of the ship produce in the water an outwards horizontal velocity on either side, while the run and stern give rise to component velocities inwards. We can also see that the same sort of effect will be produced if we remove the ship and replace it by some apparatus which supplies water where the velocities are outwards and removes it where they are inwards. This picture suggests one of the most fruitful devices in hydrodynamics, the study of the motion produced in a fluid by the presence of sources and sinks, that is points at which fluid is introduced or withdrawn at a uniform rate symmetrically round each point. Just as in the previous section we might begin with simple cases, for example a source travelling at uniform speed at a constant depth below the surface and followed at a fixed distance by an equal sink. The wave motion produced by this combination can be calculated; and we can generalize the results, with certain limitations, for any distribution of sources and sinks. We need not delay over the simpler cases, but let us see now how we may use this idea in the ship problem.

FIG. 4.

Consider the vertical section of the ship by the median plane running from bow to stern. We replace the ship by a distribution of sources and sinks over this vertical section, so arranged that the horizontal velocity outwards or inwards at each point is equal to the same component of the velocity of the corresponding element of the ship's surface at right angles to itself. This is, of course, an approximation; the chief limitation is that we must assume the lines of the ship to be fine, so that the angle between the ship's surface and the vertical median plane is small.

Without going into the details of any one problem, I shall describe now some results obtained from three sets of calculations made on these general assumptions.

Form of Water-plane Section.—Suppose that we wish to examine the relative effect of making the lines at the bow finer and increasing the beam of the ship, the displacement being constant. We shall simplify the work by assuming the draught to be infinite, which means simply that it is large compared with the wave length at the highest speed; we are not concerned with absolute values of resistance, and

this assumption is not likely to affect much the comparative values. We imagine the ship to have vertical sides and constant horizontal section; and we consider a series of models in which the length is constant, the beam and the lines altering in such a way that the area of the water-plane section is unaltered. Calculations have been made for four models in which the lines can be expressed by simple mathematical formulæ so that these conditions are satisfied (*Note* 5). Fig. 4 shows a quarter of the water-plane section for the two extreme models of the set and the Table gives some further details.

MODELS OF CONSTANT LENGTH AND DISPLACEMENT.

Model.	Beam.	Water-plane coeff.	Bow and stern lines.
A	1·0	0·667	straight
B	1·042	0·64	straight
C	1·072	0·62	Hollow
D	1·136	0·587	Hollow

The calculated curves of wave resistance for these four models are shown in Fig. 5.

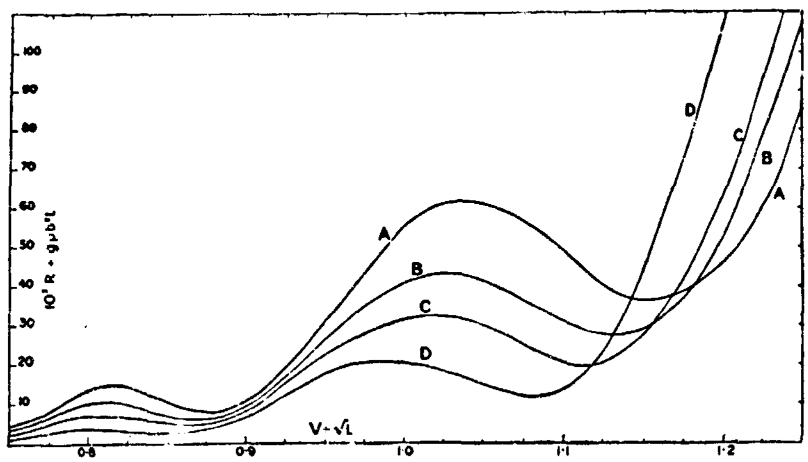

FIG. 5.

The ordinates are the wave resistance R on a certain scale, while the base is $V/\sqrt{L}$ where V is the speed in knots and L the length in feet. Look at the curve A, which belongs to the model of more normal lines. There are the typical humps and hollows due to interference, enormously exaggerated in value, but they occur at values of $V/\sqrt{L}$ which agree sufficiently well with experiment; for instance, there is a prominent hump at $V = 1{\cdot}04\sqrt{L}$. We can trace also from the set of curves the general effect of putting the displacement more amidships. The chief point of interest is the intersection of these curves in pairs of values of $V/\sqrt{L}$ ranging from 1·12 to 1·18. Now this set of

calculations corresponds to a simplified form of certain well-known sets of experiments with ship models, into the details of which we need not enter. It may be sufficient to quote one example, which is typical of the results. D. W. Taylor, referring to a series of experiments with models of the same displacement and of varying midship-section coefficients, states that the models with full midship-section coefficients drive a little easier up to $V/\sqrt{L}$ equal to 1·1 to 1·2, and

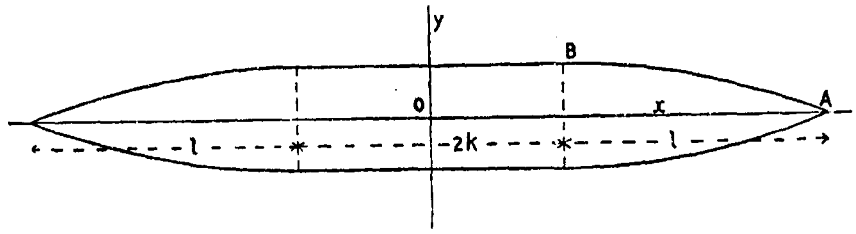

FIG. 6

the models with fine coefficients have a shade the best of it at higher speeds. The agreement with the intersections of the curves in Fig. 5 is rather striking.

Parallel Middle Body.—Take now the simple form of model A and insert varying lengths of parallel middle body between bow and stern, so that the water-plane section is like Fig. 6.

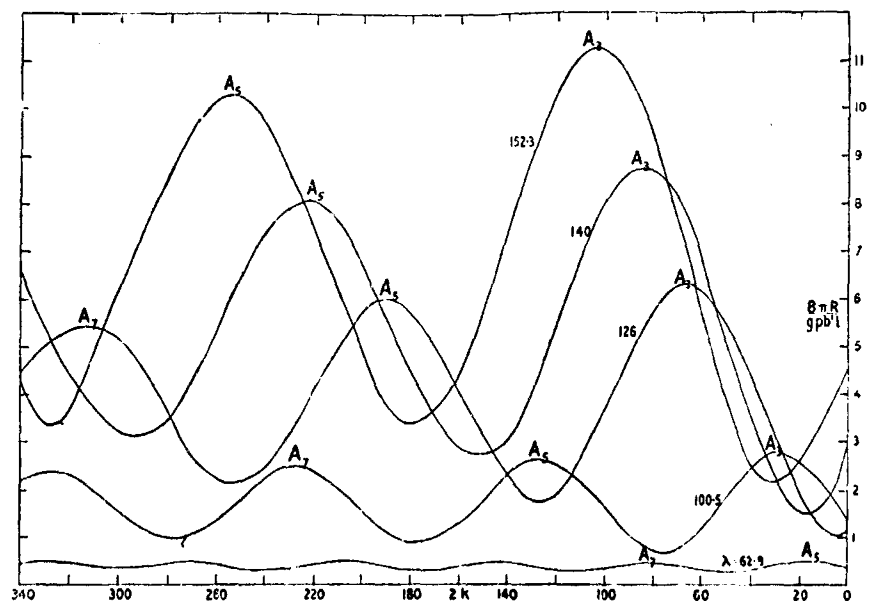

FIG. 7.

The calculations lead to curves showing how the wave resistance at a given speed varies with the length of parallel middle body (*Note* 6). Some curves are shown in Fig. 7. The base is the length ($2k$) of parallel middle body, and the entrance and run were each taken to

be 80 feet; these lengths were chosen simply because they were those adopted by W. Froude in recording the results of his original experiments on this effect. The number marking each curve is the wave-length of transverse waves at the speed for that curve.

We shall only compare these curves with experimental results in one respect, namely the positions of the maxima and minima, a matter about which there has been considerable discussion recently. There have been two interpretations of the experimental results put forward. On both of them the bow wave system is supposed to begin with a crest and the stern system with a trough, positive and negative systems as we have called them; therefore there will be a maximum on a resistance curve when there is an odd number of half wave-lengths between this crest and this trough. The difference between the two views is that in one case this distance between first bow crest and first stern trough is supposed to be constant for all speeds, while in the other it is said to increase with the speed in such a way that the increase in this distance is equal to one-quarter of the increase in the corresponding wave-length. Let us follow some particular maximum on the curves of Fig. 7, say A_3; on both views this corresponds to three half wave-lengths between the first bow crest and the first stern trough. On one theory the quantity $\frac{3}{2}\lambda - 2k$ should be independent of the speed, while on the other it should increase at the same rate as $\frac{1}{4}\lambda$ and therefore the quantity $\frac{5}{4}\lambda - 2k$ should be constant; λ is the wave-length for a given speed and $2k$ is the length of parallel middle body at which the maximum A_3 occurs at that speed. Taking the values from Fig. 7, and adding other results obtained by further calculations, we get the following table:—

$2k$	λ	$\frac{3}{2}\lambda - 2k$	$\frac{5}{4}\lambda - 2k$
0	84	126	105
28	100·5	123	98
66	126	123	92
85	140	125	90
125	167·5	126	84
171·5	201	131	81
196	218·6	132	77
244	251·3	133	70
328	314	143	65

According to this Table, neither of these quantities is constant. Calling $\frac{3}{2}\lambda - 2k$ the wave separation, it is interesting to notice that the wave separation decreases slightly at first with increasing speed;

this is an effect which we find more pronounced if we follow a higher order of maximum such as A_5 or A_7. After a range over which the wave separation is approximately constant, it ultimately increases with the speed but at a slower rate than that required by the quarter wave-length theory. Such are the results for the simplified form of model we have used; it is quite possible, of course, that different rates of variation might be obtained if the calculations could be made for forms more like actual ship models. A similar remark may be made at the same time about empirical formulæ derived from experimental results; it is not as a rule justifiable to extend these formulæ beyond the range from which they were obtained.

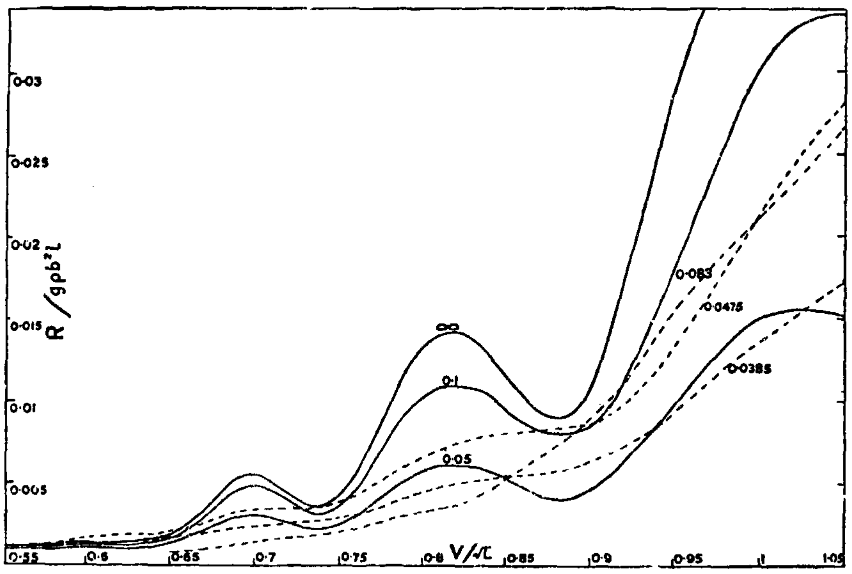

FIG. 8.

Varying Draught.—As a last example of this set of calculations let us find how the resistance of model A, without parallel body, varies when we alter the draught (*Note* 7). Hitherto we have taken the draught to be so large that it might be assumed infinite. We now cut the model off by a horizontal plane, so that it still has vertical sides and constant horizontal section; but we take the draught to be first one-tenth and then one-twentieth of the length.

Fig. 8 shows the three curves, marked with the ratio of draught to length. There is little difference at low speeds until the wave-length becomes comparable with the draught. An interesting point is that the humps and hollows occur at practically the same speeds in the three curves; one may compare this with the observed effect that for a submerged model the resistance curve gives humps and hollows at the same speeds independently of the depth at which the model is run.

The ratios one-twentieth and one-tenth cover roughly the ratios of draught to length which occur in practice. We may then compare these

curves with experimental results to see whether absolute values are reasonably of the right order of magnitude; we cannot expect more when we remember the simplified form of the model and the other limitations of the theory.

The three dotted curves in Fig. 8 are experimental curves of residuary resistance, the number marking each curve being the ratio of draught to length. The curves 0·0475 and 0·0385 have been drawn, on the scales used in Fig. 8, from those given by R. E. Froude for ships of 400 feet length of 5,390 and 4,090 tons displacement respectively; while the curve 0·083 has been deduced from some results given by J. L. Kent. We notice at once how much more prominent the interference effects are on the theoretical curves; this is probably due chiefly to the neglect of fluid friction, whose indirect effect may be equivalent to an altered distribution of velocity in the present calculations. The effect of differences of form, other than that expressed by the ratio of draught to length, is also obvious from the dotted curves. When we remember that the calculated curve, say that marked 0·05, is for a simple form not specially fitted to any actual model, the general agreement of order of magnitude over a considerable range of velocity is sufficient at least to justify the fundamental assumptions of the theory.

It is perhaps needless to add that we are very far indeed from being able to predict or to calculate in advance the wave resistance of an actual ship. Nevertheless our chief aim will have been achieved if we have gained more insight into the nature of the problem; for in this respect at least, the pursuit of theoretical investigations, even if apparently remote from practical requirements, is essential to a complete and scientific solution of the various problems of ship motion.

NOTES AND REFERENCES.

1.—The effect of a travelling surface pressure can be obtained by regarding it as a succession of applied impulses and by integrating suitably the expressions for the effect of a single impulse. Take axes Ox and Oy in the undisturbed water surface and Oz vertically upwards; let ζ be the surface elevation and let O move with uniform velocity c in the direction Ox. If the pressure distribution is symmetrical round O and is given by

$$p = \mathrm{F}(r),\ r^2 = x^2 + y^2, \qquad (1)$$

the surface elevation can be obtained in the form

$$g\rho\zeta = -\int_0^\infty e^{-\frac{1}{2}\mu u}du \int_0^\infty f(\kappa)\,\mathrm{J}_0\left[\kappa\sqrt{\left\{(x+cu)^2+y^2\right\}}\right]\sin(\kappa \mathrm{V}u)\kappa^2 d\kappa, \qquad (2)$$

where μ is to be made zero after the integrals have been evaluated; further $\mathrm{V}^2 = g/\kappa$, and

$$f(\kappa) = \int_0^\infty \mathrm{F}(a)\,\mathrm{J}_0(\kappa a)\,a\,da, \qquad (3)$$

J_0 being the Bessel Function of zero order.

From the definition of wave resistance given in the text, assuming the slope of the surface to be small, we have

$$\mathrm{R} = \int \mathrm{F}(r) \frac{\delta Z}{\delta x} dS, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the integral being taken over the whole surface. The particular case for which the calculations have been made is

$$p = \mathrm{F}(r) = \mathrm{A}f/(f^2 + r^2)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A and f being constants. It is found that the integral (4) reduces to

$$\mathrm{R} = (4\pi g^2 \mathrm{A}^2/\rho c^6) \int_0^{\frac{\pi}{2}} \sec^5 \phi \, e^{-2(gf/c^2)\sec^2\phi} \, d\phi \quad . \quad . \quad . \quad . \quad (6)$$

This integral can be expressed in terms of Bessel functions, of which tables are available, in the form

$$\mathrm{R} = \frac{\pi^2 \mathrm{A}^2 p^3 e^{-p}}{2g\rho f^3} \left\{ i \mathrm{H}_0^{(1)}(ip) - \frac{1+2p}{2p} \mathrm{H}_1^{(1)}(ip) \right\}, \quad . \quad . \quad . \quad . \quad (7)$$

where $p = gf/c^2$. This is the expression whose graph is given in Fig. 1. (*Proc. Roy. Soc. A*, 95, p. 354. (1919).

2.—With the same notation, and with h as the depth of water, instead of (6) we now have

$$\mathrm{R} = \frac{4\pi \mathrm{A}^2 c^3}{\rho} \int_{\phi_0}^{\frac{\pi}{2}} \frac{\kappa^3 e^{-2\kappa f} \sec\phi \, d\phi}{g \sec^2\phi (c^2 - gh \sec^2\phi) + \kappa^2 c^4 h}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where κ satisfies the equation

$$\kappa c^2 = g \sec^2 \phi \tanh \kappa h.$$

The lower limit ϕ_0 is to be taken zero if $c^2 < gh$, and to be the value of arc cos $\sqrt{gh/c^2}$ if $c^2 > gh$. The integral (8) was evaluated by graphical methods, the integrand being graphed on a certain base and areas taken by an Amsler planimeter. The process was carried out for the different values of the ratio h/f shown in Fig. 2. (*Proc. Roy. Soc. A.*, 100, p. 499. 1922.)

3.—With h as the distance between the centres of the two pressure systems, the integral for the wave resistance is

$$\mathrm{R} = (16\pi g^2 \mathrm{A}^2/\rho c^6) \int_0^{\frac{\pi}{2}} \sec^5 \phi \, e^{-2(gf/c^2)\sec^2\phi} \cos^2\left\{ (gh/c^2) \sec\phi \right\} d\phi \quad . \quad . \quad (9)$$

The particular case shown in Fig. 3 is for $h = 2f$, the integral being evaluated by numerical methods. (Reference as in Note 1.)

4.—For a study of some cases, with further references, see *Proc. Roy. Soc. A.*, 89, p. 489. 1914.

5.—The general expression for any distribution of sources and sinks is found by beginning with a doublet of given moment at a given depth in the liquid, with its axis parallel to Ox. The results are generalized by integration for any continuous distribution of such doublets in the plane $y = 0$, the moment per unit area in this plane being $\psi(x, z)$; this gives for the wave resistance the expression

$$\mathrm{R} = 16\pi g^2 \rho c^{-4} \int_{-\infty}^{0} dz \int_{-\infty}^{0} dz' \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \int_0^{\frac{\pi}{2}} \delta\psi/\delta x \,.\, \delta\psi'/\delta x' \,.$$

$$\times \sec^3\phi \, e^{\{g(z+z')/c^2\}\sec^2\phi} \cos\left[\left\{ g(x - x')/c^2 \right\} \sec\phi \right] d\phi \quad .. \quad . \quad (10)$$

This distribution of doublets gives over the plane $y = 0$ a normal distribution of velocity of amount $2\pi\delta\psi/\delta x$. Taking the plane $y = 0$ as the median fore and aft plane of the ship, and taking the ship's surface to be given by $y = F(x, z)$, we have, with the assumptions in the text, to substitute $2\pi\delta\psi/\delta x = c\delta F/\delta x$ in (10) to obtain the wave resistance. The curves of Fig. 4 for the form of the water-plane section are particular cases of the equation

$$y = \frac{b}{1 - \frac{1}{5}d^2}\left(1 - \frac{x^2}{l^2}\right)\left\{1 - \tfrac{1}{4}d^2\left(1 - \frac{x^2}{l^2}\right)\right\} \quad . \quad . \quad . \quad . \quad (11)$$

Here $2l$ is the constant length of the ship and $\frac{4}{3}bl$ the constant area of the water-plane section; the beam is $2b(1 - \frac{1}{4}d^2)/(1 - \frac{1}{5}d^2)$ The four models are the cases $d = 0$, 1, 1·25 and 1·5 respectively. Evaluating as far as possible the integrals in (10) for the form given in (11) we obtain

$$\begin{aligned} R = \frac{512 g\rho b^2 l}{\pi(1 - \frac{1}{5}d^2)p^3}\Big[& \tfrac{1}{3}(1 - \tfrac{1}{4}d^2)^2 + \tfrac{16}{15}(1 + 2d^2 - \tfrac{1}{3}d^4)\frac{1}{p^2} \\ & + \tfrac{512}{35}\frac{d^2}{p^4} + \tfrac{10384}{315}\frac{d^4}{p^6} + \tfrac{1}{2}(1 - 3d^2)^2 P_3 - \frac{2}{p}(1 - \tfrac{4}{3}d^2 + \tfrac{1}{3}d^4)P_4 \\ & + \frac{4}{p^2}(1 - 6d^2 + \tfrac{7}{3}d^4)P_5 - \frac{32}{p^3}d^2(1 - \tfrac{2}{3}d^2)P_6 - \frac{64}{p^4}d^2(1 - 2d^2)P_7 \\ & + \frac{128d^4}{p^5}P_8 + \frac{256d^4}{p^6}P_9\Big] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12) \end{aligned}$$

where $p = 2gl/c^2$ and the functions P are defined by

$$P_{2n}(p) = (-1)^n \int_0^{\frac{\pi}{2}} \cos^{2n}\phi \sin(p \sec\phi)d\phi$$

$$P_{2n+1}(p) = (-1)^{n+1} \int_0^{\frac{\pi}{2}} \cos^{2n+1}\phi \cos(p \sec\phi)d\phi.$$

After preliminary computation of these new functions, it was possible to calculate R from (12) for the four given values of d and for sufficient values of p in each case to give the curves of Fig. 5. (*Proc. Roy. Soc. A.*, 103, p. 571. 1923.)

6.—The equation of AB in Fig. 6 is

$$y = b\left\{1 - (x - k)^2/l^2\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In this case the integrals of (10) give, with the same notation,

$$\begin{aligned} R = \frac{512\, g\rho b^2 l}{\pi p^3}\Big[& \tfrac{1}{3} + \tfrac{32}{15}\frac{1}{p^2} - \frac{2}{p}P_4(\tfrac{1}{2}p) + \frac{4}{p^2}P_5(\tfrac{1}{2}p) \\ & + \tfrac{1}{2}P_3(p_1) - \frac{2}{p}P_4(p_1) + \frac{2}{p^2}P_5(p_1) + \frac{2}{p}P_4(p_2) \\ & - \frac{4}{p^2}P_5(p_2) + \frac{2}{p^2}P_5(p_3)\Big], \quad . \quad . \quad . \quad . \quad . \quad . \quad (14) \end{aligned}$$

where $p = 2gl/c^2$, $p_1 = g(2k + 2l)/c^2$, $p_2 = g(2k + l)/c^2$, $p_3 = 2gk/c^2$.

The curves of Fig. 7 were obtained from this formula, with $l = 80$, for the cases $g/c^2 = 0·1$, 0·0625, 0·05, 0·045, 0·04125 respectively. (*Proc. Roy. Soc. A.*, 108, p. 77. 1925).

7. The general expression (10) now gives, with $p = 2gl/c^2$, $\alpha = gd/c^2$, $\beta = d/2l$.

$$R = \frac{256\, g\rho b^2 l}{\pi p^3} \int_0^{\frac{\pi}{2}} \left(1 - e^{-\alpha \sec^2 \phi}\right)^2 \Big\{ \cos^3 \phi + \frac{4}{p^2} \cos^5 \phi$$

$$+ \left(\cos^3 \phi - \frac{4}{p^2} \cos^5 \phi\right) \cos\left(p \sec \phi\right) - \frac{4}{p} \cos^4 \phi \sin\left(p \sec \phi\right)\Big\} \, d\phi, \quad . \qquad (15)$$

After certain transformations, this expression was evaluated by approximate methods to give the curves of Fig. 8 for the cases $\beta = \infty$, 0·1 and 0·05 (*Proc. Roy. Soc. A.*, 108, p. 582. 1925.)

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The Method of Images in Some Problems of Surface Waves.

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Introduction.

1. When a circular cylinder is submerged in a uniform stream, the surface elevation may be calculated, to a first approximation, by a method due originally to Lamb for this case, and later extended to bodies of more general form: the method consists in replacing the cylinder by the equivalent doublet at its centre and then finding the fluid motion due to this doublet. In discussing the problem some years ago,* I remarked that if the solution so obtained were interpreted in terms of an image system of sources, we should then be able to proceed to further approximations by the method of successive images, taking images alternately in the surface of the submerged body and in the free surface of the stream. This is effected in the following paper for two-dimensional fluid motion, and the method is applied to the circular cylinder. It provides, theoretically at least, a process for obtaining any required degree of approximation, but, of course, the expressions soon become very complicated. It is, however, of interest to examine some cases numerically so as to obtain some idea of the degree of approximation of the first stage.

An expression is first obtained for the velocity potential of the fluid motion due to a doublet at a given depth below the surface of a stream, the doublet being of given magnitude with its axis in any direction. A transformation of this expression then gives a simple interpretation in terms of an image system. This system consists of a certain isolated doublet at the image point above the free surface, together with a line distribution of doublets on a horizontal line to the rear of this point; the moment per unit length of the line distribution is constant, but the direction of the axis rotates as we pass along the line, the period of a revolution being equal to the wave-length of surface waves for the velocity of the stream. The contribution of each part of the image system to the surface disturbance is indicated.

Before proceeding to the circular cylinder, two cases are worked out in some detail, namely, a horizontal doublet and a vertical doublet. To a first approximation these give the surface disturbance of a stream of finite depth with an obstruction in the bed of the stream; in the first case the bed of the stream is plane with a semi-circular ridge, and in the second case it has a more com-

* 'Roy. Soc. Proc.,' A, vol. 93, p. 524 (1917).

plicated form. Numerical calculations are made for both these cases, and graphs of the surface elevation are shown in figs. 1 and 2.

The second approximation for the circular cylinder is then investigated. The first stage is the surface effect due to a doublet at the centre, and the second is that due to a distribution of doublets on a certain semicircle. Expressions can be obtained for the complete surface elevation, but the calculations are limited to that part which consists of regular waves to the rear of the cylinder. The integrals are investigated and reduced to a form which permits of numerical evaluation. Calculations are carried out for various velocities for two different cases, namely, when the depth of the centre is twice, and three times, the radius. The results are tabulated for comparison, and one may estimate from these rather extreme cases the degree of approximation of the first stage. The effect of the second stage is to alter both the amplitude and the phase of the regular waves. The amplitude of the first-stage waves has a maximum for the velocity $\sqrt{(gf)}$, where f is the depth of the centre. It appears that the second stage increases the amplitude of the waves for velocities less than $\sqrt{(gf)}$ and decreases it for velocities above this value; further, the crests of the waves are moved slightly to the rear by an amount which varies with the speed. Some other possible applications of the method of images may be mentioned. For a doublet in a stream of finite depth, we can take successive images in the bed of the stream and in the free surface, and so build up the image system of a doubly infinite series of isolated doublets and of line distributions of doublets; this solution may be compared with the direct solution in finite terms which may be obtained in this case. Further, similar methods may be used for the three-dimensional fluid motion due to a doublet in a stream, and application made to the corresponding problem of a submerged sphere.

Image of Doublet in Stream.

2. We may either consider the doublet to be at rest in a uniform stream or to be moving with uniform velocity in a fluid otherwise at rest; we choose the latter alternative. Take Ox horizontal and in the undisturbed surface of the liquid, and Oy vertically upwards. Let the axes be moving with uniform velocity c in the direction of Ox, and let there be a two-dimensional doublet of moment M at the point $(0, -f)$ with its axis making an angle α with the positive direction of Ox. The velocity potential of the doublet is given by the real part of

$$\frac{\mathrm{M}e^{i\alpha}}{x+i(y+f)}. \tag{1}$$

In order to keep the various integrals convergent and so to obtain a definite result, we adopt the usual device of a small frictional force proportional to velocity and in the limit make the frictional coefficient μ' tend to zero: further, we neglect the square of the fluid velocity at the free surface.

If η is the surface elevation, the pressure equation gives the condition at the free surface,

$$\frac{\partial \phi}{\partial t} - g\eta + \mu'\phi = \text{const.}, \tag{2}$$

we have also, at the free surface,

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \phi}{\partial y}. \tag{3}$$

And as we are dealing with the fluid motion which has attained a steady state relative to the moving axes, these conditions give, in terms of the velocity potential,

$$\frac{\partial^2 \phi}{\partial x^2} + \kappa_0 \frac{\partial \phi}{\partial y} - \mu \frac{\partial \phi}{\partial x} = 0, \tag{4}$$

to be satisfied at $y = 0$. Here we have put $\kappa_0 = g/c^2$ and $\mu = \mu'/c$.

We now assume the solution to be given by

$$\phi = -i\mathrm{M}e^{i\alpha} \int_0^\infty e^{i\kappa x - \kappa (y+f)}\, d\kappa + \int_0^\infty \mathrm{F}(\kappa)\, e^{i\kappa x + \kappa y}\, d\kappa. \tag{5}$$

The first term represents the doublet (1) in an equivalent form, valid for $y + f > 0$. The function $\mathrm{F}(\kappa)$ can now be determined by means of (4), and this gives

$$\mathrm{F}(\kappa) = i\mathrm{M}e^{i\alpha}\left(1 + \frac{2\kappa_0}{\kappa - \kappa_0 + i\mu}\right)e^{-\kappa f}. \tag{6}$$

Hence the velocity potential of the image system is

$$i\mathrm{M}e^{i\alpha}\int_0^\infty e^{i\kappa x - \kappa(f-y)}\, d\kappa + 2i\kappa_0 \mathrm{M}e^{i\alpha}\int_0^\infty \frac{e^{i\kappa x - \kappa(f-y)}}{\kappa - \kappa_0 + i\mu}\, d\kappa. \tag{7}$$

By comparison with (1) and the first term in (5), it is easily seen that the first term in (7) is the velocity potential in the liquid due to an isolated doublet at the image point $(0, f)$, of moment M with its axis making an angle $\pi - \alpha$ with Ox.

To interpret the second term in (7) we put

$$\frac{i}{\kappa - \kappa_0 + i\mu} = \int_0^\infty e^{-\mu p + i(\kappa - \kappa_0)p}\, dp, \qquad \mu > 0. \tag{8}$$

We then interchange the order of integration with regard to κ and p, and integrate first with respect to κ. The second term of (7) thus becomes

$$2i\kappa_0 M e^{i\alpha} \int_0^\infty \frac{e^{-\mu p - i\kappa_0 p}}{x + p + i(f-y)} dp, \tag{9}$$

with $f - y > 0$.

By a comparison with (1), we see that the real part of (9) is the velocity potential of a line distribution of doublets along the line $y = f$, extending over the negative half of that line. The magnitude of the moment per unit length at the point $(-p, f)$ is $2\kappa_0 M e^{-\mu p}$, and the axis at that point makes with Ox an angle $\kappa_0 p - \alpha - \frac{1}{2}\pi$.

It is necessary to retain the quantity μ while manipulating the integrals, but we may put it zero ultimately and we have the following result:—The image system of the doublet M at an angle α to Ox and at depth f below the surface consists of a doublet M at the image point at height f above the surface with the axis making an angle $\pi - \alpha$ with Ox, together with a line distribution of doublets to the rear of the image point of constant line density $2\kappa_0 M$ and with the axis at a distance p in the rear making a positive angle $\kappa_0 p - \alpha$ with the downward-drawn vertical.

It is of interest to note how the parts of the image system contribute to the surface elevation. From the preceding equations we obtain

$$c\eta = \frac{2M(f\cos\alpha - x\sin\alpha)}{x^2 + f^2} + 2\kappa_0 M e^{i\alpha} \int_0^\infty \frac{e^{i\kappa x - \kappa f}}{\kappa - \kappa_0 + i\mu} d\kappa, \tag{10}$$

where the real part of the second term is to be taken.

The integral in (10) is transformed by contour integration, treating x positive and x negative separately; when μ is made zero ultimately, the complete expressions are

$$c\eta = \frac{2M(f\cos\alpha - x\sin\alpha)}{x^2 + f^2} + 2\kappa_0 M \int_0^\infty \frac{m\cos(mf - \alpha) - \kappa_0 \sin(mf - \alpha)}{m^2 + \kappa_0^2} e^{-mx} dm,$$

for $x > 0$; and

$$c\eta = \frac{2M(f\cos\alpha - x\sin\alpha)}{x^2 + f^2} + 4\pi\kappa_0 M e^{-\kappa_0 f} \sin(\kappa_0 x + \alpha)$$
$$+ 2\kappa_0 M \int_0^\infty \frac{m\cos(mf + \alpha) - \kappa_0 \sin(mf + \alpha)}{m^2 + \kappa_0^2} e^{mx} dm, \tag{11}$$

for $x < 0$.

The first term in each case represents that part of the local surface disturbance due to the doublet and the isolated image doublet. The remaining terms

are due to the semi-infinite train of doublets behind the image point. Part of the effect is the train of regular waves to the rear of the origin, evidently associated with the periodicity in the direction of the doublets along the line distribution ; and there is also a further contribution to the local surface disturbance, which we may regard as arising from the fact that the line distribution is semi-infinite and has a definite front.

Horizontal and Vertical Doublets.

3. With the axis of the doublet horizontal, we have the well-known first approximation to the submerged circular cylinder of radius a, if we take $M = ca^2$. From (11), the surface elevation can be expressed in the form

$$\eta = \frac{2a^2 f}{x^2+f^2} + 2a^2\kappa_0 P, \quad x > 0,$$

$$\eta = \frac{2a^2 f}{x^2+f^2} + 2a^2\kappa_0 P + 4\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x, \quad x < 0, \tag{12}$$

where P is the real part, for $x > 0$, of the integral

$$\int_0^\infty \frac{e^{-(x+if)m}}{m+i\kappa_0}\,dm. \tag{13}$$

Taking the axis of the doublet to be vertically upwards, we have $\alpha = \pi/2$ in the general formulæ ; and, putting $M = ca^2$ in this case also, we obtain

$$\eta = -\frac{2a^2 x}{x^2+f^2} - 2a^2\kappa_0 Q, \quad x > 0,$$

$$\eta = -\frac{2a^2 x}{x^2+f^2} + 2a^2\kappa_0 Q + 4\pi\kappa_0 a^2 e^{-\kappa_0 f}\cos\kappa_0 x, \quad x < 0, \tag{14}$$

where Q is the imaginary part of the integral (13). This integral may be expressed formally in terms of $li\,(e^{f-ix})$, where li denotes the logarithmic integral, and may be expanded in various forms. For the numerical calculations which follow, it was found simplest to use the series

$$\int_0^\infty \frac{e^{-(\alpha+i\beta)u}}{u+i}\,du = -(A+iB)\,e^{-(\beta-i\alpha)},$$

$$A = \gamma + \log r + \sum_1^\infty \frac{r^n}{n!}\cos n\theta,$$

$$B = \pi - \theta - \sum_1^\infty \frac{r^n}{n!}\sin n\theta, \tag{15}$$

where

$$r = (\alpha^2+\beta^2)^{\frac{1}{2}}, \qquad \tan\theta = \alpha/\beta, \qquad \text{and} \qquad \gamma = 0\cdot 57721.$$

The series is sufficiently simple for calculation, though in some of the cases it was necessary to take a large number of terms.

For both the horizontal and vertical doublets we take

$$\mathrm{M} = ca^2, \qquad f = 2a, \qquad \kappa_0 f = 4. \tag{16}$$

This means that we take the velocity to be such that the wave-length of the regular waves is $\frac{1}{2}\pi f$. We are assuming, in each case, a given doublet at depth f below the surface of deep water. The only restrictions so far are the general ones due to neglecting the square of the fluid velocity at the free surface, and the consequent limitation to waves of small height. From this point of view the data of (16) are rather extreme; but, this being understood, it may be permissible to use them for a comparison of the two cases. With the values in (16), the calculations are comparatively simple, and lead to graphs which can be drawn suitably on the same scale throughout; these are shown in figs. 1 and 2, where the unit of length is the quantity a.

In fig. 1, there is a horizontal doublet at C; the arrow shows the direction

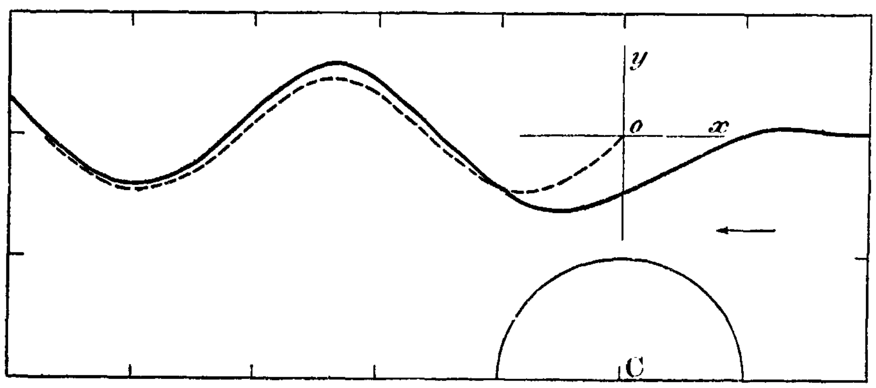

FIG. 1.

of the stream assuming the doublet to be stationary, and Ox is in the undisturbed surface. The surface elevation was calculated from (12) for the case (16). The broken curve shows the regular sine waves to which the disturbance approximates as we pass to the rear. This solution is also the first approximation for a submerged cylinder of radius a; or, again, to the same order, it gives the effect caused by a semicircular ridge on the bed of a stream of depth twice the radius. From this point of view the diagram may be compared with that given by Kelvin* for a small obstruction on the bed of a stream of finite depth.

* Kelvin, 'Math. and Phys. Papers,' vol. 4, p. 295.

Fig. 2 shows the corresponding curves for a vertical doublet, calculated from (14) for the case (16); the doublet is at the point C. Here, again, the broken curve shows the cosine term of the solution to which the disturbance approximates.

We may also regard this as an approximate solution for the flow of a stream

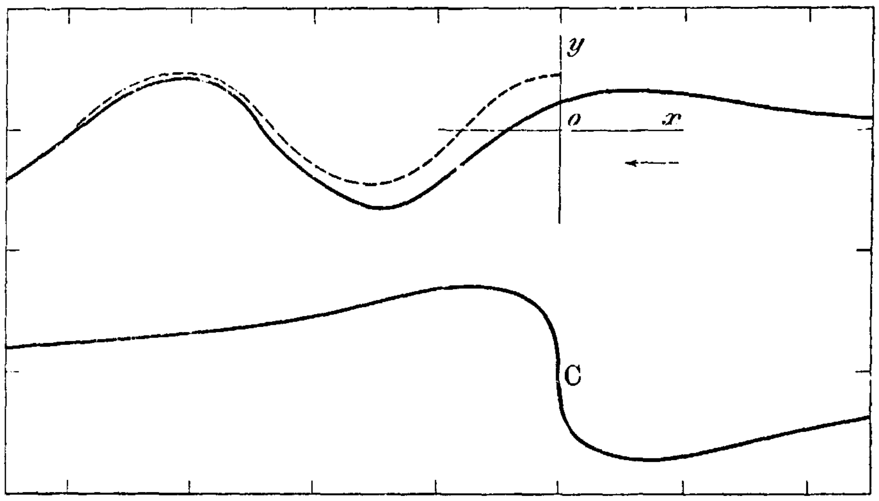

FIG. 2.

over a bed of a certain form. This is obtained by taking the zero stream-line for the combination of the uniform stream and a vertical doublet at C under the conditions given in (16); the equation of this curve is

$$(y + 2a)\{x^2 + (y + 2a)^2\} + a^2x = 0, \tag{17}$$

and its form is shown in the figure. Fig. 2 may be compared with a graph given by Wien* for the case of a sudden small rise in the bed of a stream.

It is interesting to note the general similarity of the surface elevation in the two cases shown in figs. 1 and 2; although the regular waves are given by a sine curve in one case and a cosine curve in the other, that is only because of the different position of the origin relative to the general form of the obstacle.

Second Approximation for Circular Cylinder.

4. We may now carry out further approximations for a circular cylinder in a uniform stream by the method of successive images. Reference may be made to fig. 3, which is not drawn exactly to scale.

The image of the stream in the circle is a horizontal doublet M at the centre C. The image of M in the free surface is a doublet $-$ M at the image point C_1

* W. Wien, 'Hydrodynamik,' p. 206.

together with a trail of doublets to the rear of C_1. The image of this system in the circle gives a doublet $-Ma^2/4f^2$ at C_2, together with a certain line distribution of doublets on the semicircle on CC_2. So the process could be carried on, but we shall stop at this stage.

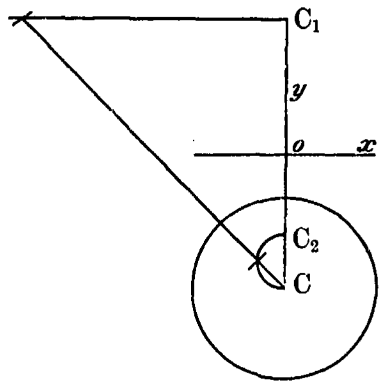

FIG. 3.

From the results already given, we could build up complete expressions for the velocity potential and surface elevation for each stage. It would be of interest to work these out graphically to compare with fig. 1; but the expressions soon become complicated and their evaluation difficult, especially for the immediate vicinity of the origin. We shall therefore limit the study to the regular waves established in the rear of the cylinder. We have seen that the regular waves of the first approximation, due to the doublet ca^2 at C, are given by

$$\eta = 4\pi\kappa_0 a^2 e^{-\kappa_0 f} \sin \kappa_0 x\,; \qquad x<0. \tag{18}$$

We take the next stage in two parts. First we have an isolated horizontal doublet of moment $-ca^4/4f^2$ at C_2, whose co-ordinates are $(0, -f+a^2/2f)$. From (11) it follows that the contribution of this doublet to the regular waves is

$$\eta = -\pi\kappa_0 a^4 f^{-2} e^{-\kappa_0(f-a^2/2f)} \sin \kappa_0 x\,; \qquad x<0. \tag{19}$$

Next we consider the line distribution of doublets to the rear of C_1 and its image in the circle. Referring to the results in § 2, there is at the point $(-p, f)$ an elementary doublet of moment $2\kappa_0 ca^2\,dp$, with its axis making an angle $\kappa_0 p - \frac{1}{2}\pi$ with the positive direction of Ox. The image of this in the circle is a doublet at the point whose co-ordinates are

$$-\frac{a^2 p}{p^2+4f^2}, \quad -f+\frac{2a^2 f}{p^2+4f^2}; \tag{20}$$

the moment of the doublet is $2\kappa_0 ca^4\,.\,dp/(p^2+4f^2)$, and its axis makes with Ox the angle

$$2\tan^{-1}(p/2f) - \kappa_0 p + \tfrac{1}{2}\pi. \tag{21}$$

From (11) we can now write down the waves due to this doublet. It should be noted that the expression will hold for

$$x + \frac{a^2 p}{p^2+4f^2} < 0.$$

If, therefore, we wish to obtain the complete expression for this part of the surface elevation at a point in the range $-a^2/4f < x < 0$, we should have to integrate with respect to p between appropriate variable limits. We shall consider only points to the rear of this range, so that the limits for p are 0 and ∞. This being understood, the distribution of doublets on the semicircle CC_2 contributes to the regular waves a part given by

$$\eta = 8\pi\kappa_0{}^2 a^4 e^{-\kappa_0 f} \int_0^\infty e^{\frac{2a^2\kappa_0 f}{p^2+4f^2}} \cos\left\{\kappa_0\left(x + \frac{a^2 p}{p^2+4f^2}\right) + 2\tan^{-1}\frac{p}{2f} - \kappa_0 p\right\}\frac{dp}{p^2+4f^2}. \quad (22)$$

Putting $p = 2f \tan \frac{1}{2}\theta$, this becomes

$$\eta = 2\pi\kappa_0{}^2 a^4 f^{-1} e^{-\kappa_0 f + \kappa_0 a^2/4f}(\mathrm{A} \cos \kappa_0 x - \mathrm{B} \sin \kappa_0 x), \quad (23)$$

where

$$\mathrm{A} = \int_0^\pi e^{h\cos\theta} \cos(\theta + h \sin\theta - k \tan \tfrac{1}{2}\theta)\, d\theta,$$

$$\mathrm{B} = \int_0^\pi e^{h\cos\theta} \sin(\theta + h \sin\theta - k \tan \tfrac{1}{2}\theta)\, d\theta,$$

with $h = \kappa_0 a^2/4f$ and $k = 2\kappa_0 f$.

5. In the applications to be made, h and k are positive, h is less than unity and is usually a small fraction. In these circumstances, the integrals may be evaluated by expansion in power series of h. It can be shown, after a little reduction, that we have

$$\mathrm{A} = 2\sum_0^\infty \frac{h^n}{n!}\mathrm{L}_{n+1}; \qquad \mathrm{B} = 2\sum_0^\infty \frac{h^n}{n!}\mathrm{M}_{n+1}; \quad (24)$$

where

$$\mathrm{L}_r = \int_0^{\pi/2} \cos(2r\phi - k\tan\phi)\, d\phi$$

$$\mathrm{M}_r = \int_0^{\pi/2} \sin(2r\phi - k\tan\phi)\, d\phi. \quad (25)$$

The quantities L and M may be evaluated in terms of known functions by a reduction formula. It can readily be shown that

$$(r+1)\,\mathrm{L}_{r+1} = k\mathrm{L}_r'' - k\mathrm{L}_r' + r\mathrm{L}_r, \quad (26)$$

the accents denoting differentiation with respect to k; or denoting this operation by D, we have

$$r!\,\mathrm{L}_r = (k\mathrm{D}^2 - k\mathrm{D} + r - 1)(k\mathrm{D}^2 - k\mathrm{D} + r - 2)\ldots(k\mathrm{D}^2 - k\mathrm{D})\,\mathrm{L}_0. \quad (27)$$

The quantity M satisfies similar relations.

Further, we have

$$L_0 = \int_0^{\pi/2} \cos(k \tan\phi)\, d\phi = \tfrac{1}{2}\pi e^{-k}$$

$$M_0 = -\int_0^{\pi/2} \sin(k \tan\phi)\, d\phi = -\tfrac{1}{2}\{e^{-k} li(e^k) - e^k li(e^{-k})\}. \tag{28}$$

We shall find it necessary to go as far as the sixth term in numerical calculation of A and B; we therefore record to this order explicit expressions for L and M obtained from (27) and (28).

$$\begin{aligned}
L_1 &= \pi k e^{-k},\\
L_2 &= -\pi k(1-k)e^{-k},\\
L_3 &= \tfrac{1}{3}\pi k(3-6k+2k^2)e^{-k},\\
L_4 &= -\tfrac{1}{3}\pi k(3-9k+6k^2-k^3)e^{-k},\\
L_5 &= \tfrac{1}{15}\pi k(15-60k+60k^2-20k^3+2k^4)e^{-k},\\
L_6 &= -\tfrac{1}{45}\pi k(45-225k+300k^2-150k^3+30k^4-2k^5)e^{-k},\\
M_1 &= -ke^{-k}li(e^k)+1,\\
M_2 &= k(1-k)e^{-k}li(e^k)+k,\\
M_3 &= -\tfrac{1}{3}k(3-6k+2k^2)e^{-k}li(e^k)+\tfrac{1}{3}(1-4k+2k^2),\\
M_4 &= \tfrac{1}{3}k(3-9k+6k^2-k^3)e^{-k}li(e^k)+\tfrac{1}{3}k(5-5k+k^2),\\
M_5 &= -\tfrac{1}{15}k(15-60k+60k^2-20k^3+2k^4)e^{-k}li(e^k)\\
&\qquad +\tfrac{1}{15}(3-28k+44k^2-18k^3+2k^4),\\
M_6 &= \tfrac{1}{45}k(45-225k+300k^2-150k^3+30k^4-2k^5)e^{-k}li(e^k)\\
&\qquad +\tfrac{1}{45}k(93-198k+124k^2-28k^3+2k^4).
\end{aligned}$$

6. The first case we shall examine is that already discussed in § 3, a cylinder whose centre is at a depth of twice the radius. It has been remarked that this is an extreme case, but it has the advantage, as far as the calculations are concerned, of magnifying the difference between the first and second approximations and so of lightening the numerical work involved. In the notation of the previous sections, we have

$$f = 2a;\quad k = 2\kappa_0 f = 4\pi f/\lambda_0;\quad h = \kappa_0 a^2/4f = k/32. \tag{29}$$

Collecting the terms in (18), (19) and (23), the regular waves established to the rear of the cylinder are given by

$$\eta/a = \pi k e^{-\frac{1}{2}k}\sin\kappa_0 x - \tfrac{1}{16}\pi k e^{-\frac{15}{32}k}\sin\kappa_0 x + \tfrac{1}{16}\pi k^2 e^{-\frac{7}{16}k}(A\cos\kappa_0 x - B\sin\kappa_0 x). \tag{30}$$

The first term is the first approximation, and the amplitude in this case has a maximum at $k = 2$, or when the velocity is such that the wave-length is $2\pi f$.

We shall calculate the value of (30) for k equal to 10, 8, 6, 4, 2, 1 and 0·5, given in order of increasing velocity. Omitting the intermediary steps for the numerical values of the L and M functions, the following table gives the values of A and B, calculated from (24), for these values of k and with $h = k/32$ in each case.

k	10	8	6	4	2	1	0·5
A	0·021	0·064	0·204	0·646	1·805	2·311	1·891
B	−0·418	−0·522	−0·716	−0·950	−0·596	0·668	1·742

The simplest form in which to show the difference made by the second approximation is to express (30) in each case in the form

$$\eta/a = \mathrm{D} \sin \kappa_0 (x + \xi), \tag{31}$$

and compare it with the first approximation

$$\eta/a = \mathrm{C} \sin \kappa_0 x. \tag{32}$$

A comparison of D and C gives the alteration in the amplitude of the waves; further, there is an alteration in phase expressed as a displacement of the crests to the rear by an amount ξ.

In this form the final numerical values, for $f = 2a$, are given in the following table:—

$c/\sqrt{(ga)}$.	C	D	ξ/a.
0·63	0·212	0·263	0·006
0·71	0·460	0·568	0·017
0·82	0·939	1·159	0·050
1·0	1·701	2·046	0·148
1·41	2·312	2·396	0·468
2·0	1·906	1·721	0·669
2·83	1·223	1·081	0·595

We see that the second approximation makes a considerable difference in the amplitude in this case; but it should be noted that, in addition to the depth being only twice the radius, the velocities are relatively large, the wave-length at the lowest velocity being about $1\frac{1}{4}$ times the depth.

The amplitude C has a maximum at the speed $\sqrt{(2ga)}$; and it appears from the table that the second approximation increases the amplitude below this velocity and diminishes it at higher velocities. It seems that the rearward displacement, given by ξ, also has a maximum, amounting to about two-thirds of the radius of the cylinder.

7. It is clear, from the form of the expressions for the surface elevation, that the accuracy of the first approximation increases rapidly as the depth of the cylinder is increased or as we take relatively smaller velocities. Without pursuing the calculations in this direction, we shall take one other case which is not quite so extreme as in the previous section. We take the depth of the centre to be three times the radius; the data are now

$$f = 3a\,;\ k = 2\kappa_0 f\,;\ h = \kappa_0 a^2/4f = k/72. \tag{33}$$

In this case, instead of (30), we have

$$\begin{aligned}\eta/a = \tfrac{2}{3}\pi k e^{-\frac{1}{2}k} \sin \kappa_0 x - \tfrac{1}{54}\pi k e^{-\frac{17}{18}k} \sin \kappa_0 x \\ + \tfrac{1}{54}\pi k^2 e^{-\frac{17}{18}k} (\mathrm{A} \cos \kappa_0 x - \mathrm{B} \sin \kappa_0 x).\end{aligned} \tag{34}$$

The following table shows the values of A and B, with $h = k/72$, calculated for convenience at the same values of k as before :—

k	10	8	6	4	2	1	0·5
A	0·008	0·033	0·137	0·540	1·747	2·311	1·898
B	−0·324	−0·428	−0·626	−0·911	−0·644	0·663	1·732

With the same notation as in (31) and (32), the results are given in the following table :—

$c/\sqrt{(ga)}$.	C	D	ξ/a.
0·77	0·141	0·149	0·001
0·87	0·307	0·329	0·006
1·0	0·626	0·677	0·024
1·22	1·134	1·222	0·088
1·73	1·541	1·558	0·295
2·45	1·270	1·214	0·409
3·46	0·816	0·802	0·324

The calculations were made for the same values of k; and as we have taken $\sqrt{(ga)}$ as the unit of velocity, we get a different set of velocities, but they cover much the same range. We notice that the decrease of the ratio a/f from $\frac{1}{2}$ to $\frac{1}{3}$ has diminished considerably the difference between C and D, and also the displacement ξ. The results have the same general character as we noted in the previous case.

In any given case there are two significant quantities involved: one is the ratio of the radius to the depth and the other is the ratio of the wave-length to the depth. It would require a more elaborate numerical study than has been attempted here to enable us to state precisely the degree of accuracy of the first approximation for given values of these ratios.

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Wave Resistance.

By T. H. HAVELOCK, F.R.S.

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Introduction.

1. The object of this paper is to give more direct proofs of certain expressions for wave resistance which have been used in previous calculations; further, in view of other possible applications, the expressions are generalised so that one can obtain the wave resistance for any set of doublets in any positions or directions in a uniform stream, or for any continuous distribution of doublets or equivalent sources and sinks. The only limitation is the usual one that the additional velocities at the surface are small compared with the velocity of the stream. One might take a simple source as the unit, but to avoid certain minor difficulties it would be necessary to assume an equal sink at some other point. The possible applications are to bodies either wholly, or with certain limitations partially, submerged. The image system in such a case consists of a distribution of sources and sinks of zero aggregate strength, and so may be replaced by an equivalent distribution of doublets. Hence it is simpler to use the doublet as the unit from the beginning.

The wave resistance of a submerged sphere was obtained previously both by direct calculation of pressures on the sphere and by an analogy with the effect of a certain surface distribution of pressure. The latter method was then generalised to give the wave resistance of any distribution of horizontal doublets in a vertical plane parallel to the direction of the stream. In a recent paper Lamb* has supplied a method for calculating wave resistance which avoids the

* H. Lamb, 'Roy. Soc. Proc.,' A, vol. 111, p. 14 (1926).

comparison with an equivalent surface pressure: it consists in calculating the rate of dissipation of energy by a certain integral taken over the free surface when, as is usual in these problems, a small frictional force has been introduced into the equations of motion of the fluid. Lamb, however, deals only with a single doublet, to which a submerged body is equivalent to a first approximation, and so does not obtain the interference effects which arise from an extended distribution of doublets; further, he carries out the necessary calculation by analysing first the surface distribution of velocity potential or in effect analysing the wave pattern. In the following paper it is shown that this intermediate analysis may be avoided by a direct application of the Fourier double integral theorem in two dimensions. This step simplifies the extension of the calculation to any distribution of doublets in any positions and directions; various cases, which it is hoped to use later, are given in some detail for deep water, and one case of a single doublet in a stream of finite depth.

Two-dimensional Motion.

2. The results for a two-dimensional doublet are well-known, but there are one or two points of interest in the calculation. We shall suppose the liquid to be at rest, and the doublet to be moving with uniform velocity c. Let the doublet be of moment M, with its axis horizontal, at a depth f. Take the origin in the free surface, with Ox in the direction of motion and Oz vertically upwards. If ζ is the surface elevation, and if there is a frictional force proportional to velocity, the pressure condition at the free surface gives

$$\frac{\partial \phi}{\partial t} - g\zeta + \mu'\phi = \text{constant}, \tag{1}$$

ϕ being the velocity potential. Since, at the free surface $\partial\zeta/\partial t = -\partial\phi/\partial z$, we have for the steady motion relative to the moving axes,

$$\frac{\partial^2 \phi}{\partial x^2} + \kappa_0 \frac{\partial \phi}{\partial z} - \mu \frac{\partial \phi}{\partial x} = 0, \tag{2}$$

to be satisfied at $z = 0$, with $\kappa_0 = g/c^2$ and $\mu = \mu'/c$. The conditions of the problem are satisfied by

$$\phi = \frac{\text{M}}{x + i(z+f)} - \frac{\text{M}}{x + i(z-f)} + 2i\kappa_0 \text{M} \int_0^\infty \frac{e^{i\kappa x - \kappa(f-z)}}{\kappa - \kappa_0 + i\mu} d\kappa, \tag{3}$$

where the real part is to be taken.

If R is the equivalent wave resistance, Rc is equal to the rate of dissipation of energy; this gives, following Lamb,

$$R = -\frac{\mu'\rho}{c}\int \phi \frac{\partial \phi}{\partial n}\, dS, \tag{4}$$

taken over the free surface. Thus in the two-dimensional case we have

$$R = \lim_{\mu \to 0} \mu\rho \int_{-\infty}^{\infty} \phi \frac{\partial \phi}{\partial z}\, dx, \tag{5}$$

with $z = 0$.

The surface values of ϕ and $\partial\phi/\partial z$ can be obtained from (3); after applying well-known transformations we obtain, at $z = 0$,

$$\phi = 2\kappa_0 M \int_0^{\infty} \frac{(m+\mu)\sin mf + \kappa_0 \cos mf}{(m+\mu)^2 + \kappa_0^2} e^{-mx}\, dm,$$

$$\frac{\partial \phi}{\partial z} = -\frac{4Mxf}{(x^2+f^2)^2} - 2\kappa_0 M \int_0^{\infty} \frac{(m+\mu)\cos mf - \kappa_0 \sin mf}{(m+\mu)^2 + \kappa_0^2} me^{-mx}\, dm,$$

for $x > 0$; and

$$\phi = 4\pi\kappa_0 M e^{\mu x - \kappa_0 f} \cos(\kappa_0 x + \mu f)$$
$$- 2\kappa_0 M \int_0^{\infty} \frac{(m-\mu)\sin mf - \kappa_0 \cos mf}{(m-\mu)^2 + \kappa_0^2} e^{mx}\, dm,$$

$$\frac{\partial \phi}{\partial z} = 4\pi\kappa_0 M e^{\mu x - \kappa_0 f}\{\kappa_0 \cos(\kappa_0 x + \mu f) + \mu \sin(\kappa_0 x + \mu f)\}$$
$$- \frac{4Mxf}{(x^2+f^2)^2} + 2\kappa_0 M \int_0^{\infty} \frac{(m-\mu)\cos mf - \kappa_0 \sin mf}{(m-\mu)^2 + \kappa_0^2} me^{mx}\, dm, \tag{6}$$

for $x < 0$.

These expressions are continuous at $x = 0$. It is easily seen that the only terms which give any contribution to (5) in the limit are the first terms in the expressions for ϕ and $\partial\phi/\partial z$ when x is negative. These are the terms which arise from the train of regular waves established in the rear of the moving doublet and so this method is connected with the alternative calculation of wave resistance by means of group velocity. The dissipation of energy when there is a frictional term is represented in the limit, when μ is made zero, by the propagation of energy away from the system in the train of regular waves. To complete the calculation from (5) and (6) we have

$$R = \lim \mu\rho \int_{-\infty}^{0} 16\pi^2\kappa_0^3 M^2 e^{2\mu x - 2\kappa_0 f} \cos^2 \kappa_0 x\, dx$$
$$= 4\pi^2\rho\kappa_0^3 M^2 e^{-2\kappa_0 f}. \tag{7}$$

We may obtain this result without analysing the expressions for ϕ and $\partial\phi/\partial z$. We obtain from (3) the following complete expressions in real terms, at $z = 0$,

$$\phi = 2\kappa_0 \mathrm{M} \int_0^\infty \frac{\mu \cos \kappa x - (\kappa - \kappa_0) \sin \kappa x}{(\kappa - \kappa_0)^2 + \mu^2} e^{-\kappa f} d\kappa, \tag{8}$$

$$\frac{\partial\phi}{\partial z} = 2\mathrm{M} \int_0^\infty \frac{\mu\kappa_0 \cos \kappa x - \{\kappa(\kappa - \kappa_0) + \mu^2\} \sin \kappa x}{(\kappa - \kappa_0)^2 + \mu^2} e^{-\kappa f} \kappa \, d\kappa. \tag{9}$$

To carry out the integration of the product over the surface, we use the following theorem: if

$$f(x) = \int_0^\infty (\mathrm{A}_1 \cos \kappa x + \mathrm{B}_1 \sin \kappa x) \, d\kappa,$$

$$\psi(x) = \int_0^\infty (\mathrm{A}_2 \cos \kappa x + \mathrm{B}_2 \sin \kappa x) \, d\kappa,$$

where A_1, A_2, B_1, B_2 are functions of κ, then

$$\int_{-\infty}^\infty f(x) \, \psi(x) \, dx = \pi \int_0^\infty (\mathrm{A}_1 \mathrm{A}_2 + \mathrm{B}_1 \mathrm{B}_2) \, d\kappa. \tag{10}$$

This theorem is derived from the Fourier double integral

$$\phi(x) = \frac{1}{\pi} \int_0^\infty d\kappa \int_{-\infty}^\infty \phi(\alpha) \cos \kappa (x - \alpha) \, d\alpha, \tag{11}$$

and is subject to the same conditions.

In the present case, comparing (8) and (11) we have

$$\int_{-\infty}^\infty \phi(\alpha) \cos \kappa\alpha \, d\alpha = \frac{2\pi\kappa_0 \mathrm{M} \mu e^{-\kappa f}}{(\kappa - \kappa_0)^2 + \mu^2},$$

$$\int_{-\infty}^\infty \phi(\alpha) \sin \kappa\alpha \, d\alpha = -\frac{2\pi\kappa_0 \mathrm{M} (\kappa - \kappa_0) e^{-\kappa f}}{(\kappa - \kappa_0)^2 + \mu^2}. \tag{12}$$

Hence we have

$$\int_{-\infty}^\infty \phi \frac{\partial\phi}{\partial z} dx = 4\pi\kappa_0 \mathrm{M}^2 \int_0^\infty \frac{\kappa^2 e^{-2\kappa f}}{(\kappa - \kappa_0)^2 + \mu^2} d\kappa,$$

$$\mathrm{R} = \lim_{\mu \to 0} 4\pi\kappa_0 \mathrm{M}^2 \rho\mu \int_0^\infty \frac{\kappa^2 e^{-2\kappa f}}{(\kappa - \kappa_0)^2 + \mu^2} d\kappa$$

$$= 4\pi^2 \rho \kappa_0^3 \mathrm{M}^2 e^{-2\kappa_0 f}. \tag{13}$$

Horizontal Doublet.

3. To consider three-dimensional fluid motion, take first a horizontal doublet of moment M at the point $(0, 0, -f)$. Assume that the velocity potential can be expressed in the form

$$\phi = -\frac{i\mathrm{M}}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty}\kappa e^{-\kappa(z+f)+i\kappa(x\cos\theta+y\sin\theta)}\cos\theta\,d\theta\,d\kappa$$

$$+\int_{-\pi}^{\pi}\int_{0}^{\infty}\kappa\mathrm{F}(\theta,\kappa)\,e^{-\kappa(f-z)+i\kappa(x\cos\theta+y\sin\theta)}\cos\theta\,d\theta\,d\kappa, \quad (14)$$

where real parts are to be taken, and where the first term is the velocity potential of the given doublet in a form valid for $z+f>0$.

The surface condition is equation (2) as before; applying this, we obtain

$$\mathrm{F}(\theta,\kappa) = \frac{i\mathrm{M}}{2\pi}\,\frac{\kappa+\kappa_0\sec^2\theta+i\mu\sec\theta}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}. \quad (15)$$

Hence from (14) and (15) the surface values of ϕ and $\partial\phi/\partial z$ are

$$\phi = \frac{i\kappa_0\mathrm{M}}{\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{e^{-\kappa f+i\kappa(x\cos\theta+y\sin\theta)}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\kappa\sec\theta\,d\theta\,d\kappa, \quad (16)$$

$$\frac{\partial\phi}{\partial z} = \frac{i\mathrm{M}}{\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty}\frac{e^{-\kappa f+i\kappa(x\cos\theta+y\sin\theta)}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\kappa^2(\kappa+i\mu\sec\theta)\cos\theta\,d\theta\,d\kappa. \quad (17)$$

Taking real parts of these expressions we obtain

$$\phi = \int_{-\pi}^{\pi}\int_{0}^{\infty}\{\mathrm{F}_1(\theta,\kappa)\cos(\kappa x\cos\theta)\cos(\kappa y\sin\theta)$$
$$+\mathrm{F}_2(\theta,\kappa)\sin(\kappa x\cos\theta)\cos(\kappa y\sin\theta)\}\,\kappa\,d\kappa\,d\theta, \quad (18)$$

and a similar form for $\partial\phi/\partial z$ with G instead of F, with

$$\begin{aligned}
\mathrm{F}_1 &= \mathrm{M}\kappa_0\mu e^{-\kappa f}\sec^2\theta./\mathrm{D}\\
\mathrm{F}_2 &= -\mathrm{M}\kappa_0(\kappa-\kappa_0\sec^2\theta)\,e^{-\kappa f}\sec\theta./\mathrm{D}\\
\mathrm{G}_1 &= \mathrm{M}\mu\kappa_0\kappa e^{-\kappa f}\sec^2\theta./\mathrm{D}\\
\mathrm{G}_2 &= -\mathrm{M}\{\kappa(\kappa-\kappa_0\sec^2\theta)+\mu^2\sec^2\theta\}\,\kappa e^{-\kappa f}\cos\theta./\mathrm{D}\\
\mathrm{D} &= \pi\{(\kappa-\kappa_0\sec^2\theta)^2+\mu^2\sec^2\theta\}.
\end{aligned} \quad (19)$$

We now apply a theorem in two dimensions corresponding to that given in (10). The Fourier integral theorem is

$$\mathrm{F}(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{F}(s,t)\cos u(x-s)\cos v(y-t)\,ds\,dt.$$

Putting $u = \kappa \cos\theta$, $v = \kappa \sin\theta$, this may be written

$$F(x, y) = \int_{-\pi}^{\pi} d\theta \int_0^{\infty} \{F_1 \cos(\kappa x \cos\theta) \cos(\kappa y \sin\theta)$$

$$+ F_2 \sin(\kappa x \cos\theta) \cos(\kappa y \sin\theta) + F_3 \cos(\kappa x \cos\theta) \sin(\kappa y \sin\theta)$$

$$+ F_4 \sin(\kappa x \cos\theta) \sin(\kappa y \sin\theta)\} \, \kappa \, d\kappa, \tag{20}$$

where

$$F_1 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(s, t) \cos(\kappa s \cos\theta) \cos(\kappa t \sin\theta) \, ds \, dt, \tag{21}$$

with similar expressions for F_2, F_3, F_4.

If $G(x, y)$ is another function given as a double integral in the form (20), it follows as in the one-dimensional case that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y) \, G(x, y) \, dx \, dy = 4\pi^2 \int_{-\pi}^{\pi} d\theta \int_0^{\infty} (F_1G_1 + F_2G_2 + F_3G_3 + F_4G_4) \, \kappa d\kappa. \tag{22}$$

It is assumed that the various integrals are convergent.

For the particular case given in (18) and (19), we find that $F_1G_1 + F_2G_2$ reduces to a simple expression, and we obtain

$$R = \lim_{\mu \to 0} \mu\rho \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \frac{\partial \phi}{\partial z} \, dx \, dy$$

$$= \lim_{\mu \to 0} 16\rho\kappa_0 M^2 \mu \int_0^{\pi/2} \int_0^{\infty} \frac{\kappa^3 e^{-2\kappa f} \, d\kappa \, d\theta}{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta}$$

$$= 16\pi\rho\kappa_0^4 M^2 \int_0^{\pi/2} \sec^5 \; e^{-2\kappa_0 f \sec^2\theta} \, d\theta$$

$$= 4\pi\rho\kappa_0^4 M^2 e^{-\kappa_0 f} \left\{ K_0(\kappa_0 f) + \left(1 + \frac{1}{2\kappa_0 f}\right) K_1(\kappa_0 f) \right\}, \tag{23}$$

where K_n is the Bessel Function defined by

$$K_n(x) = \int_0^{\infty} e^{-x \cosh u} \cosh nu \, du.$$

Horizontal Doublets in Vertical Plane.

4. This method allows easily an extension to any distribution of doublets. Consider first two horizontal doublets M and M′ at the points $(h, 0, -f)$ and $(h', 0, -f')$ respectively. The surface value of ϕ is now given by (16) with

$x - h$ instead of x, together with a similar expression in M′ and $x - h'$. Taking the real part we have

$$\phi = \frac{\kappa_0}{\pi}\int_{-\pi}^{\pi} \sec\theta\, d\theta \int_0^\infty \frac{\cos(\kappa y \sin\theta)}{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta}\, \mathrm{P}\kappa\, d\kappa,$$

$$\begin{aligned}\mathrm{P} = {} & \mu \sec\theta\,[\mathrm{M}e^{-\kappa f} \cos\{\kappa(x-h)\cos\theta\} + \mathrm{M}'e^{-\kappa f'}\cos\{\kappa(x-h')\cos\theta\}] \\ & - (\kappa - \kappa_0\sec^2\theta)\,[\mathrm{M}e^{-\kappa f}\sin\{\kappa(x-h)\cos\theta\} + \mathrm{M}'e^{-\kappa f'}\sin\{\kappa(x-h')\cos\theta\}]\end{aligned} \tag{24}$$

There is a similar expression for the surface value of $\partial\phi/\partial z$. We now write both these in the form (18), omitting terms which from symmetry give zero when integrated with respect to θ. We find again that we have only to form the quantity $F_1G_1 + F_2G_2$ and that this simplifies considerably; the wave resistance, after this reduction, is given by

$$\begin{aligned}\mathrm{R} &= \lim_{\mu\to 0} 16\rho\kappa_0\mu \int_0^{\pi/2} d\theta \int_0^\infty \frac{\mathrm{M}^2 e^{-2\kappa f} + \mathrm{M}'^2 e^{-2\kappa f'} + 2\mathrm{M}\mathrm{M}' e^{-\kappa(f+f')}\cos\{\kappa(h-h')\cos\theta\}}{(\kappa-\kappa_0\sec^2\theta)^2 + \mu^2\sec^2\theta}\,\kappa^3\, d\kappa \\ &= 16\pi\rho\kappa_0^4 \int_0^{\pi/2} [\mathrm{M}^2 e^{-2\kappa_0 f\sec^2\theta} + \mathrm{M}'^2 e^{-2\kappa_0 f'\sec^2\theta} \\ &\qquad + 2\mathrm{M}\mathrm{M}' e^{-\kappa_0(f+f')\sec^2\theta}\cos\{\kappa_0(h-h')\sec\theta\}]\sec^5\theta\, d\theta.\end{aligned} \tag{25}$$

The first two terms give the resistance due to the two doublets separately, while the third term represents the interference effects. This expression was obtained formerly from the analogy between the waves produced by a submerged sphere and those due to a certain surface distribution of pressure; it was then generalised for any distribution of horizontal doublets in the vertical plane $y = 0$.* The method given here can also obviously be extended by integration for any such continuous distribution, and confirms the general expression used in previous calculations; if M (h, f) is the moment per unit area at the point $(h, 0, -f)$, then (25) generalises to give

$$\begin{aligned}\mathrm{R} = 16\pi\rho\kappa_0^4 \int_0^\infty df \int_0^\infty df' \int_{-\infty}^\infty dh \int_{-\infty}^\infty dh' \int_0^{\pi/2} \mathrm{M}(h,f)\,\mathrm{M}(h',f') \times \\ e^{-\kappa_0(f+f')\sec^2\theta}\cos\{\kappa_0(h-h')\sec\theta\}\sec^5\theta\, d\theta.\end{aligned} \tag{26}$$

General Distribution.

5. We can use the same method for doublets with their axes in any directions, for we can always obtain the surface values of ϕ and $\partial\phi/\partial z$ in the form (20) and so can integrate over the surface by means of (24). Beginning with a single

* 'Roy. Soc. Proc.,' A, vol. 95, p. 363 (1919); also A, vol. 108, p. 78 (1925).

doublet at the point $(0, 0, -f)$, let the direction cosines of its axis be (l, m, n). By the same process as for the horizontal doublet in (16) and (17), the surface values of ϕ and $\partial\phi/\partial z$ are found to be

$$\phi = \frac{\kappa_0 M}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty \frac{(il\cos\theta + im\sin\theta - n)\sec^2\theta}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} Q\kappa\, d\kappa,$$

$$\frac{\partial\phi}{\partial z} = \frac{M}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^\infty \frac{\kappa(il\cos\theta + im\sin\theta - n)(\kappa + i\mu\sec\theta)}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} Q\kappa\, d\kappa, \qquad (27)$$

with $Q = e^{-\kappa f + i\kappa(x\cos\theta + y\sin\theta)}$.

With the same notation as before,

$$F_1 = \kappa_0\{\mu l - n(\kappa - \kappa_0\sec^2\theta)\} D\sec^2\theta,$$
$$F_2 = -\kappa_0\{\mu n\sec\theta + l(\kappa - \kappa_0\sec^2\theta)\cos\theta\} D\sec^2\theta,$$
$$F_3 = -\kappa_0 m(\kappa - \kappa_0\sec^2\theta) D\sin\theta\sec^2\theta,$$
$$F_4 = -\kappa_0\mu m D\sin\theta\sec^3\theta,$$
$$G_1 = \kappa[\mu\kappa_0 l\sec^2\theta - n\{\kappa(\kappa - \kappa_0\sec^2\theta) + \mu^2\sec^2\theta\}] D,$$
$$G_2 = -\kappa[l\{\kappa(\kappa - \kappa_0\sec^2\theta) + \mu^2\sec^2\theta\}\cos\theta + \mu n\kappa_0\sec^3\theta] D,$$
$$G_3 = -\kappa m\{\kappa(\kappa - \kappa_0\sec^2\theta) + \mu^2\sec^2\theta\} D\sin\theta,$$
$$G_4 = -\mu m\kappa_0\kappa D\sin\theta\sec^3\theta,$$
$$D = (M/\pi)e^{-\kappa f}/\{(\kappa - \kappa_0\sec^2\theta)^2 + \mu^2\sec^2\theta\}. \qquad (28)$$

We find that Σ FG simplifies very much even if we take the expressions as they stand; since we are only concerned ultimately with μ zero, we could further simplify the work by omitting superfluous terms. The expression for R reduces to the limit of an integral of the same type as in (23), and the result is

$$R = 16\pi\rho\kappa_0^4 M^2\int_0^{\pi/2}(l^2\cos^2\theta + m^2\sin^2\theta + n^2)e^{-2\kappa_0 f\sec^2\theta}\sec^7\theta\, d\theta$$
$$= 4\pi\rho\kappa_0^4 M^2 e^{-\alpha}\Big[l^2\Big\{K_0(\alpha) + \Big(1 + \frac{1}{2\alpha}\Big)K_1(\alpha)\Big\}$$
$$+ \frac{m^2}{4\alpha}\Big\{K_0(\alpha) + \Big(1 + \frac{2}{\alpha}\Big)K_1(\alpha)\Big\}$$
$$+ n^2\Big\{\Big(1 + \frac{1}{4\alpha}\Big)K_0(\alpha) + \Big(1 + \frac{3}{4\alpha} + \frac{1}{2\alpha^2}\Big)K_1(\alpha)\Big\}\Big], \qquad (29)$$

where $\alpha = \kappa_0 f = gf/c^2$.

6. The only further stage to which we need carry the calculation is for two **doublets in any positions:** M at the point $(h, k, -f)$ with its axis in the direction

(l, m, n), and M′ at $(h', k', -f')$ in the direction (l', m', n'). We have simply to put the surface values of ϕ and $\partial\phi/\partial z$ in the standard form, and evaluate the quantity Σ FG. The reduction need not be reproduced here; omitting terms in μ which make no contribution in the limit, we obtain

$$\pi^2 \Sigma \text{ FG} . \{(\kappa - \kappa_0 \sec^2 \theta)^2 + \mu^2 \sec^2 \theta\}/\kappa_0 \kappa^2 \sec^2 \theta$$
$$= [(l \cos\theta \sin \text{P} \cos \text{Q} + m \sin\theta \cos \text{P} \sin \text{Q} - n \cos \text{P} \cos \text{Q}) \text{M} e^{-\kappa f}$$
$$+ (l' \cos\theta \sin \text{P}' \cos \text{Q}' + m' \sin\theta \cos \text{P}' \sin \text{Q}' - n \cos \text{P}' \cos \text{Q}') \text{M}' e^{-\kappa f'}]^2$$
$$+ [(l \cos\theta \cos \text{P} \cos \text{Q} - m \sin\theta \sin \text{P} \sin \text{Q} + n \sin \text{P} \cos \text{Q}) \text{M} e^{-\kappa f}$$
$$+ (l' \cos\theta \cos \text{P}' \cos \text{Q}' - m' \sin\theta \sin \text{P}' \sin \text{Q}' + n' \sin \text{P}' \cos \text{Q}') \text{M}' e^{-\kappa f'}]^2$$
$$+ [(l \cos\theta \sin \text{P} \sin \text{Q} - m \sin\theta \sin\theta \cos \text{P} \cos \text{Q} - n \cos \text{P} \sin \text{Q}) \text{M} e^{-\kappa f}$$
$$+ (l' \cos\theta \sin \text{P}' \sin \text{Q}' - m' \sin\theta \cos \text{P}' \cos \text{Q}' - n' \cos \text{P}' \sin \text{Q}') \text{M}' e^{-\kappa f'}]^2$$
$$+ [(l \cos\theta \cos \text{P} \sin \text{Q} + m \sin\theta \sin \text{P} \cos \text{Q} + n \sin \text{P} \sin \text{Q}) \text{M} e^{-\kappa f}$$
$$+ (l' \cos\theta \cos \text{P}' \sin \text{Q}' + m' \sin\theta \sin \text{P}' \cos \text{Q}' + n' \sin \text{P}' \sin \text{Q}') \text{M}' e^{-\kappa f'}]^2, \tag{30}$$

where $\text{P} = \kappa h \cos\theta$, $\text{Q} = \kappa k \sin\theta$, and similarly P′ and Q′. Carrying out the rest of the calculation for R, the wave resistance is given by

$$\text{R} = 16\pi\rho\kappa_0{}^4 \int_0^{\pi/2} \Big[(l^2 \cos^2\theta + m^2 \sin^2\theta + n^2) \text{M}^2 e^{-2\kappa_0 f \sec^2\theta}$$
$$+ (l'^2 \cos^2\theta + m'^2 \sin^2\theta + n'^2) \text{M}'^2 e^{-2\kappa_0 f' \sec^2\theta}$$
$$+ 2 \{(ll' \cos^2\theta + mm' \sin^2\theta + nn') \cos \text{A} \cos \text{B}$$
$$- (lm' + l'm) \sin\theta \cos\theta \sin \text{A} \sin \text{B} + (nm' - n'm) \sin\theta \cos \text{A} \sin \text{B}$$
$$+ (nl' - n'l) \cos\theta \sin \text{A} \cos \text{B}\} \text{MM}' e^{-\kappa_0 (f+f') \sec^2\theta} \Big] \sec^7\theta \, d\theta, \tag{31}$$

where $\text{A} = \kappa_0 (h - h') \sec\theta$, $\text{B} = \kappa_0 (k - k') \sin\theta \sec^2\theta$. The various terms represent the contributions of the three components of each doublet and their mutual interference in pairs.

Water of Finite Depth.

7. For water of finite depth h, we shall consider only the simplest case of a horizontal doublet of moment M at depth f. It is clear that the same surface integral can be used for evaluating the wave resistance.

We now assume the velocity potential in the form

$$\phi = -\frac{i\text{M}}{2\pi} \int_{-\pi}^{\pi} \cos\theta \, d\theta \int_0^{\infty} \{e^{-\kappa f} + e^{-\kappa(2h-f)}\} e^{-\kappa z + i\kappa(x\cos\theta + y \sin\theta)} \kappa \, d\kappa$$
$$+ \frac{i\text{M}}{2\pi} \int_{-\pi}^{\pi} \cos\theta \, d\theta \int_0^{\infty} \text{F}(0, \kappa) \cosh \kappa (z + h) \, e^{i\kappa(x\cos\theta + y\sin\theta)} \kappa \, d\kappa. \tag{32}$$

This satisfies the condition $\partial\phi/\partial z = 0$ at $z = -h$; we note that the first term represents the original doublet and its image in the bed in an analytical form valid for $z + f > 0$, and therefore suitable for applying the boundary condition (2) at the free surface. This yields

$$\mathrm{F}(\theta, \kappa) = \frac{2e^{-\kappa h}\cosh\kappa(h-f)}{\cosh\kappa h}\,\frac{\kappa + \kappa_0\sec^2\theta + i\mu\sec\theta}{\kappa - \kappa_0\sec^2\theta\tanh\kappa h + i\mu\sec\theta}, \quad (33)$$

consequently the required surface values are given by the real parts of

$$\phi = \frac{i\kappa_0 \mathrm{M}}{\pi}\int_{-\pi}^{\pi}\sec\theta\, d\theta\int_0^\infty \frac{e^{-\kappa h}\cosh\{\kappa(h-f)\}(1+\tanh\cdot\kappa h)}{\kappa - \kappa_0\sec^2\theta\tanh\kappa h + i\mu\sec\theta}e^{i\kappa\varpi}\,\kappa\, d\kappa,$$

$$\frac{\partial\phi}{\partial z} = \frac{i\mathrm{M}}{\pi}\int_{-\pi}^{\pi}\cos\theta\, d\theta\int_0^\infty \frac{e^{-\kappa h}\cosh\{\kappa(h-f)\}(1+\tanh\kappa h)(\kappa + i\mu\sec\theta)}{\kappa - \kappa_0\sec^2\theta\tanh\kappa h + i\mu\sec\theta}e^{i\kappa\varpi}\,\kappa^2 d\kappa, \quad (34)$$

where $\varpi = x\cos\theta + y\sin\theta$.

Comparing these with the corresponding values for deep water given in (16) and (17), we can write down the expression for the wave resistance as

$$\mathrm{R} = \lim_{\mu\to 0} 16\rho\kappa_0\mathrm{M}^2\mu\int_0^{\pi/2} d\theta\int_0^\infty \frac{\kappa^3 e^{-2\kappa h}\cosh^2\{\kappa(h-f)\}(1+\tanh\kappa h)^2}{(\kappa-\kappa_0\sec^2\theta\tanh\kappa h)^2+\mu^2\sec^2\theta}\,d\kappa. \quad (35)$$

There are two points to notice in evaluating this limit. The result is only different from zero when

$$\kappa - \kappa_0\sec^2\theta\tanh\kappa h = 0 \quad (36)$$

has a real positive root; and this occurs only for $\kappa_0 h\sec^2\theta > 1$. Further we must introduce in the denominator $d.(\kappa - \kappa_0\sec^2\theta\tanh\kappa h)/d\kappa$. We may sum up the result in this form

$$\mathrm{R} = 16\pi\rho\kappa_0\mathrm{M}^2\int_{\theta_0}^{\pi/2}\frac{\kappa^3 e^{-2\kappa h}\cosh^2\{\kappa(h-f)\}(1+\tanh\kappa h)^2}{1-\kappa_0 h\sec^2\theta\,\mathrm{sech}^2\kappa h}\cosh\theta\, d\theta, \quad (37)$$

where κ is the positive root of (36); further, the lower limit θ_0, is given by

$$\theta_0 = 0, \quad \text{for} \quad \kappa_0 h > 1, \quad \text{or} \quad c^2 < gh,$$
$$\theta_0 = \arccos\sqrt{(\kappa_0 h)}, \quad \text{for} \quad c^2 > gh.$$

We may note that the change in the lower limit occurs at the so-called critical velocity $\sqrt{(gh)}$ for the given depth. From (37), R may be graphed as a function of the velocity for various ratios of f to h; the calculations may be carried out by numerical and graphical methods. A similar expression in the case of a certain distribution of surface pressure was examined in detail in a previous paper,* and it may be anticipated that (37) would give somewhat similar curves.

* 'Roy. Soc. Proc.,' A, vol. 100, p. 503 (1922).

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The Wave Pattern of a Doublet in a Stream.

By T. H. HAVELOCK, F.R.S.

(Received September 18, 1928.)

1. The following paper is a study of the surface waves caused by a doublet in a uniform stream, and in particular the variation in the pattern with the velocity of the stream or the depth of the doublet. In most recent work on this subject attention has been directed more to the wave resistance, which can be evaluated with less difficulty than is involved in a detailed study of the waves ; in fact, it would seem that it is not necessary for that purpose to know the surface elevation completely, but only certain significant terms at large distances from the disturbance. Recent experimental work has shown considerable agreement between theoretical expressions for wave resistance and results for ship models of simple form, and attempts have been made at a similar comparison for the surface elevation in the neighbourhood of the ship. In the latter respect it may be necessary to examine expressions for the surface elevation with more care, as they are not quite determinate ; any suitable free disturbance may be superposed upon the forced waves. For instance, it is well known that in a frictionless liquid a possible solution is one which gives waves in advance as well as in the rear of the ship, and the practical solution is obtained by superposing free waves which annul those in advance, or by some equivalent artifice. This process is simple and definite for an ideal point disturbance, but for a body of finite size or a distributed disturbance the complete surface elevation in the neighbourhood of the body requires more careful specification as regards the local part due to each element. It had been intended to consider some expressions specially from this point of view, but as the matter stands at present it would entail a very great amount of numerical calculation, and the present paper is limited to a much simpler problem although also involving considerable computation.

A horizontal doublet of given moment is at a depth f below the surface of a stream of velocity c; the surface effect may be described as a local disturbance symmetrical fore and aft of the doublet together with waves to the rear. Two points are made in the following work. One is the variation of the local disturbance with the depth of the doublet, or rather with its relation to the velocity. Roughly, it may be said that the local surface effect changes from a depression to an elevation at a certain speed, which we might have

anticipated, to be somewhere about the speed $\sqrt{(gf)}$. A line doublet is first examined, and the surface elevation immediately over the doublet is calculated; it is found to be zero at approximately a speed $0 \cdot 86 \sqrt{(gf)}$. To illustrate the difference for speeds greater or less than this value, curves are shown in fig. 1 for the complete surface elevation when gf/c^2 has the values 4 and $0 \cdot 5$. A three-dimensional doublet is then considered and a similar calculation for the surface elevation immediately over the doublet gives a critical speed of about $0 \cdot 84 \sqrt{(gf)}$.

The second point is the variation of the wave pattern. We may compare it with the pattern due to an ideal point disturbance of the surface of the stream. In that case the approximate evaluation of the integrals by the method of stationary phase gives the system of transverse and diverging waves established in the rear. But in our case there is a variable amplitude factor for the constituent harmonic terms of the integrals, and we notice that the velocity $\sqrt{(gf)}$ has here also a special significance; for the amplitude factor itself possesses an additional stationary value, a maximum, when the velocity exceeds $\sqrt{(gf)}$. The difference this makes in the wave pattern is examined; roughly, at lower speeds the pattern consists chiefly of transverse waves, while at higher speeds the diverging waves become of increasing relative importance. A direct numerical study has been made of the integral for this part of the surface elevation for two values of gf/c^2, namely, 4 and $0 \cdot 5$; graphs are given in figs. 3 and 4 for the surface elevation along various radial lines from the origin, including some outside the limits of the ideal wave pattern.

2. Take Ox in the undisturbed surface of the stream, and Oy vertically upwards, and let the velocity of the stream be c in the negative direction of Ox. Let there be a two-dimensional horizontal doublet of moment M at the point $(0, -f)$. The solution of the problem is familiar as the first approximation for the effect of a submerged cylinder of radius a, if we take $\mathrm{M} = ca^2$. We quote here the complete expression for the surface elevation η in the form used in previous calculations*

$$\eta = \frac{2\mathrm{M}f}{c\,(x^2+f^2)} + \frac{2\kappa_0\mathrm{M}}{c}\int_0^\infty \frac{m\cos mf - \kappa_0 \sin mf}{m^2+\kappa_0^2}\, e^{-mx}\, dm, \tag{1}$$

for $x > 0$, and

$$\eta = \frac{2\mathrm{M}f}{c\,(x^2+f^2)} + \frac{2\kappa_0\mathrm{M}}{c}\int_0^\infty \frac{m\cos mf - \kappa_0 \sin mf}{m^2+\kappa_0^2}\, e^{mx}\, dm + (4\pi\kappa_0\mathrm{M}/c)\, e^{-\kappa_0 f} \sin \kappa_0 x, \tag{2}$$

for $x < 0$, with $\kappa_0 = g/c^2$.

* 'Roy. Soc. Proc.,' A, vol. 115, p. 271 (1927).

The last term in (2) gives the regular waves to the rear, and the remaining terms the local disturbance which is symmetrical before and behind the origin. The integral in (1) is the real part of

$$\int_0^\infty \frac{e^{-(\alpha+i\beta)u}}{u+i}\,du, \tag{3}$$

where $\alpha = \kappa_0 x$, $\beta = \kappa_0 f$. Asymptotic expansions may be obtained for large values of the parameters, or the integral may easily be evaluated directly by numerical methods when α is not small. For small or moderate values of α and β, (3) may be calculated from

$$-(\mathrm{A}+i\mathrm{B})\,e^{-(\beta-i\alpha)},$$

$$\mathrm{A} = \gamma + \log r + \sum_1^\infty \frac{r^n}{n\,.\,n\,!}\cos n\theta,$$

$$\mathrm{B} = \pi - \theta - \sum_1^\infty \frac{r^n}{n\,.\,n\,!}\sin n\theta,$$

$$r = (\alpha^2+\beta^2)^{\frac{1}{2}}, \qquad \tan\theta = \alpha/\beta, \qquad \gamma = 0\cdot 5772. \tag{4}$$

Consider the surface elevation at the origin ($x = 0$). Since we have

$$\int_0^\infty \frac{u\cos\beta u - \sin\beta u}{1+u^2}\,du = -\,e^{-\beta}\,\mathrm{li}\,(e^{\beta}), \tag{5}$$

for $\beta > 0$, where li is the logarithmic integral, we have at the origin

$$\eta = \frac{2\mathrm{M}}{cf}\{1 - \beta e^{-\beta}\,\mathrm{li}\,(e^{\beta})\}. \tag{6}$$

Using tables of these functions, we find that η is zero when β is approximately $1\cdot35$, or when $c = 0\cdot86\sqrt{(gf)}$; when c is less, the value of (6) is negative, while at greater speeds it is positive. To illustrate this point, the surface elevation has been calculated from the complete expressions (1) and (2) for two different cases, $\kappa_0 f = 4$ and $\kappa_0 f = 0\cdot5$. The graphs are shown in fig. 1, A being for the smaller value of $\kappa_0 f$ and B for the larger. The ordinates are to the same scale assuming M and f constant and c to be the variable; the abscissæ are in wave-lengths, or more strictly the values of $\kappa_0 x$.

3. Consider now the three-dimensional problem. Take Ox and Oy in the surface of the stream, the current being in the negative direction of Ox and take Oz vertically upwards. For a horizontal doublet of moment M at the

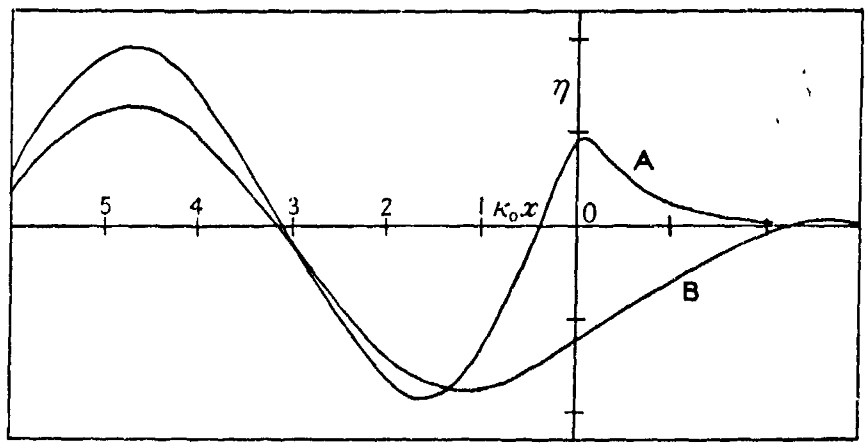

FIG. 1.

point (0, 0, $-f$), we take the velocity potential from a previous paper in the form*

$$\phi = cx - \frac{i\mathrm{M}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} \kappa e^{-\kappa(z+f)+i\kappa\varpi} \cos\theta \, d\theta \, d\kappa$$

$$+ \frac{i\mathrm{M}}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} \kappa \mathrm{F}(\theta, \kappa) \, e^{-\kappa(f-z)+i\kappa\varpi} \cos\theta \, d\theta \, d\kappa,$$

with

$$\mathrm{F}(\theta, \kappa) = \frac{\kappa + \kappa_0 \sec^2\theta + i\mu\sec\theta}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta},$$

$$\varpi = x\cos\theta + y\sin\theta. \tag{7}$$

The real part of the expression is to be taken, and further the limiting value as $\mu \to 0$. The surface elevation is obtained from

$$c\frac{\partial\zeta}{\partial x} = \frac{\partial\phi}{\partial z}.$$

After some reduction, ζ is obtained in the form

$$\zeta = \frac{2\mathrm{M}f}{c(x^2+y^2+f^2)^{3/2}} + \frac{\kappa_0\mathrm{M}}{\pi c}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sec^2\theta\,d\theta\int_0^{\infty}\left\{\frac{e^{i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\right.$$

$$\left. + \frac{e^{-i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta - i\mu\sec\theta}\right\} e^{-\kappa f}\kappa\,d\kappa. \tag{8}$$

Transforming the integral with respect to κ in (8), and taking the limiting value, we obtain

$$2\int_0^{\infty}\frac{\kappa_0\sec^2\theta\cos mf + m\sin mf}{m^2+\kappa_0^2\sec^4\theta}e^{-m\varpi}\,m\,dm, \text{ for } \varpi > 0; \tag{9}$$

* 'Roy. Soc. Proc.,' A, vol. 118, p. 28 (1928).

$$4\pi\kappa_0 \sec^2\theta\, e^{-\kappa_0 f \sec^2\theta} \sin(\kappa_0 \varpi \sec^2\theta)$$
$$+ 2\int_0^\infty \frac{\kappa_0 \sec^2\theta \cos mf + m \sin mf}{m^2 + \kappa_0{}^2 \sec^4\theta} e^{m\varpi}\, m\,dm, \quad \text{for } \varpi < 0. \tag{10}$$

We have now to integrate with respect to θ, subject to the conditions in (9) and (10). The form of the surface is symmetrical with respect to Ox, so we may write down only the expressions for y positive; and we shall put

$$x = r\cos\theta', \qquad y = r\sin\theta'. \tag{11}$$

We find that the value of ζ can be given by one expression, valid for $0 \leqslant \theta' \leqslant \pi$, namely,

$$\zeta = \frac{2Mf}{c(r^2+f^2)^{3/2}}$$
$$+ \frac{2\kappa_0 M}{\pi c}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{\kappa_0 \sec^2\theta \cos mf + m \sin mf}{m^2 + \kappa_0{}^2 \sec^4\theta} e^{-mr\,|\cos(\theta-\theta')|}\, m\,dm$$
$$+ \frac{4\kappa_0{}^2 M}{c}\int_{-\frac{1}{2}\pi}^{\theta'-\frac{1}{2}\pi} \sec^4\theta\, e^{-\kappa_0 f \sec^2\theta} \sin\{\kappa_0 r \cos(\theta-\theta')\sec^2\theta\}\, d\theta. \tag{12}$$

This expression is exact, apart from the usual limitation that, at the surface of the stream, we neglect the squares of the additional fluid velocities.

4. The first two terms in (12) represent the local effect which is only of importance in the neighbourhood of the origin. A few preliminary calculations show that, as in the two-dimensional case, it changes from a depression to an elevation about the value $\kappa_0 f = 1$. Considering the elevation at the origin, we have from (12) with $r = 0$,

$$\zeta_0 = \frac{2M}{cf^2} + \frac{4\kappa_0 M}{\pi c f}\int_0^{\frac{1}{2}\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{p\cos u + u \sin u}{p^2 + u^2}\, u\, du, \tag{13}$$

where $p = \kappa_0 f \sec^2\theta = \beta \sec^2\theta$.

The integration with respect to u can be expressed in terms of the logarithmic integral, and we obtain finally

$$\zeta_0 = \frac{4M}{\pi c f^2}\left[\frac{\pi}{2} + \int_0^{\frac{1}{2}\pi} p\{1 - pe^{-p}\,\mathrm{li}(e^p)\}\, d\theta\right]. \tag{14}$$

The integral in (14) was evaluated approximately for certain values of $\kappa_0 f$ ranging between 1 and 2. The integrand was calculated in each case for a sufficient number of values of p and was then graphed on a base of θ; the value was found by taking values from the graph and using Simpson's rule. In this way it was estimated that ζ_0 is zero at about $\kappa_0 f = 1{\cdot}4$, or $c = 0{\cdot}84\sqrt{(gf)}$. It was also verified that at lower speeds ζ_0 is negative,

and at higher speeds positive. For comparison of the maximum local effect with the rest of the surface elevation in two cases discussed later, it may be noted that

$$\pi g^{1/2} f^{5/2} \zeta_0 / 4\mathrm{M} = -1 \cdot 346, \text{ for } \kappa_0 f = 4$$
$$= +0 \cdot 860, \text{ for } \kappa_0 f = 0 \cdot 5. \tag{15}$$

The local effect at points other than the origin was not calculated, although rough estimates were made for the central line, $\theta' = 0$, from (12) to verify that it falls off in much the same way as in the two-dimensional case; for this purpose the second term in (12) was put into the form

$$(4\kappa_0 \mathrm{M}/\pi c f) \int_0^{\frac{1}{2}\pi} \mathrm{J} \sec^2 \theta \, d\theta, \tag{16}$$

where

$$\mathrm{J} = \int_0^\infty \frac{p \cos u + u \sin u}{p^2 + u^2} e^{-\alpha u/\sqrt{(\beta p)}} u \, du$$
$$= \frac{p^2}{\rho^2} - pe^{-p} [\mathrm{P} \cos \{\alpha \sqrt{(p/\beta)}\} - \mathrm{Q} \sin \{\alpha \sqrt{(p/\beta)}\}],$$
$$\mathrm{P} = \gamma + \log \rho + \sum_1^\infty \frac{\rho^n}{n \, . \, n!} \cos n\phi,$$
$$\mathrm{Q} = \pi - \phi - \sum_1^\infty \frac{\rho^n}{n \, . \, n!} \sin n\phi ; \tag{17}$$

with $\rho^2 = p^2 + p\alpha^2/\beta$, $\tan \phi = \alpha/\sqrt{(\beta p)}$, $\alpha = \kappa_0 r$, $p = \kappa_0 f \sec^2 \theta$, $\beta = \kappa_0 f$.

5. Consider now the third term in (12). For computation, we alter the form slightly. We take $\theta' = \pi - \phi$, so that ϕ is the angle the radius vector makes with the negative axis of x, and further we put

$$t' = \cot \phi ; \qquad t = \tan \theta. \tag{18}$$

Then this part of the surface elevation, which we may denote by $\zeta - \zeta_l$ is given by

$$\zeta - \zeta_l = -\frac{4\kappa_0{}^2 \mathrm{M}}{c} e^{-\beta} \int_{-\infty}^{t'} (1 + t^2) e^{-\beta t^2} \sin [\alpha (t' - t) \{(1 + t^2)/(1 + t'^2)\}^{\frac{1}{2}}] \, dt. \tag{19}$$

In this form α, or $\kappa_0 r$, is a positive quantity, r being the distance from the origin. The axis of x in front of the disturbance is given by $t' = -\infty$, and (19) is then zero; for the axis of x in the rear of the origin $t' = +\infty$. The usual first approximation to the integral in (19) consists in assuming α large enough so that the only appreciable contributions come from the groups of terms near the positions of stationary phase of the harmonic constituents; this leads to the familiar pattern of transverse and diverging waves contained within radial lines making angles of about 19° 26′ on either side of the negative

axis of x, or lines for which $t' = \pm 2\sqrt{2}$. Within the range $\infty > t' > 2\sqrt{2}$, there are two values of t for which the phase is stationary, namely, the roots of

$$2t^2 - t't + 1 = 0, \tag{20}$$

the smaller root corresponding to the transverse wave and the larger to the diverging wave at each point. In the elementary ideal case the constituent harmonic waves have equal amplitudes, but in (19) we have the amplitude factor $(1 + t^2)\, e^{-\beta t^2}$. If $\beta > 1$, this function has a maximum at $t = 0$, and diminishes steadily to zero as t increases. But if $\beta < 1$, there is a minimum at $t = 0$ and a maximum at $t = \{(1 - \beta)/\beta\}^{\frac{1}{2}}$. We may expect then a difference in the wave pattern according as c^2 is greater or less than gf. When $\beta > 1$ and α has moderate values, the main part of (19) comes from small values of t; further, when α is large and the typical wave pattern should be developed, we see that the diverging waves will be relatively small. On the other hand, when $\beta < 1$, there is increasing importance of the diverging waves; and in particular, there will be a value of t', that is a certain radial line, for which the maximum of the amplitude factor coincides with the greater root of (20) for which the phase is stationary. As we are not calculating the wave pattern at large distances we need not put down the general first approximations to (19) by the stationary phase method; we may, however, note the particular cases for $t' = \infty$ and $t' = 2\sqrt{2}$, that is for radial lines along the rearward axis and along the line of cusps of the so-called isophasal lines. For these cases (19) gives, by the usual methods,

$$\zeta - \zeta_l = -\frac{2^{\frac{1}{2}}\pi^{\frac{1}{2}}}{(\kappa_0 r)^{\frac{1}{2}}} \cdot \frac{4\kappa_0{}^2 \mathrm{M}}{c} \cdot e^{-\kappa_0 f} \cos\left(\kappa_0 r - \frac{\pi}{4}\right),$$

for $t' = \infty$, and

$$\zeta - \zeta_l = -\frac{2^{\frac{1}{2}}3^{\frac{1}{3}}\pi}{(k_0 r)^{\frac{1}{3}}\,\Gamma\left(\frac{2}{3}\right)} \cdot \frac{4\kappa_0{}^2 \mathrm{M}}{c} \cdot e^{-\frac{3}{2}\kappa_0 f} \sin\left(\tfrac{1}{2}\kappa_0 r\sqrt{3}\right), \tag{21}$$

for $t' = 2\sqrt{2}$. We note here the additional factor $e^{-\frac{1}{2}\kappa_0 f}$ in the second case, so far as variation of the amplitude with the depth is concerned; we see that the relative prominence of the so-called cusp waves is only a feature of the limiting case of a point surface disturbance.

6. Returning to the exact integral (19) for this part of the surface elevation, it seemed of interest to make some numerical calculations directly from the integral for points near to the origin, or for moderate values of α. Instead of following the isophasal lines, which are not significant in this region, we have calculated the surface elevation from (19) along certain radial lines. We take in turn the values $t' = \infty$, 3, $2\sqrt{2}$, 2, 1 and zero; these are shown in fig. 2 as A, B, C, D, E and F respectively.

For a given value of t', the value of (19) was found for about a dozen values of α, so that the graph could be obtained with sufficient accuracy for our

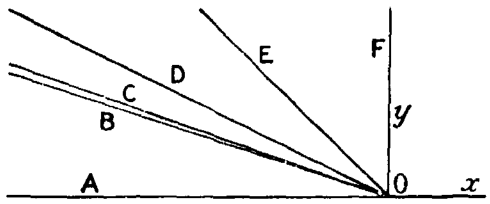

Fig. 2.

purpose over a range of α, that is of $\kappa_0 r$, extending from 0 to 18. In each case the value of (19) was obtained by evaluating the integrand at intervals of 0·1 for t for a sufficient range of t until, by reason of the exponential factor, the remaining terms became negligible. Sets of calculations were made for two values of β, that is of $\kappa_0 f$, namely, 4·0 and 0·5; in the latter case it was necessary to take 40 or more values of the integrand in each case, but a smaller number sufficed in the former case. The value of the integral was obtained finally by using Simpson's rule. The collected results are shown in the graphs of figs. 3 and 4, the curves being lettered in agreement with the radial lines of fig. 2.

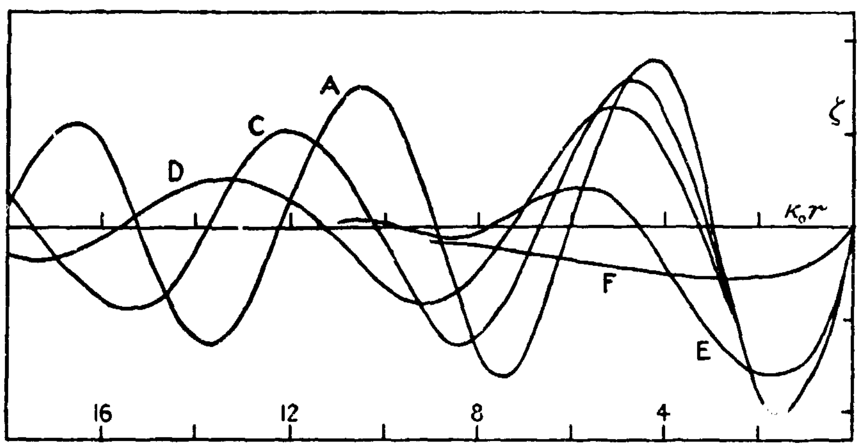

Fig. 3.

Fig. 3 is for $\kappa_0 f = 4$, or $c = \frac{1}{2}\sqrt{(gf)}$. Consider first the radial lines within the limits of the ideal wave pattern, namely, A, B, C. In this case, though B and C were calculated separately, there was not sufficient difference to show on the graph without confusion and so B has been omitted; A is the central line and C, at an angle of 19° 26′, would be the cusp line of the simple theory. We may picture the waves in the present case as chiefly transverse waves,

slightly curved, and diminishing in height from the central line outwards. The remaining curves are for radial lines outside the usual pattern and show how the wave disturbance is continued in this region. D is for an angle of about 26° 26′ with the rearward central line; it shows an appreciable wave effect, but there are indications that it decreases more rapidly with distance from the origin than for the previous curves. A similar effect, more pronounced, is shown in E and F, for radial lines at 45° and 90° respectively.

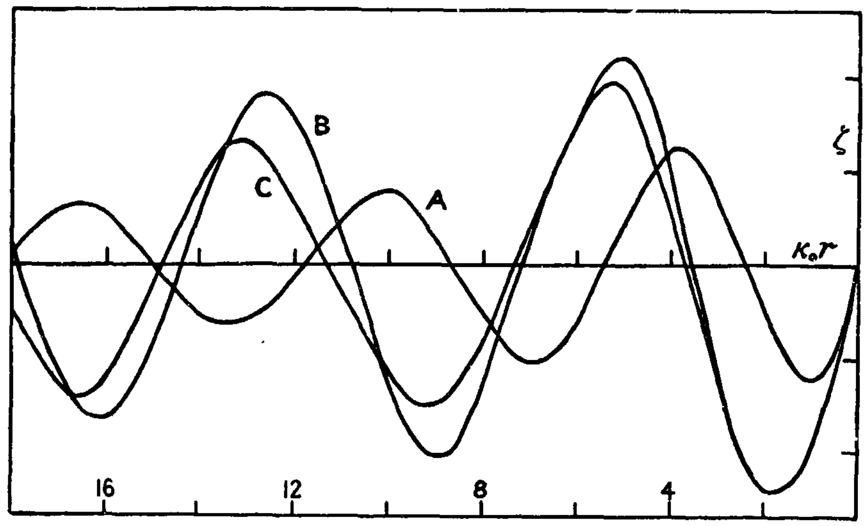

Fig. 4.

Fig. 4 is for $\kappa_0 f = 0 \cdot 5$, or $c = \sqrt{(2gf)}$. Here, on account of the labour involved in the calculations, only three curves have been drawn; but they bring out the points made in the general discussion. The amplitude along the cusp line C is now greater than along the central line A. Moreover, the greatest amplitude is along the line B, inside the cusp line C, and shows evidence of superimposed diverging waves. The radial line B is given by $t' = 3$. Now for $\kappa_0 f = \beta = \frac{1}{2}$, the maximum of the amplitude factor in (19) occurs at $t = 1$. But from (20), when $t' = 3$ the positions of stationary phase occur at $t = \frac{1}{2}$ and $t = 1$; the latter coincides with the maximum of the amplitude factor and so in this case we should expect a prominent wave along the radial line $t' = 3$, or the line B in the diagram.

A comparison of the curves in figs. 3 and 4 enables us to form some picture of the wave disturbance due to the doublet and the changes that occur as the doublet is brought nearer the surface; in the limit, as far as the wave pattern is concerned, the effect would approximate to the ideal case of a concentrated point disturbance at the surface.

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The Vertical Force on a Cylinder Submerged in a Uniform Stream.

By T. H. HAVELOCK, F.R.S.

(Received November 28, 1928.)

1. The horizontal force on a circular cylinder immersed in a stream is familiar as an example of wave resistance. The following note supplies a similar calculation for the resultant vertical force. The problem was suggested in a consideration of the forces on a floating body in motion, the horizontal and vertical forces and the turning moment; but the case of a partially immersed body presents great difficulties. It seemed, however, of sufficient interest to compare the resultant horizontal and vertical forces for a simple case of complete immersion for which the calculations can be carried out. The horizontal force, or wave resistance, has usually been obtained indirectly from considerations of energy, but a different method is adopted here for both components of force and the turning moment. In a former paper the method of successive images was applied to the problem of the circular cylinder, taking images alternately in the surface of the cylinder and in the free surface of the stream. Using these results to the required stage of approximation, the complete force on the cylinder is now obtained as the resultant of forces between the sources and sinks within the cylinder and those external to it. The same method can be applied to any submerged body for which the image sytems are known, and the resultant force and couple calculated in the same manner.

The proposition used in this method is that for a body in a fluid, the motion of which is due to given sources and sinks, the resultant force and couple on the body are the same as if the sources and their images attract in pairs according to a simple law of force, inverse distance for the two-dimensional case and inverse square of the distance for point sources. This fairly obvious proposition follows directly from a contour integration in the two-dimensional case; and, in view of the application, the extension is given in § 2 when the flow is due to a distribution of doublets. In § 3 the horizontal and vertical force on a circular cylinder are obtained by this method, the former agreeing with the usual expression for the wave resistance. The different variation of the two components with velocity is of interest, and the expressions are graphed on the same scale. The additional vertical force due to velocity changes direction at a certain speed, and is clearly associated more with the surface elevation immediately over the centre of the cylinder. In § 4 reference is made to the

couple on the cylinder. This should, of course, be zero for a complete solution; it is verified that the method used here gives zero moment up to the stage of approximation in terms of the ratio of the radius of the cylinder to the depth of its centre.

2. Consider steady two-dimensional flow of a liquid of density ρ past a solid body, the motion being irrotational and there being no field of force. The motion being specified in the usual manner by a function w of the complex variable $x + iy$, the resultant force (X, Y) on the solid and the moment M about the origin are given by

$$X - iY = \tfrac{1}{2}\rho i \int \left(\frac{dw}{dz}\right)^2 dz, \tag{1}$$

$$M = -\tfrac{1}{2}\rho \int z \left(\frac{dw}{dz}\right)^2 dz, \tag{2}$$

where in (2) the real part is to be taken. In each case the integration is taken round the contour of the rigid body, or indeed round any contour enclosing the body but excluding any external sources and sinks.

Now suppose the motion to be given by

$$w = -\Sigma\, m_r \log(z - z_r) - \Sigma\, m_s \log(z - z_s), \tag{3}$$

where the suffix s refers to the given distribution in the liquid, and r to the image system within the surface of the body, m_r and m_s being real.

Forming $(dw/dz)^2$, we see that this quantity has simple poles at the points z_r within the contour of integration; and we obtain at once from the theory of residues

$$X - iY = -2\pi\rho\, \Sigma \frac{m_r m_s}{z_r - z_s}, \tag{4}$$

the summation extending over the external and internal sources taken in pairs. Hence we obtain

$$X = 2\pi\rho\, \Sigma\, m_r m_s (x_s - x_r)/R_{rs}^2,$$
$$Y = 2\pi\rho\, \Sigma m_r m_s (y_s - y_r)/R_{rs}^2. \tag{5}$$

It follows that the resultant force is the same as if each pair of external and internal sources attracted each other with a force $2\pi\rho m_r m_s/R_{rs}$, where R_{rs} is the distance between them.

It may easily be verified in the same way from (2) that the moment M is accounted for by the same forces acting at the internal sources. It is convenient to have a similar analysis for doublets. If M is the moment of a

doublet making an angle α with the axis of x, we have with the same notation as before,

$$w = \Sigma \frac{M_r e^{i\alpha_r}}{z - z_r} + \Sigma \frac{M_s e^{i\alpha_s}}{z - z_s}. \tag{6}$$

Forming $(dw/dz)^2$ we see that again the only terms which give any contribution to the integral (1) are the product terms in r and s, and for a typical term we have

$$\int \frac{dz}{(z - z_r)^2 (z - z_s)^2} = -\frac{4\pi i}{(z_r - z_s)^3}. \tag{7}$$

Thus we obtain

$$X - iY = -4\pi\rho \Sigma \frac{M_r M_s e^{i(\alpha_r + \alpha_s)}}{(z_s - z_r)^3}, \tag{8}$$

and the contribution to the total force due to M_r and M_s is

$$\begin{aligned} X_{rs} &= -4\pi\rho M_r M_s \cos(\alpha_r + \alpha_s - 3\theta_{rs}) / R_{rs}^3, \\ Y_{rs} &= 4\pi\rho M_r M_s \sin(\alpha_r + \alpha_s - 3\theta_{rs}) / R_{rs}^3, \end{aligned} \tag{9}$$

θ_{rs} being the angle between Ox and the vector R_{rs} drawn from the doublet (r) to the doublet (s). Further, calculating the total moment M from (2), the product terms $M_r M_s$ are the only terms which give any value, and the corresponding contour integral is

$$\int \frac{z\,dz}{(z - z_r)^2 (z - z_s)^2} = 2\pi i \frac{z_r + z_s}{(z_s - z_r)^3}. \tag{10}$$

Hence we obtain

$$M = -2\pi\rho i \Sigma M_r M_s e^{i(\alpha_r + \alpha_s)} (z_r + z_s)(z_s - z_r)^{-3}, \tag{11}$$

the real part to be taken.

On reduction it is seen that this consists of the sum of the moments of the forces given by (9) acting at the internal doublets, together with a couple for each pair of internal and external doublets of amount

$$2\pi\rho M_r M_s \sin(\alpha_r + \alpha_s - 2\theta_{rs}) \,.\, / R_{rs}^2. \tag{12}$$

The contribution to the forces and moment on the body when the external field includes also a uniform flow can easily be obtained in the same manner.

3. We now apply these expressions to a circular cylinder of radius a submerged in a uniform stream. Take Ox in the undisturbed surface of the stream Oy vertically upwards; and let the stream velocity be c in the negative direction of Ox. Let the centre C of the circle be at the point $(0, -f)$. Then the image of the stream in the circle is a horizontal doublet at C of moment ca^2. The

image system of this doublet in the free surface of the stream consists of a horizontal doublet of moment $-ca^2$ at C_1 the image point $(0, f)$, together with a continuous line distribution of doublets along a horizontal line through C_1 in the negative direction of Ox. At a point $(-p, f)$ on this line the moment of an elementary doublet is $2\kappa_0 ca^2 dp$, where $\kappa_0 = g/c^2$, and the axis of the doublet makes an angle $\kappa_0 p - \frac{1}{2}\pi$ with the positive direction of Ox.* We may stop at this stage meantime.

In the notation of the previous section the external system (s) consists of the uniform stream and the image system just specified; the internal system (r) is the doublet ca^2 at the point $(0, -f)$.

For the wave resistance R, we have from (9)

$$\mathrm{R} = -\mathrm{X} = 8\pi\rho\kappa_0 c^2 a^4 \int_0^\infty \frac{\cos(\kappa_0 p - \frac{1}{2}\pi - 3\theta)}{(p^2 + 4f^2)^{3/2}}\, dp, \tag{13}$$

where

$$\sin\theta = 2f/(p^2 + 4f^2)^{\frac{1}{2}}; \quad \cos\theta = -p/(p^2 + 4f^2)^{\frac{1}{2}}.$$

This gives

$$\mathrm{R} = -8\pi\rho\kappa_0 c^2 a^4 \int_0^\infty \frac{p(p^2 - 12f^2)\sin\kappa_0 p + 2f(3p^2 - 4f^2)\cos\kappa_0 p}{(p^2 + 4f^2)^3}\, dp. \tag{14}$$

$$\int_0^\infty \frac{p\sin\kappa_0 p + 2f\cos\kappa_0 p}{p^2 + 4f^2}\, dp = \pi\epsilon^{-2\kappa_0 f}, \tag{15}$$

by differentiating twice with respect to f; and we obtain

$$\mathrm{R} = 4\pi^2 g\rho\kappa_0^2 a^4 e^{-2\kappa_0 f}, \tag{16}$$

in agreement with other methods.

Turning to the vertical force, if we calculate it from the expressions in (9) we shall obtain the hydrodynamic part depending upon the velocity. There is also the hydrostatic part $g\rho\pi a^2$, arising from the term $g\rho y$ in the expression for the pressure; and in addition there is the weight of the cylinder itself. We may assume the cylinder to be of the same density as the liquid, and then the calculation will give the total vertical force.

Measuring Y vertically upwards, the contribution of the two finite doublets at C and C_1 is, from (9),

$$-\pi\rho c^2 a^4/2f^3, \tag{17}$$

* 'Roy. Soc. Proc.,' A, vol. 115, p. 271 (1927).

The part arising from the interaction of the line distribution of doublets and the doublet at C is

$$8\pi\rho\kappa_0 c^2 a^4 \int_0^\infty \frac{\sin(\kappa_0 p - \frac{1}{2}\pi - 3\theta)}{(p^2+4f^2)^{3/2}}\,dp, \tag{18}$$

which reduces to

$$8\pi\rho\kappa_0 c^2 a^4 \int_0^\infty \frac{p\,(p^2-12f^2)\cos\kappa_0 p - 2f\,(3p^2-4f^2)\sin\kappa_0 p}{(p^2+4f^2)^3}\,dp. \tag{19}$$

This integral may be evaluated by differentiating twice with respect to f the integral

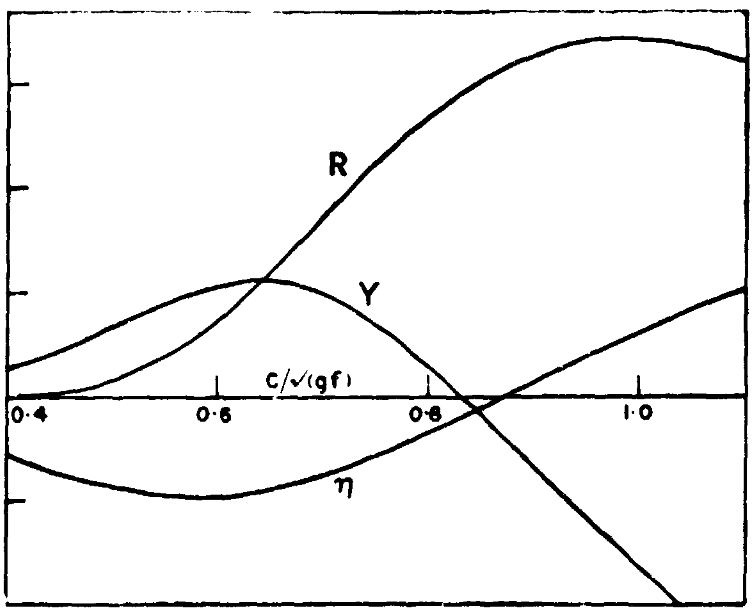
$$\int_0^\infty \frac{p\cos\kappa_0 p - 2f\sin\kappa_0 p}{p^2+4f^2}\,dp = -\,e^{-2\kappa_0 f}\,li\,(e^{2\kappa_0 f}), \tag{20}$$

where li is the logarithmic integral.

Collecting the terms from (17) and (19) we obtain finally

$$\mathrm{Y} = -\frac{\pi\rho c^2 a^4}{2f^3}\{1 + 2\kappa_0 f + 4\kappa_0^2 f^2 - 8\kappa_0^3 f^3 e^{-2\kappa_0 f}\,li\,(e^{2\kappa_0 f})\}. \tag{21}$$

This vertical force changes from upwards to downwards at a certain velocity. For when c is small, that is $\kappa_0 f$ large, using the asymptotic expansion of the logarithmic integral we find that Y approximates to $\pi\rho c^2 a^4/2f^3$; on the other hand, when c is large, Y is approximately $-\pi\rho c^2 a^4/2f^3$. The value of (21) can be calculated readily from tables and it is of interest to compare the relative values of R and Y and their variation with velocity.



The figure shows R and Y graphed on the same scale on a base of $c/\sqrt{(gf)}$. R is very small at low velocities and then increases rapidly to its maximum at

about $c = \sqrt{(gf)}$. On the other hand Y is relatively large at lower velocities, and changes sign at about $c = 0{\cdot}84\sqrt{(gf)}$. The wave resistance arises from the flow of energy in the regular waves to the rear of the cylinder, while the vertical force is associated more with the surface elevation near the cylinder. The surface elevation immediately above the centre of the cylinder is given by*

$$\eta = (2a^2/f)\{1 - \kappa_0 f e^{-\kappa_0 f}\, li\,(e^{\kappa_0 f})\}, \tag{22}$$

and for comparison this is shown on the figure with an arbitrary scale for the ordinates. Doubtless the variation in the vertical force with the velocity is connected with the mean curvature of the lines of flow in the neighbourhood of the cylinder. It may be noted that the usual approximation for the pressure condition at the free surface involves neglecting the square of the slope of the surface; this would not affect the present approximation but would come into the next stage involving higher powers of the ratio a/f.

4. Obviously in a complete solution the fluid pressures on the cylinder cannot give rise to any couple. As the method of successive images amounts to an expansion of the solution in ascending powers of a/f, it is worth while verifying that the couple is zero at each stage of the approximation. With the images specified up to the present the couple comes from the interaction of the doublet at C with the line distribution to the rear of C_1. Using the result (12), this gives a moment

$$4\pi\kappa_0\rho c^2 a^4 \int_0^\infty \frac{\sin\left(\kappa_0 p - \frac{1}{2}\pi - 2\theta\right)}{p^2 + 4f^2}\, dp, \tag{23}$$

which, on substituting for θ, gives

$$4\pi\kappa_0\rho c^2 a^4 \int_0^\infty \frac{4pf \sin\kappa_0 p - (p^2 - 4f^2)\cos\kappa_0 p}{(p^2 + 4f^2)^2}\, dp. \tag{24}$$

This can be evaluated, and its value is not zero. But it can be seen that we shall get a contribution of the same order, in the radius a, from the next stage of successive images. The next set of images is internal to the cylinder and consists of a horizontal doublet of moment $-ca^4/4f^2$ at the inverse point C_2 whose co-ordinates are $(0, -f + a^2/2f)$, together with a continuous distribution of doublets on a semi-circle described on CC_2 as diameter. At a point on this semi-circle whose co-ordinates are

$$-\frac{a^2 p}{p^2 + 4f^2},\quad -f + \frac{2a^2 f}{p^2 + 4f^2},$$

* 'Roy. Soc. Proc.,' A, vol. 121, p. 517 (1928).

the moment of an elementary doublet is $2\kappa_0 ca^4\, dp/(p^2+4f^2)$, and its axis makes with Ox an angle

$$2\tan^{-1}(p/2f) - \kappa_0 p + \tfrac{1}{2}\pi.$$

Now at this stage the only additional terms of order a^4 for the turning moment arise from the interaction of the external uniform stream and the semi-circular distribution of doublets just described.

It can easily be seen that the amount due to the uniform stream and an internal doublet of moment M_r at an angle α_r to Ox is

$$-2\pi c\rho M_r \sin\alpha_r.$$

Therefore the additional couple of the specified order is

$$-4\pi\kappa_0\rho c^2 a^4 \int_0^\infty \frac{\sin\{2\tan^{-1}(p/2f) - \kappa_0 p + \pi/2\}}{p^2+4f^2}\, dp \tag{25}$$

On reduction (25) comes to precisely the expression (24) with a minus sign and therefore the couple on the cylinder is zero to the order specified.

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Forced Surface-Waves on Water.

By T. H. HAVELOCK, F.R.S.

1. THE following notes deal with some problems of forced waves on the surface of water, the waves being forced in that the normal fluid velocity has an assigned value at every point of a given vertical surface; the problems treated here are the elementary cases when the given surface is an infinite plane or a circular cylinder. The motion of the water surface consists in general of travelling waves together with a local disturbance, and the type of solution is one which may have possible application to the waves produced in water by the small oscillations of a solid body.

2. Consider first deep water, and take the origin in the free surface with Ox horizontally and Oz vertically downwards. The velocity potential satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0. \qquad (1)$$

Neglecting the square of the fluid velocity at the free surface, and omitting the effect of capillarity, the condition at the free surface is

$$\frac{\partial^2\phi}{\partial t^2}-g\frac{\partial\phi}{\partial z}=0, \qquad (2)$$

and the surface elevation ζ is given by

$$\zeta=\frac{1}{g}\left(\frac{\partial\phi}{\partial t}\right)_{z=0} \qquad (3)$$

For simple harmonic motion we assume a time-factor $e^{i\sigma t}$, and (2) gives

$$\kappa_0\phi+\frac{\partial\phi}{\partial z}=0, \text{ at } z=0, \qquad (4)$$

with $\kappa_0=\sigma^2/g$.

Suppose now that we are also given

$$-\frac{\partial\phi}{\partial x}=f(z)\,e^{i\sigma t}, \text{ at } x=0, \qquad (5)$$

where $f(z)$ is given for all positive values of z; and we require a solution of (1), (4) and (8) suitable for positive values of x.

The solution can be obtained by various methods; for example, by combining suitable elementary solutions of (1) and (4). The usual solution for free progressive waves is found from

$$\phi = e^{-\kappa_0 z - i\kappa_0 x}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

There is also another elementary solution,

$$\phi = e^{-\kappa x}(\kappa \cos \kappa z - \kappa_0 \sin \kappa z), \quad . \quad . \quad . \quad . \quad (7)$$

where κ may have any real positive value.

We can generalize these solutions by means of the following integral theorem, which may easily be verified:

$$f(z) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(\alpha)\frac{(\kappa \cos \kappa z - \kappa_0 \sin \kappa z)(\kappa \cos \kappa\alpha - \kappa_0 \sin \kappa\alpha)}{\kappa^2 + \kappa_0^2}\, d\kappa\, d\alpha$$
$$+ 2\kappa_0 e^{-\kappa_0 z}\int_0^\infty f(\alpha) e^{-\kappa_0 \alpha}\, d\alpha. \quad . \quad . \quad . \quad . \quad (8)$$

Here $f(z)$ is given for all positive values of z, and it should be remarked that the proof involves the Fourier integral theorem, and that $f(z)$ is subject to suitable limitations.

We may now write down a solution which satisfies the condition (5). It is clear that, on the forced vibrations so obtained, we may superpose any free oscillations for which $\partial\phi/\partial x$ is zero over the plane $x=0$; we shall choose the latter so that the complete solution represents waves travelling outwards for large positive values of x. This solution is given by

$$\phi = 2e^{-\kappa_0 z}\sin(\sigma t - \kappa_0 x)\int_0^\infty f(\alpha)e^{-\kappa_0\alpha}d\alpha + \frac{2}{\pi}\cos\sigma t \times$$
$$\times \int_0^\infty \int_0^\infty f(\alpha)\frac{(\kappa \cos \kappa z - \kappa_0 \sin \kappa z)\times(\kappa \cos \kappa\alpha - \kappa_0 \sin \kappa\alpha)}{\kappa(\kappa^2 + \kappa_0^2)} e^{-\kappa x} d\kappa\, d\alpha. \quad . \quad (9)$$

This gives a normal velocity $f(z)\cos\sigma t$ over the plane $x=0$, and reduces to a positive wave for large positive values of x. The corresponding surface elevation is

$$\zeta = \frac{2\sigma}{g}\cos(\sigma t - \kappa_0 x)\int_0^\infty f(\alpha)e^{-\kappa_0\alpha}\, d\alpha$$
$$- \frac{2\sigma}{\pi g}\sin\sigma t\int_0^\infty \int_0^\infty f(\alpha)\frac{\kappa \cos \kappa\alpha - \kappa_0 \sin \kappa\alpha}{\kappa^2 + \kappa_0^2} e^{-\kappa x} d\kappa\, d\alpha. \quad (10)$$

The first term of (10) is a plane progressive wave of the

same wave-length as the free wave of the same frequency, while the second term may be called the local oscillation.

If we take $f(z)=e^{-\kappa_0 z}$, the second term in (9) and (10) vanishes, and we regain the expressions for a simple progressive wave.

If we take, more generally,

$$f(z)=\mathrm{A}e^{-pz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

over the whole range for z, the second integral in (10) can be evaluated explicitly in terms of Cosine and Sine integrals, and we obtain

$$\zeta=\frac{2\sigma\mathrm{A}}{\kappa_0+p}\cos(\sigma t-\kappa_0 x)-\frac{2\sigma\mathrm{A}\sin\sigma t}{\pi g(\kappa_0+p)}\times$$
$$\times\left[\mathrm{Ci}(px)\cos px-\mathrm{Ci}(\kappa_0 x)\cos\kappa_0 x+\mathrm{Si}(px)\sin px\right.$$
$$\left.-\mathrm{Si}(\kappa_0 x)\sin\kappa_0 x-\frac{\pi}{2}(\sin px-\sin\kappa_0 x)\right]. \quad . \quad (12)$$

As we make p smaller we approach the limiting case of constant normal velocity over the whole of the plane $x=0$. It is of interest to note that the amplitude of the travelling wave remains finite in the limit, but that the amplitude of the local oscillation becomes logarithmically infinite at $x=0$.

3. A problem of some interest is the decay of the vertical oscillations of a floating body due to the propagation of waves outwards from it, but a direct attack upon the problem is difficult. We may perhaps obtain a rough estimate by applying the preceding analysis to a simplified form of the problem. Imagine a log of rectangular section floating in water with the sides vertical; let b be the breadth and d the depth immersed. Now suppose the log made to execute small vertical oscillations of amplitude a and frequency σ. Let one of the sides of the log lie in the plane $x=0$; then the disturbance in the water on that side may be regarded as due to a certain oscillating distribution of normal fluid velocity over the plane $x=0$ from $z=d$ to $z=\infty$. If we make the assumption that this is of the form

$$f(z)\cos\sigma t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

then, from continuity of flow, we have

$$2\int_d^\infty f(z)dz=\sigma ab. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Without attempting to solve the actual problem, let us assume

$$f(z)=\mathrm{A}e^{-pz}; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

then from (14) we have

$$A=\tfrac{1}{2}\sigma abpe^{pd}. \qquad (16)$$

From (10) we find that the amplitude of the waves travelling out on either side of the log would be

$$\frac{\sigma^2 abp}{g}e^{pd}\int_d^{\infty}e^{-(\kappa_0+p)\alpha}\,d\alpha$$

$$=\frac{\sigma^2 abp}{g(\kappa_0+p)}e^{\kappa_0 d}. \qquad (17)$$

A large value of p would correspond to a concentration of the outward flow round the lower edges of the log; hence this estimate gives, as an upper limit for the amplitude,

$$(\sigma^2 ab/g)e^{-\sigma d/g}. \qquad (18)$$

4. If, in the general problem of §2, the normal velocity at $x=0$ is a function of y as well as of z, the solution of the three-dimensional motion can be obtained by an additional Fourier synthesis.

Assume first that ϕ is proportional to $\cos\kappa'(y-\beta)$, then the potential equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}-\kappa'^2\phi=0. \qquad (19)$$

The time entering as a simple harmonic factor, the boundary condition at $z=0$ is given by (4).

We have now the following elementary solutions, omitting the factors in y and t:

$$\phi=e^{-\kappa_0 z-ix(\kappa_0^2-\kappa'^2)^{\frac{1}{2}}}, \text{ for } \kappa'<\kappa_0;$$

$$\phi=e^{-\kappa_0 z-x(\kappa'^2-\kappa_0^2)^{\frac{1}{2}}} \text{ for } \kappa'>\kappa_0;$$

$$\phi=e^{-x(\kappa^2+\kappa'^2)^{\frac{1}{2}}}(\kappa\cos\kappa z-\kappa_0\sin\kappa z). \qquad (20)$$

The theorem (8) may be generalized, with suitable limitations on the function $f(y,z)$, to

$$f(y,z)=\frac{2\kappa_0}{\pi}e^{-\kappa_0 z}\int_0^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\int_0^{\infty}f(\alpha,\beta)e^{-\kappa_0\alpha}\cos\kappa'(y-\beta)d\kappa'$$

$$+\frac{2}{\pi^2}\int_0^{\infty}d\alpha\int_0^{\infty}d\kappa\int_{-\infty}^{\infty}d\beta\int_0^{\infty}f(\alpha,\beta)\times$$

$$(\kappa\cos\kappa z-\kappa_0\sin\kappa z)\times$$

$$\times\frac{\times(\kappa\cos\kappa\alpha-\kappa_0\sin\kappa\alpha)}{\kappa^2+\kappa_0^2}\cos\kappa'(y-\beta)d\kappa'. \qquad (21)$$

Suppose that at $x=0$ we are given

$$-\frac{\partial\phi}{\partial x}=f(y, z)\cos\sigma t. \quad . \quad . \quad . \quad . \quad (22)$$

Then from (20) and (21) we obtain an expression for ϕ, valid for positive values of x, and adjusted so that it represents progressive waves at large values of x; we find

$$\begin{aligned}\phi=&\frac{2\kappa_0}{\pi}e^{-\kappa_0 z}\int_0^\infty d\alpha\int_{-\infty}^{\infty}d\beta\int_0^{\kappa_0}f(\alpha,\beta)\sin\left(\sigma t-x\sqrt{\kappa_0^2-\kappa'^2}\right)\times\\&\times\frac{e^{-\kappa_0\alpha}\cos\kappa'(y-\beta)}{(\kappa_0^2-\kappa'^2)^{\frac{1}{2}}}d\kappa'+\frac{2\kappa_0}{\pi}e^{-\kappa_0 z}\cos\sigma t\int_0^\infty d\alpha\int_{-\infty}^{\infty}d\beta\times\\&\times\int_{\kappa_0}^\infty f(\alpha,\beta)e^{-x(\kappa'^2-\kappa_0^2)^{\frac{1}{2}}-\kappa_0\alpha}\frac{\cos\kappa'(y-\beta)}{(\kappa'^2-\kappa_0^2)^{\frac{1}{2}}}d\kappa'+\frac{2}{\pi^2}\cos\sigma t\times\\&\times\int_0^\infty d\alpha\int_0^\infty d\kappa\int_{-\infty}^{\infty}d\beta\int_0^\infty f(\alpha,\beta)e^{-x(\kappa^2+\kappa'^2)^{\frac{1}{2}}}\cos\kappa'(y-\beta)\\&\times\frac{(\kappa\cos\kappa z-\kappa_0\sin\kappa z)(\kappa\cos\kappa\alpha-\kappa_0\sin\kappa\alpha)}{(\kappa^2+\kappa_0^2)(\kappa^2+\kappa'^2)^{\frac{1}{2}}}d\kappa'. \quad . \quad . \quad (23)\end{aligned}$$

A particular case which would illustrate the spreading of plane waves emerging from a canal into an infinite sea is obtained by taking

$$f(y, z)=(ga\kappa_0/\sigma)e^{-\kappa_0 z}\cos\sigma t, \quad . \quad . \quad . \quad (24)$$

over the whole range for z and between the limits $\pm b$ for y, the function being zero outside these limits for y. Substituting in (23), the third term disappears, and also the integrations with respect to β can be effected in the remaining terms. We find that the surface elevation for this case is given by

$$\begin{aligned}\zeta=&\frac{2\kappa_0 a}{\pi}\int_0^{\kappa_0}\frac{\sin\kappa' b\cos\kappa' y\cos\{\sigma t-x(\kappa_0^2-\kappa'^2)^{\frac{1}{2}}\}}{\kappa'(\kappa_0^2-\kappa'^2)^{\frac{1}{2}}}d\kappa'\\&-\frac{2\kappa_0 a}{\pi}\cos\sigma t\int_{\kappa_0}^\infty\frac{\sin\kappa' b\cos\kappa' y e^{-x(\kappa'^2-\kappa_0^2)^{\frac{1}{2}}}}{\kappa'(\kappa'^2-\kappa_0^2)^{\frac{1}{2}}}d\kappa'. \quad . \quad . \quad (24)\end{aligned}$$

The form of the surface could be studied by approximate evaluation of these integrals as in similar diffraction problems.

5. We return to plane waves, and suppose now that the water is of finite depth h. We have the additional condition

$$\frac{\partial\phi}{\partial z}=0, \text{ for } z=h. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The corresponding elementary solutions are

$$\phi = e^{i(\sigma t - \kappa_0 x)} \cosh \kappa_0 (z-h), \quad . \quad . \quad . \quad . \quad (26)$$

where κ_0 is the real positive root of

$$g\kappa_0 \tanh \kappa_0 h = \sigma^2 ; \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and

$$\phi = e^{i\sigma t - \kappa x} \cos \kappa (z-h), \quad . \quad . \quad . \quad . \quad (28)$$

where κ is any real positive root of

$$g\kappa \tan \kappa h + \sigma^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

This equation has an infinite sequence of real roots, together with an imaginary root $i\kappa_0$. We assume then the possibility of expanding a function $f(z)$ in the range $0 \overline{<} z \overline{<} h$, in the form

$$f(z) = \mathrm{A} \cosh \kappa_0 (z-h) + \Sigma \mathrm{B} \cos \kappa (z-h), \quad . \quad (30)$$

where the summation extends over the real positive roots of the equation (29). We find that the coefficients are given by

$$\mathrm{A} = \frac{4\kappa_0}{2\kappa_0 h + \sinh 2\kappa_0 h} \int_0^h f(\alpha) \cosh \kappa_0 (\alpha - h)\, d\alpha,$$

$$\mathrm{B} = \frac{4\kappa}{2\kappa h + \sin 2\kappa h} \int_0^h f(\alpha) \cos \kappa (\alpha - h)\, d\alpha. \quad . \quad . \quad (31)$$

If at $x=0$ we are given

$$-\frac{\partial \phi}{\partial x} = f(z) \cos \sigma t, \quad . \quad . \quad . \quad . \quad (32)$$

then the velocity potential for positive values of x, such that the motion at a distance is a plane progressive wave, is given by

$$\begin{aligned} \phi = {} & \mathrm{A}\kappa_0^{-1} \cosh \kappa_0 (z-h) \sin (\sigma t - \kappa_0 x) \\ & + \Sigma \mathrm{B} \kappa^{-1} e^{-\kappa x} \cos \kappa (z-h) \cos \sigma t. \quad . \quad (33) \end{aligned}$$

Suppose, for instance, that one end of a long tank is made to execute simple harmonic vibrations of small amplitude a, then we have $f(z) = \sigma a$. The values of A and B follow from (31), and from (33) we deduce the surface elevation in this case :

$$\begin{aligned} \zeta = {} & \frac{2\sigma^2 a \sinh 2\kappa_0 h}{g\kappa_0 (2\kappa_0 h + \sinh 2\kappa_0 h)} \cos (\sigma t - \kappa_0 x) \\ & - \sin \sigma t \Sigma \frac{2\sigma^2 a e^{-\kappa x} \sin 2\kappa h}{g\kappa (2\kappa h + \sin 2\kappa h)}. \quad . \quad . \quad (34) \end{aligned}$$

6. The same analysis may be applied to circular waves, and we limit consideration here to symmetry round the origin. The normal fluid velocity is supposed to be assigned over a vertical cylindrical surface; for example, we take

$$-\frac{\partial\phi}{\partial r}=f(z)\cos\sigma t,\ \text{for } r=a. \quad . \quad . \quad . \quad (35)$$

The velocity potential satisfies

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0. \quad . \quad . \quad . \quad . \quad (36)$$

The condition at the free surface is the same as before, and we assume the water to be deep. Elementary solutions of the required form are found in terms of suitable Bessel functions. The solution

$$\phi=e^{i\sigma t-\kappa_0 z}\mathrm{H}_0^{(2)}(\kappa_0 r) \quad . \quad . \quad . \quad . \quad . \quad (37)$$

represents diverging waves for large values of r; while in the solution

$$\phi=e^{i\sigma t}(\kappa\cos\kappa z-\kappa_0\sin\kappa z)\,\mathrm{K}_0(\kappa r), \quad . \quad . \quad (38)$$

$\mathrm{K}_0(\kappa r)$ tends exponentially to zero for large distances from the origin.

Generalizing as before, we obtain the solution

$$\phi=2e^{i\sigma t-\kappa_0 z}\frac{\mathrm{H}_0^{(2)}(\kappa_0 r)}{\mathrm{H}_1^{(2)}(\kappa_0 a)}\int_0^\infty f(\alpha)e^{-\kappa_0\alpha}d\alpha-\frac{2}{\pi}e^{i\sigma t}\times$$

$$\times\int_0^\infty\int_0^\infty f(\alpha)\frac{\mathrm{K}_0(\kappa r)}{\kappa\mathrm{K}_0'(\kappa a)}\,\frac{(\kappa\cos\kappa z-\kappa_0\sin\kappa z)\times(\kappa\cos\kappa\alpha-\kappa_0\sin\kappa\alpha)}{\kappa^2+\kappa_0^2}\,d\kappa\,d\alpha, \quad (39)$$

where the real part is to be taken.

The surface elevation at a great distance from the origin is given by

$$\zeta\sim-\frac{2i\sigma}{g}\left(\frac{2}{\pi\kappa_0 r}\right)^{\frac{1}{2}}\frac{e^{i(\sigma t-\kappa_0 r+\frac{1}{4}\pi)}}{\mathrm{J}_0'(\kappa_0 a)-i\mathrm{Y}_0'(\kappa_0 a)}\int_0^\infty f(\alpha)e^{-\kappa_0\alpha}d\alpha, \quad (40)$$

or, in real terms, this gives

$$\zeta-\frac{2\sigma}{g}\left(\frac{2}{\pi\kappa_0 r}\right)\int_0^\infty f(\alpha)\,e^{-\kappa_0\alpha}d\alpha\times$$

$$\times\frac{\mathrm{J}_0'(\kappa_0 a)\sin(\sigma t-\kappa_0 r+\frac{1}{4}\pi)+\mathrm{Y}_0'(\kappa_0 a)\cos(\sigma t-\kappa_0 r+\frac{1}{4}\pi)}{\mathrm{J}_0'^2(\kappa_0 a)+\mathrm{Y}_0'^2(\kappa_0 a)} \quad . \quad . \quad . \quad . \quad (41)$$

This expression might be used, as in §3, to give an estimate for the energy propagated outwards from a circular cylinder immersed to a given depth, and making small vertical oscillations of given frequency.

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The Wave Resistance of a Spheroid.

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1. A method which has been used to calculate the wave resistance of a submerged solid is to replace the solid by a distribution of sources and sinks, or of doublets, the distribution being the image system for the solid in a uniform stream. The cases which have been solved hitherto have been limited to those in which the image system is either a single doublet or a distribution of doublets lying in a vertical plane parallel to the direction of motion. It is shown here how to obtain the solution for an ellipsoid moving horizontally at given depth below the surface of the water, and with its axes in any assigned directions. The present paper deals specially with prolate and oblate spheroids moving end-on and broadside-on, the general case of an ellipsoid with unequal axes being left for a subsequent paper.

In § 2 it is shown that the image system for an ellipsoid in a uniform stream is a certain surface distribution of parallel doublets over the elliptic focal conic, the direction of the doublets being in general inclined to the direction of motion; if the motion is parallel to a principal axis, the doublets are in the same direction. For a spheroid the image system reduces to either a line distribution or to a surface distribution over a certain circle; explicit expressions are given in § 3 for prolate and oblate spheroids when moving either in the direction of the axis of symmetry or at right angles to that axis.

The calculation of the wave resistance is considered in § 4. An expression has been given previously for the wave resistance associated with two doublets at any points in the liquid with their axes in any assigned directions; this can be generalised to cover continuous line, surface or volume distributions of doublets. Incidentally, it is shown how by integration we may pass from a three-dimensional doublet, corresponding to a submerged sphere, to a two-dimensional doublet, corresponding to a circular cylinder. In § 5 expressions for the wave resistance are developed for the particular cases of moving spheroids of § 3. In the final section these results are illustrated by numerical and graphical calculations for certain series of models. In each case the axis of the spheroid is supposed horizontal, and to make the calculations definite the depth of the axis is taken to be twice the radius of the central circular section. The models consist of a sphere, radius b; an oblate spheroid

with semi-axis $a = 4b/5$; and prolate spheroids with $a = 5b/4$, $5b/2$ and $5b$ respectively. Graphs are given for the variation of wave resistance with velocity for these five models (i) when moving in the direction of the axis of revolution, (ii) when moving at right angles to that axis; these illustrate respectively the effect of increased length, and the effect of increased beam and area of cross-section. It is of interest to note that increase of length gives diminished resistance at low speeds, with a subsequent rapid increase; while increasing beam in the second series gives increased resistance at all speeds.

2. Consider the motion of a solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

in an infinite liquid, the velocity being u parallel to Ox.

It is well known that if V is the gravitational potential of a uniform solid of unit density bounded by (1), then the velocity potential of the fluid motion is given by

$$\phi = \frac{u}{2\pi(2 - \alpha_0)} \frac{\partial V}{\partial x}, \tag{2}$$

where

$$\alpha_0 = abc \int_0^\infty \frac{du}{(a^2 + u)^{3/2} (b^2 + u)^{1/2} (c^2 + u)^{1/2}}. \tag{3}$$

Since

$$V = \iiint \frac{dx'\, dy'\, dz'}{\{(x - x')^2 + (y - y')^2 + (z - z')^2\}^{1/2}}, \tag{4}$$

taken throughout the ellipsoid, it follows from (2) and (4) that the velocity potential of the fluid motion is that due to a uniform volume distribution of doublets throughout the ellipsoid, with their axes parallel to Ox, and of moment per unit volume equal to $u/2\pi(2 - \alpha_0)$.

Similarly for motion parallel to Oy or Oz we have a like result with a corresponding quantity β_0 or γ_0 taking the place of α_0. For motion in any other direction we resolve the velocity along the three axes and combine the component doublet systems.

The gravitational potentials of two solid homogeneous ellipsoids, bounded by confocals, at any point external to both are proportional to their masses. Hence in the hydrodynamical problem we may replace the distribution of doublets throughout the ellipsoid (1) by a uniform distribution through any interior confocal, increasing the moment per unit volume by the factor

$$abc/\sqrt{\{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)\}}, \tag{5}$$

where λ is the parameter of the confocal.

In particular, we obtain the simplest system by taking the confocal which reduces in the limit to the elliptic focal conic

$$z = 0\,;\ \frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \tag{6}$$

with $a > b > c$. In this case the volume distribution of doublets reduces to a surface distribution over the plane area bounded externally by (6). The moment per unit area is found by putting $\lambda^2 = -c^2 + \delta$ and taking limiting values as $\delta \to 0$, taking into account the factor (5) and the limiting thickness of the confocal at each point. We may refer to the distribution found in this way as the image system for an ellipsoid in a uniform stream.

If the motion is parallel to Ox, the doublets are parallel to Ox and are distributed over (6) with a moment per unit area given by

$$\frac{abcu}{\pi\,(2 - \alpha_0)\,(a^2 - c^2)^{1/2}\,(b^2 - c^2)^{1/2}}\left(1 - \frac{x^2}{a^2 - c^2} - \frac{y^2}{b^2 - c^2}\right)^{1/2}. \tag{7}$$

There are similar expressions for motion parallel to Oy, Oz with β_0, γ_0 respectively in place of α_0.

3. We shall specify now the particular results for spheroids, using the known values of α_0, β_0, γ_0. We take Ox to be the axis of symmetry, with $c = b$; and consider first motion parallel to the axis of symmetry.

For a prolate spheroid, the focal conic reduces to the line joining the foci of the generating ellipse. The image system reduces to a line distribution along Ox, from $x = -ae$ to $x = ae$, of moment per unit length

$$\mathrm{A}u\,(a^2e^2 - x^2), \tag{7}$$

where

$$\mathrm{A}^{-1} = 4e/(1 - e^2) - 2\log\{(1 + e)/(1 - e)\}, \tag{8}$$

with $e^2 = 1 - b^2/a^2$.

For an oblate spheroid under the same conditions, the system is a surface distribution of doublets parallel to Ox, over the circle

$$x = 0\,;\ \ y^2 + z^2 = b^2e'^2, \tag{9}$$

where $e'^2 = 1 - a^2/b^2$; and the moment per unit area is

$$\mathrm{B}u\,(b^2e'^2 - y^2 - z^2)^{1/2}, \tag{10}$$

with

$$\mathrm{B}^{-1} = 2\pi\left(\sin^{-1} e' - e'\sqrt{1 - e'^2}\right). \tag{11}$$

For motion at right angles to the axis of symmetry, we take Oy as the direction

of motion. For a prolate spheroid the system is a line distribution along Ox between $x = \pm ae$, with axes parallel to Oy, and of moment per unit length

$$A'u\,(a^2e^2 - x^2), \tag{12}$$

where

$$A'^{-1} = 2e\,(2e^2 - 1)/(1 - e^2) + \log\{(1+e)/(1-e)\}. \tag{13}$$

For an oblate spheroid the system consists of doublets parallel to Oy, over the circle

$$x = 0\,; \quad y^2 + z^2 = b^2e'^2, \tag{14}$$

and of moment per unit area

$$B'u\,(b^2e'^2 - y^2 - z^2)^{\frac{1}{2}}, \tag{15}$$

where

$$B'^{-1} = \pi\{e'(1+e'^2)/(1-e'^2)^{1/2} - \sin^{-1} e'\}. \tag{16}$$

For $e = 0$, all these distributions reduce to the finite doublet at the origin appropriate to the motion of a sphere.

4. Consider now the wave resistance when an ellipsoid is wholly immersed at some depth in water and is moving with constant horizontal velocity; we obtain the first approximation for the resistance by replacing the ellipsoid by the image system which was discussed in the preceding section. The resistance is derived from the doublet system by expressions which have been given previously; in particular, reference may be made to an expression for the wave resistance corresponding to two doublets at any points in the water with their axes in any given directions.* We shall not quote the general result, as we require here only the case in which the doublets have their axes parallel to the direction of motion. Take the origin O in the free surface of the water, Oz vertically upwards; for a doublet of moment M at the point $(h, k, -f)$ and a doublet M′ at $(h', k', -f')$, both axes being parallel to Ox, the direction of motion, the wave resistance is given by

$$R = 16\pi\rho\kappa_0^4 \int_0^{\pi/2} \{M^2 e^{-2\kappa_0 f \sec^2\theta} + M'^2 e^{-2\kappa_0 f' \sec^2\theta} + 2MM'\,e^{-\kappa_0 (f+f')\sec^2\theta} \cos A \cos B\} \sec^5\theta\, d\theta, \tag{17}$$

with

$$\kappa_0 = g/u^2; \quad A = \kappa_0 (h - h') \sec\theta\,; \quad B = \kappa_0 (k - k') \sin\theta \sec^2\theta.$$

This can easily be extended to continuous distributions. For distributions

* 'Proc. Roy. Soc.,' A, vol. 118, p. 32 (1928).

in a vertical plane parallel to the direction of motion, to which previous work has been limited, we have

$$R = 16\pi\rho\kappa_0^4 \int_0^\infty df \int_0^\infty df' \int_{-\infty}^\infty dh \int_{-\infty}^\infty dh' \int_0^{\pi/2} M(h, f)\, M(h', f')$$
$$\times\, e^{-\kappa_0 (f+f') \sec^2\theta} \cos\{\kappa_0 (h - h') \sec\theta\} \sec^5\theta\, d\theta, \tag{18}$$

where we have taken $y = 0$ as the plane of distribution. This expression can be written as

$$R = 16\pi\rho\kappa_0^4 \int_0^{\pi/2} (P^2 + Q^2) \sec^5\theta\, d\theta, \tag{19}$$

where

$$P + iQ = \int_0^\infty df \int_{-\infty}^\infty dh \,.\, M(h, f) \,.\, e^{-\kappa_0 f \sec^2\theta + i\kappa_0 h \sec\theta}. \tag{20}$$

When the distribution is in a plane perpendicular to the direction of motion, say the plane $x = 0$, it is easily seen that we have the same expression (19) for R, but now

$$P + iQ = \int_0^\infty df \int_{-\infty}^\infty dk \,.\, M(k, f) \,.\, e^{-\kappa_0 f \sec^2\theta + i\kappa_0 k \sin\theta \sec^2\theta}. \tag{21}$$

If the doublets are distributed along a line, the suitable forms for R may readily be deduced from these expressions.

Before proceeding to apply these results to spheroids, we may notice a simple case of (21). The first problem in wave resistance to be solved was that of a two-dimensional doublet corresponding to the motion of a circular cylinder with its axis horizontal and moving at right angles to the axis; the next problem was the three-dimensional doublet for the motion of a sphere. By means of (19) and (21) we may pass from the second problem to the first by integration.

Write down the velocity potential of a uniform distribution of three-dimensional doublets of moment M per unit length over a straight line of finite length, the axes of the doublets being at right angles to this line; evaluate the expression in the limit when the length of the distribution becomes infinite, and we obtain the velocity potential of a two-dimensional doublet of moment 2M. Consider now the expression for the wave resistance for the same process; if $2l$ is the length of the distribution, (21) gives

$$P + iQ = \int_{-l}^{l} M e^{-\kappa_0 f \sec^2\theta + i\kappa_0 k \sin\theta \sec^2\theta}\, dk. \tag{22}$$

Evaluating the integral and using (19) we have

$$R = 64\pi\rho\kappa_0{}^2 M^2 \int_0^{\pi/2} \frac{\sin^2(\kappa_0 l \sin\theta \sec^2\theta)}{\sin^2\theta\cos\theta} e^{-2\kappa_0 f\sec^2\theta} d\theta. \tag{23}$$

The wave resistance for the corresponding two-dimensional doublet is for unit length perpendicular to the plane of motion, and should be given by $\lim (R/2l)$ as $l \to \infty$. From (23), this is

$$\operatorname*{Lim}_{l\to\infty} 32\pi\rho\kappa_0{}^2 M^2 e^{-2\kappa_0 f}\int_0^\infty \sin^2(\kappa_0 u\sqrt{1+u^2/l^2})\, e^{-2\kappa_0 f u^2/l^2} u^{-2}\sqrt{1+u^2/l^2}\, du$$
$$= 16\pi^2\rho\kappa_0{}^3 M^2 e^{-2\kappa_0}, \tag{24}$$

and this is the known expression for the wave resistance of a two-dimensional doublet of the corresponding moment 2M.

5. We proceed now to the wave resistance of a submerged spheroid, taking in each case the axis of the spheroid to be horizontal and at a depth f below the surface of the water.

Prolate Spheroid in Direction of Axis.—From (7) and (20) we have

$$(P + iQ)/Aue^{-\kappa_0 f\sec^2\theta} = \int_{-ae}^{ae} (a^2e^2 - h^2)\, e^{i\kappa_0 h\sec\theta}\, dh$$
$$= (8\pi a^3e^3/\kappa_0{}^3 \sec^3\theta)^{1/2} J_{3/2}(\kappa_0 ae \sec\theta), \tag{25}$$

where J denotes the usual Bessel function. Hence from (19),

$$R = 128\pi^2 g\rho a^3e^3A^2 \int_0^{\pi/2} e^{-2\kappa_0 f\sec^2\theta}\{J_{3/2}(\kappa_0 ae\sec\theta)\}^2 \sec^2\theta\, d\theta, \tag{26}$$

a result which was obtained previously by a different method.* For purposes of numerical calculation it is convenient to change the variable in the integration from θ to $\tan\theta$; we then have

$$R = 128\pi^2 g\rho a^3e^3A^2e^{-p}\int_0^\infty e^{-pt^2}\{J_{3/2}(\kappa_0 ae\sqrt{1+t^2})\}^2\, dt, \tag{27}$$

where $p = 2\kappa_0 f = 2gf/u^2$, and A is given in (8).

Oblate Spheroid in Direction of Axis.—Here we have a surface distribution given by (10), and remembering that the centre of the circular distribution is at a depth f, (21) gives

$$(P + iQ)/Bue^{-\kappa_0 f\sec^2\theta} = \iint (b^2e'^2 - y^2 - z^2)^{1/2}\, e^{\kappa_0 z\sec^2\theta + i\kappa_0 y\sin\theta\sec^2\theta}\, dy\, dz, \tag{28}$$

taken over the circle $y^2 + z^2 = b^2e'^2$.

* 'Proc. Roy. Soc.,' A, vol. 95, p. 365 (1919).

Taking the integration with respect to y first, we obtain, after integration by parts,

$$(\kappa_0 \sin \theta \sec^2 \theta)^{-1} \int_{-(b^2e'^2-z^2)^{1/2}}^{(b^2e'^2-z^2)^{1/2}} y\,(b^2e'^2 - y^2 - z^2)^{-1/2} \sin(\kappa_0 y \sin \theta \sec^2 \theta)\, dy$$

$$= \frac{\pi\,(b^2e'^2 - z^2)^{1/2}}{\kappa_0 \sin \theta \sec^2 \theta} J_1\left(\kappa_0 \sqrt{b^2e'^2 - z^2} \sin \theta \sec^2 \theta\right). \quad (29)$$

The integration with respect to z now becomes

$$\int_{-be'}^{be'} (b^2e'^2 - z^2)^{1/2} e^{\kappa_0 z \sec^2 \theta} J_1\left(\kappa_0 \sqrt{b^2e'^2 - z^2} \sin \theta \sec^2 \theta\right) dz, \quad (30)$$

and this is equivalent to evaluating

$$2 \int_0^{\pi/2} \cosh(\alpha \cos \phi)\, J_1(\beta \sin \phi) \sin^2 \phi\, d\phi, \quad (31)$$

where $\alpha = \kappa_0 be' \sec^2 \theta$, $\beta = \kappa_0 be' \sin \theta \sec^2 \theta$.

The integral (31) may be evaluated as a special case of Sonine's integral, or by expanding $\cosh(\alpha \cos \phi)$ in powers of $\cos \phi$, integrating term by term, and summing the resulting series. The latter expression for (31) is found to be

$$2 \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2n!} \frac{2^{n-1/2}\,\Gamma(n + \frac{1}{2})}{\beta^{n+1/2}} J_{n+3/2}(\beta). \quad (32)$$

Noting that in the present problem, $\alpha \gg \beta$, the value of (32), or of the integral (31), is

$$2 \left(\frac{\pi}{2}\right)^{1/2} \frac{\beta}{(\alpha^2 - \beta^2)^{3/4}} I_{3/2}\{(\alpha^2 - \beta^2)^{1/2}\}, \quad (33)$$

where the Bessel function is given by

$$I_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\cosh x - \frac{\sinh x}{x}\right). \quad (34)$$

Collecting these results, we obtain

$$(P + iQ)/Bu\, e^{-\kappa_0 f \sec^2 \theta} = 2(\pi^3 b^3 e'^3 / 2\kappa_0^3 \sec^3 \theta)^{1/2} I_{3/2}(\kappa_0 be' \sec \theta). \quad (35)$$

Finally, from (19) we find

$$R = 32\,\pi^4 \rho \kappa_0 b^3 e'^3 B^2 u^2 \int_0^{\pi/2} e^{-2\kappa_0 f \sec^2 \theta} \{I_{3/2}(\kappa_0 be' \sec \theta)\}^2 \sec^2 \theta\, d\theta. \quad (36)$$

or in the same form as (27),

$$R = 32\,\pi^4 g \rho b^3 e'^3 B^2 e^{-p} \int_0^{\infty} e^{-pt^2} \{I_{3/2}(\kappa_0 b e' \sqrt{1 + t^2})\}^2 dt, \quad (37)$$

where B is given in (11).

Prolate Spheroid at Right Angles to its Axis.—The distribution is given in (12), and in this case we use (21) instead of (20); apart from this, the calculation follows the same course and we obtain finally

$$\mathrm{R} = 128\pi^2 g\rho a^3 e^3 \mathrm{A}'^2 \int_0^{\pi/2} e^{-2\kappa_0 f \sec^2\theta} \{\mathrm{J}_{3/2}(\kappa_0 ae \sin\theta \sec^2\theta)\}^2 \cos\theta \, d\theta/\sin^3\theta$$

$$= 128\pi^2 g\rho a^3 e^3 \mathrm{A}'^2 e^{-p} \int_0^{\infty} e^{-pt^2} \{\mathrm{J}_{3/2}(\kappa_0 aet \sqrt{1+t^2})\}^2 t^{-3} \, dt, \tag{38}$$

with A′ given in (13).

Oblate Spheroid at Right Angles to its Axis.—The distribution given in (15) lies in a plane parallel to the direction of motion, so we now use (20); the integrals are, however, of the same type as those already discussed and the analysis need not be given in detail. Using (15), (19) and (20), we obtain after some reduction

$$\mathrm{R} = 32\pi^4 g\rho b^3 e'^3 \mathrm{B}'^2 e^{-p} \int_0^{\infty} e^{-pt^2} \{\mathrm{I}_{3/2}(\kappa_0 be't \sqrt{1+t^2})\}^2 t^{-3} \, dt. \tag{39}$$

where B′ is given in (16).

Sphere.—It may easily be verified that in the limit when e, or e', becomes zero, all these expressions (27), (37), (38) and (39), reduce to the known result for a sphere, namely

$$\mathrm{R} = 4\pi g\rho\kappa_0{}^3 b^6 e^{-p} \int_0^{\infty} (1+t^2)^{3/2} e^{-pt^2} \, dt$$

$$= \pi g\rho\kappa_0{}^3 b^6 e^{-\frac{1}{2}p} \left\{ \mathrm{K}_0(\tfrac{1}{2}p) + \left(1 + \frac{1}{p}\right) \mathrm{K}_1(\tfrac{1}{2}p) \right\}, \tag{40}$$

where K_n is the Bessel function defined by

$$\mathrm{K}_n(x) = \int_0^{\infty} e^{-x\cosh u} \cosh nu \, du. \tag{41}$$

6. The resistances for prolate and oblate spheroids have been worked out independently in the preceding section. It is of interest to note that the results have the same analytical form and may, in fact, be deduced from each other by taking the eccentricity to be imaginary instead of real. For the prolate spheroid, $e^2 = 1 - b^2/a^2$; while for the oblate spheroid, $e'^2 = 1 - a^2/b^2$. It may be verified that if in (27) we write $e = ie'b/a$, the expression transforms precisely into (37); and the same relation holds between (38) and (39).

7. The integrals in the various expressions can be transformed into alternative forms, or expressed in infinite series in several ways; but either the series do not converge rapidly enough for the values of the parameters which are of interest, or else the functions involved have not been tabulated. It

has been found simpler to make numerical calculations directly from the integrals as given, although a considerable amount of work is involved in any case.

The calculations have been carried out for a set of five spheroids, including the sphere, the radius b of the central circular section being supposed constant and the semi-axis a varied. The following are the data for the series :—A, oblate, $a = 4b/5$, $e' = 0\cdot6$; B, sphere, $a = b$; C, prolate, $a = 5b/4$, $e = 0\cdot6$; D, prolate, $a = 5b/2$, $e = 0\cdot9165$; E, prolate, $a = 5b$, $e = 0\cdot9798$. The axial sections of these forms are shown in fig. 1, drawn to scale, the diagram showing one quarter of the section in each case.

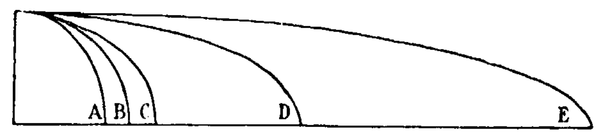

FIG. 1.

We suppose the axis horizontal in each case and at the same depth f below the free surface. To make a definite case for numerical calculation we take

$$f = 2b, \tag{42}$$

that is, the depth twice the radius of the central circular section. We consider the models in two series, (i) with the axis in the direction of motion, (ii) with the axis at right angles to the motion. Our object is to show the variation of wave-resistance with velocity for each model, and to see how the graph varies, in (i) with increasing length, and in (ii) with increasing beam. To give one example of the calculations, when $a = 5b/2$, (27) gives

$$R = 22\cdot70\pi g\rho b^3 e^{-p}\int_0^\infty e^{-pt^2}\left\{J_{3/2}(0\cdot5728\,p\sqrt{1+t^2})\right\}^2 dt. \tag{43}$$

For velocities which are of special interest, the parameter p ranges from about 1 to 8. A graph of the Bessel function $J_{3/2}$ was drawn on a large scale and values were taken from it, except for small values of the argument when they were calculated from tables of $J_{1/2}$ and $J_{-1/2}$. Values of the integrand were calculated for values of t at intervals of $0\cdot1$, and the numerical integration carried out by the usual methods. Owing to the exponential factor, it was unnecessary to go beyond $t = 2$ in any case ; and for the larger values of p, a smaller range of t was sufficient. This process was carried out for seven or eight values of p, and so a graph could be drawn for the variation of R with p, that is, with velocity u.

A similar method was used for the integrals in (37) and (38). For (39), the Bessel function was expanded in powers of $(1+t^2)$, and integration carried out term-by-term; the integrals involved are then of the form

$$\int_0^{\infty}(1+t^2)^{\frac{2n+1}{2}}e^{-pt^2}\,dt, \tag{44}$$

which can be expressed in terms of the Bessel function K_n defined in (41). By recurrence formulæ, the terms can be reduced to expressions involving K_0 and K_1, and tables of these functions are available. In all these calculations no attempt was made to obtain any high degree of numerical accuracy; the object was to obtain sufficient values to enable graphs to be drawn showing the nature of the results and the main differences between the two series.

The graphs are shown in figs. 2 and 3; the scale is the same throughout, the ordinates being $R/\pi g\rho b^3$, and the abscissæ $u/\sqrt{(gf)}$.

The nature of the results is obvious from the graphs. Fig. 2 shows the curves for the end-on motion. The curve B, which is the same in both

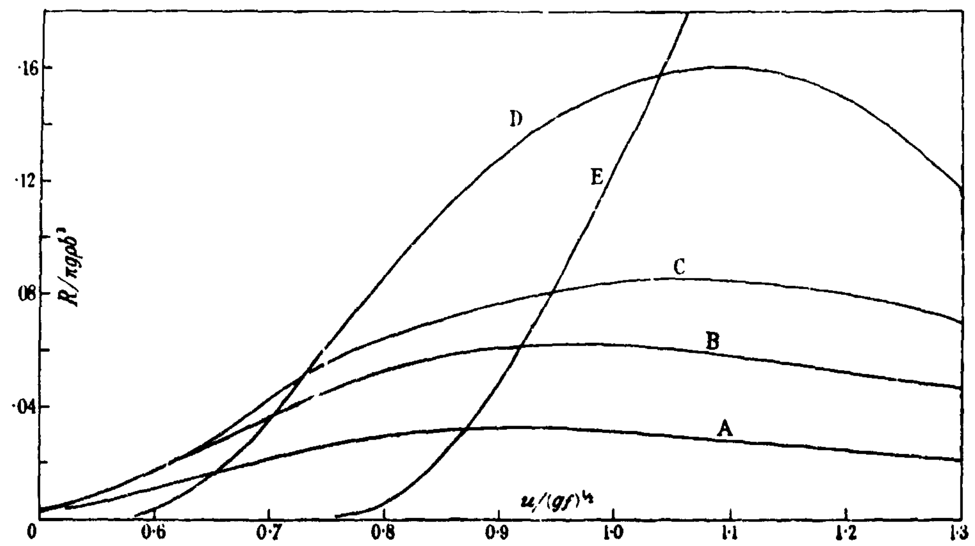

Fig. 2.

diagrams, is for the sphere and shows the maximum just before the velocity $(gf)^{1/2}$. The graphs for C, D, E show how much the resistance is diminished at the lower velocities by increasing length in this way; but this is followed by a rapid increase at higher velocities. The latter effect may be described, roughly, as due to the final interference between bow and stern system giving a prominent hump on the resistance curve; the interference effects at lower

speeds were found in the calculations for curve E, but could not be shown on the scale of the diagram.

The graphs in fig. 3 for the broadside motion are in striking contrast to those in fig. 2. Here we have increased resistance at all velocities as we go up the series of models; the values for E were calculated, but could not be shown on

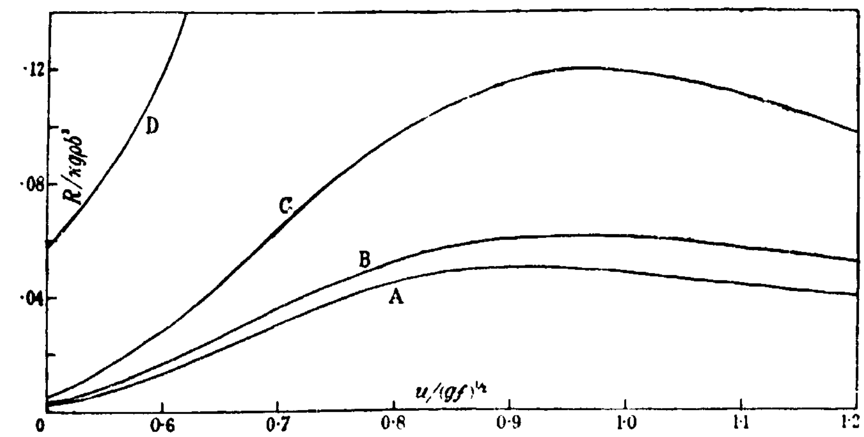

FIG. 3.

the scale of the diagram. It appears from the curves of fig. 3 that a rough empirical rule for this series is that the resistance per unit volume of displacement is proportional to the area of the midship section.

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The Wave Resistance of an Ellipsoid

By T. H. HAVELOCK, F.R.S.

(Received May 9, 1931.)

1. In a recent paper* it was shown how to obtain, to the usual approximation, the wave resistance of a solid of ellipsoidal form submerged at a constant depth below the surface of water and moving horizontally with any orientation of the axes; and explicit calculations were made for prolate and oblate spheroids moving end-on and broadside-on. The present note is a brief study of an ellipsoid with unequal axes, moving in the direction of the longest axis. It had been intended to examine numerically in some detail the effect of different ratios of the axes upon the resistance-velocity curve; but the necessary computations would have been lengthy, and the main results of interest may be seen from the form of the expressions obtained for the wave resistance. In the discussion attention is directed specially to cases in which the ratios of the axes are similar to the corresponding ratios for a ship.

2. It is convenient first to evaluate some integrals which occur in the analysis.

Consider the integral

$$A = \iint\left(1 - \frac{x^2}{m^2} - \frac{y^2}{n^2}\right)^{1/2} \cos \alpha x \cos \beta y \, dx \, dy, \tag{1}$$

taken over the ellipse

$$\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1. \tag{2}$$

Putting $x = m \sin \phi \cos \theta$, $y = n \cos \phi$, we obtain

$$A = mn \int_0^\pi \int_0^\pi \sin^3 \phi \sin^2 \theta \cos (m\alpha \sin \phi \cos \theta) \cos (n\beta \cos \phi) \, d\theta \, d\phi. \tag{3}$$

Integrating first with respect to θ, this gives

$$A = (\pi n/\alpha) \int_0^\pi \cos (n\beta \cos \phi) \, J_1 (m\alpha \sin \phi) \sin^2 \phi \, d\phi$$

$$= \left(\frac{2\pi^3 n^3 \beta}{\alpha}\right)^{1/2} \int_0^{\pi/2} J_1 (m\alpha \sin \phi) \, J_{-1/2} (n\beta \cos \phi) \sin^2 \phi \cos^{1/2} \phi \, d\phi. \tag{4}$$

* 'Proc. Roy. Soc.,' A, vol. 131, p. 275 (1931).

This is a particular case of Sonine's integral,* and we obtain finally

$$A = 2^{1/2}\pi^{3/2} mn \frac{J_{3/2}\{(m^2\alpha^2 + n^2\beta^2)^{1/2}\}}{(m^2\alpha^2 + n^2\beta^2)^{3/4}}. \tag{5}$$

A similar integral which we require is

$$B = \iint \left(1 - \frac{x^2}{m^2} - \frac{y^2}{n^2}\right)^{1/2} e^{\beta y} \cos \alpha x \, dx \, dy, \tag{6}$$

taken over the ellipse (2).

This may be evaluated in the same manner. To avoid possible ambiguity we distinguish between various cases according to the relative magnitude of $m\alpha$ and $n\beta$. We find

$$\begin{aligned} B &= 2^{1/2}\pi^{3/2} mn \frac{J_{3/2}\{(m^2\alpha^2 - n^2\beta^2)^{1/2}\}}{(m^2\alpha^2 - n^2\beta^2)^{3/4}}, \qquad m\alpha > n\beta\,; \\ &= 2^{1/2}\pi^{3/2} mn \frac{I_{3/2}\{(n^2\beta^2 - m^2\alpha^2)^{1/2}\}}{(n^2\beta^2 - m^2\alpha^2)^{3/4}}, \qquad m\alpha < n\beta\,; \\ &= \tfrac{2}{3}\pi mn, \qquad m\alpha = n\beta\ ; \end{aligned} \tag{7}$$

where I denotes the usual modified Bessel function.

3. Consider a solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{8}$$

moving with uniform speed u in the direction of Ox, the axis Ox being horizontal and at a depth f below the surface of the water, while the axis Oy is vertical.

We shall consider first the case $a > b > c$.

The image of a uniform stream in the ellipsoid is a distribution of doublets over the plane area bounded by the elliptic focal conic

$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \qquad z = 0\,; \tag{9}$$

the axes of the doublets are parallel to Ox, and the moment per unit area is

$$\frac{abcu}{\pi(2-\alpha_0)(a^2-c^2)^{1/2}(b^2-c^2)^{1/2}}\left(1 - \frac{x^2}{a^2-c^2} - \frac{y^2}{b^2-c^2}\right)^{1/2},$$

where

$$\alpha_0 = abc\int_0^\infty \frac{du}{(a^2+u)^{3/2}(b^2+u)^{1/2}(c^2+u)^{1/2}}. \tag{10}$$

For numerical calculation α_0 may be expressed in terms of elliptic integrals.

* G. N. Watson, "Bessel Functions," p. 376 (1st edn., 1922).

From (12) of the previous paper, the wave resistance is given by

$$R = 16\pi\rho\kappa_0^4 \int_0^{\pi/2} P^2 \sec^5\theta\, d\theta, \tag{11}$$

$$P = \frac{abcue^{-\kappa_0 f \sec^2\theta}}{\pi(2-\alpha_0)(a^2-c^2)^{1/2}(b^2-c^2)^{1/2}} \iint \left(1 - \frac{x^2}{a^2-c^2} - \frac{y^2}{b^2-c^2}\right)^{1/2} e^{\kappa_0 y \sec^2\theta} \times \cos(\kappa_0 x \sec\theta)\, dx\, dy, \tag{12}$$

the integral for P being taken over the ellipse (9), and $\kappa_0 = g/u^2$.

Comparing with (6) and (7), we obtain for the integral in the expression for P the value

$$\{2\pi^3(a^2-c^2)(b^2-c^2)\}^{1/2} \frac{J_{3/2}[\kappa_0 \sec\theta\{a^2-c^2-(b^2-c^2)\sec^2\theta\}^{1/2}]}{\kappa_0^{3/2}\sec^{3/2}\theta\{a^2-c^2-(b^2-c^2)\sec^2\theta\}^{3/4}}, \tag{13}$$

when $\cos\theta > \sqrt{\{(b^2-c^2)/(a^2-c^2)\}}$, and a similar expression when $\cos\theta$ is less than this value. Collecting these expressions, and for comparison with previous results, putting $\tan\theta = t$, we obtain finally

$$\frac{(2-\alpha_0)^2(a^2-b^2)^{3/2}e^{2\kappa_0 f}}{32\pi^2 g\rho a^2b^2c^2} R$$
$$= \int_0^{1/\alpha} \frac{[J_{3/2}\{\kappa_0^2(a^2-b^2)(1+t^2)(1-\alpha^2t^2)\}^{1/2}]^2}{(1-\alpha^2t^2)^{3/2}} e^{-2\kappa_0 f t^2} dt$$
$$+ \int_{1/\alpha}^{\infty} \frac{[I_{3/2}\{\kappa_0^2(a^2-b^2)(1+t^2)(\alpha^2t^2-1)\}^{1/2}]^2}{(\alpha^2t^2-1)^{3/2}} e^{-2\kappa_0 f t^2} dt, \tag{14}$$

where $\alpha^2 = (b^2-c^2)/(a^2-b^2)$.

This expression is for an ellipsoid moving horizontally in the direction of the longest axis, and having the least axis horizontal and the mean axis vertical; or, we may say, with the beam less than the draught.

4. We consider now the case when the beam is greater than the draught; that is, keeping the axes Ox, Oy, Oz as before, we have $a > c > b$. The elliptic focal conic is now in the horizontal plane and is given by

$$\frac{x^2}{a^2-b^2} + \frac{z^2}{c^2-b^2} = 1, \qquad y = 0. \tag{15}$$

The doublet system is distributed over the area bounded by (15), the axes being parallel to Ox and the moment per unit area being given by

$$M(x, z) = \frac{abcu}{\pi(2-\alpha_0)(a^2-b^2)^{1/2}(c^2-b^2)^{1/2}}\left(1 - \frac{x^2}{a^2-b^2} - \frac{z^2}{c^2-b^2}\right)^{1/2}. \tag{16}$$

For a distribution of this type the expression* for the wave resistance of any two doublets generalises into

$$R = 16\pi\rho\kappa_0^4 \int_0^{\pi/2} P^2 e^{-2\kappa_0 f \sec^2\theta} \sec^5\theta \, d\theta, \tag{17}$$

where

$$P^2 = \iiiint M(x, z) \, M(x', z') \cos\{\kappa_0 (x - x') \sec\theta\} \cos\{\kappa_0 (z - z') \sin\theta \sec^2\theta\} \, dx\, dz\, dx'\, dz'. \tag{18}$$

From the symmetry of the distribution specified in (15) and (16) we see that

$$P = \iint M(x, z) \cos(\kappa_0 x \sec\theta) \cos(\kappa_0 z \sin\theta \sec^2\theta) \, dx\, dz, \tag{19}$$

where M is given in (16) and the integration extends over the ellipse (15).

Comparing with (1) and (5), we obtain

$$P = \frac{(2\pi)^{1/2} abcu}{2 - \alpha_0} \cdot \frac{J_{3/2}[\kappa_0 \sec\theta \{a^2 - b^2 + (c^2 - b^2)\tan^2\theta\}^{1/2}]}{\kappa_0^{3/2} \sec^{3/2}\theta \{a^2 - b^2 + (c^2 - b^2)\tan^2\theta\}^{3/4}}. \tag{20}$$

From (17), after putting $\tan\theta = t$, we deduce

$$\frac{(2 - \alpha_0)^2 (a^2 - b^2)^{3/2} e^{2\kappa_0 f}}{32\pi^2 g\rho a^2 b^2 c^2} R$$
$$= \int_0^\infty \frac{[J_{3/2}\{\kappa_0^2 (a^2 - b^2)(1 + t^2)(1 + \alpha^2 t^2)\}^{1/2}]^2}{(1 + \alpha^2 t^2)^{3/2}} e^{-2\kappa_0 f t^2} dt, \tag{21}$$

where $\alpha^2 = (c^2 - b^2)/(a^2 - b^2)$.

The cases $c < b$ and $c > b$ have been worked out separately; however, on comparing (14) and (21), we see that the results could both be included in the same formal expression with a suitable interpretation of the integrand when α^2 and $1 + \alpha^2 t^2$ are negative.

5. A numerical examination of these results could be made for different ratios of the axes a, b, c; certain points of interest may, however, be seen from the form of the expressions, keeping in view the analogy with the wave resistance of a ship. We note in the first place that the exponential factor exp. $(-2\kappa_0 f t^2)$ in the integrand means in practice that the greater part of the value of the integrals arises from small values of the variable t.

An interesting feature of curves of wave resistance and velocity is the occurrence of so-called humps and hollows which, on a simple theory, arise from interference between bow and stern wave systems. In (14) and (21)

* 'Proc. Roy. Soc.,' A. vol. 118. p. 32 (1928).

these oscillations are due to the Bessel function J in the integrands, the modified Bessel function I being non-oscillatory; and one might trace the relative importance and positions of these humps and hollows with variation of the quantity α^2, that is, $(b^2 - c^2)/(a^2 - b^2)$. For instance, in (14) the second integral is non-oscillatory; and, as one would expect, it becomes of less relative importance as the ratio of a to b is increased. Or, again, consider the positions of the humps and hollows. The maxima on the resistance-velocity curve will be in the neighbourhood of the maxima and minima of

$$J_{3/2}\,(\kappa_0\sqrt{a^2 - b^2}),$$

while the minima will be near the zeros of this function. Suppose, as an example, we take $a = 5b$ and compare ellipsoids with different ratios of c to b. When c lies between zero and b, the factor $(1 - \alpha^2 t^2)$ in the integrand of (14) lies between $1 - \frac{1}{24}t^2$ and unity; further, if in (21) we take c as much as $2b$, the corresponding factor is $1 + \frac{1}{8}t^2$. It is clear, without further calculation, that the positions of the interference maxima and minima will be altered only very slightly by such a variation in beam when the ratio of length to draught is five or more. It appears in fact, that when the beam and draught are of the same order of magnitude and the length is of the order of 10 times either of these quantities, the form of the resistance-velocity curve is comparatively insensitive to changes in beam. This consideration may, perhaps, account partly for the measure of agreement which has been obtained between calculated values of the wave resistance of ship models and experimental results; the theory, of course, fails in many details, but the agreement in general character is better than might have been anticipated in view of the simplifying assumptions which have to be made.

6. The calculations for ship models are usually made from Michell's formula for the wave resistance. That expression holds for a model with a longitudinal vertical plane of symmetry, and is derived from an assigned distribution of horizontal velocity at right angles to that plane; it is, in fact, the same as can be obtained from a distribution of sources and sinks, or of horizontal doublets, in the vertical plane. In applying the expression to a ship there are two approximations, which probably involve the same limitation; one is in extending the distribution right up to the surface of the water, and the other is in obtaining the equivalent distribution from the slope of the ship's surface. The latter approximation could, of course, be examined quite independently of the wave phenomena, but it is of interest to compare the expressions for the

wave resistance in one or two definite cases. In a former paper* the comparison was made for a submerged prolate spheroid, and from the formulæ given then numerical calculations were made later by Wigley† in connection with an experimental investigation. We may make now a similar comparison for a flat ellipsoid moving in the direction of the greatest axis, that is, for the case $a > b > c$ worked out in § 3 above; it has, moreover, been found possible to put all the expressions into the same analytical form, and we can see from inspection the difference between them.

Michell's formula for wave resistance is

$$\mathrm{R} = \frac{4\rho u^4}{\pi g} \int_{g/u^2}^{\infty} (\mathrm{P}^2 + \mathrm{Q}^2) \frac{m^2 dm}{(m^2u^4/g^2 - 1)^{1/2}}, \tag{22}$$

where

$$\mathrm{P} + i\mathrm{Q} = \iint \frac{\partial z}{\partial x} e^{-m^2u^2y/g + imx} dx dy. \tag{23}$$

The integration in (23) is taken over the vertical longitudinal section of the model, that is, in the present notation, over the section by the xy-plane; and $\partial z/\partial x$ is derived from the equation to the surface. Applying this to the model specified by (8), with Ox at a depth f below the surface, and putting the expressions into the form used in § 3, we obtain after some reduction

$$\mathrm{R} = 4\pi^{-1} g\rho\kappa_0{}^3 u^2 \int_0^{\pi/2} \mathrm{A}^2 e^{-2\kappa_0 f \sec^2\theta} \sec^5\theta \, d\theta, \tag{24}$$

$$\mathrm{A} = \iint \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} e^{\kappa_0 y \sec^2\theta} \cos(\kappa_0 x \sec\theta) \, dx \, dy, \tag{25}$$

the integration in (25) being extended over the area of the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

Carrying out the integrations in (25), we obtain finally

$$\frac{4a^3 e^{2\kappa_0}}{32\pi^2 g\rho a^2 b^2 c^2} \mathrm{R} = \int_0^{1/\beta} \frac{\left[\mathrm{J}_{3/2}\{\kappa_0{}^2 (a^2-b^2)(1+t^2)(1-\beta^2t^2)\}^{1/2}\right]^2}{(1-\beta^2t^2)^{3/2}} e^{-2\kappa_0 f t^2} dt$$
$$+ \int_{1/\beta}^{\infty} \frac{\left[\mathrm{I}_{3/2}\{\kappa_0{}^2 (a^2-b^2)(1+t^2)(\beta^2t^2-1)\}^{1/2}\right]^2}{(\beta^2t^2-1)^{3/2}} e^{-2\kappa_0 f t^2} dt, \tag{26}$$

where $\beta = b\sqrt{a^2-b^2}$.

* 'Proc. Roy. Soc.,' A, vol. 103, p. 574 (1923).

† W. C. S. Wigley, 'Trans. Inst. Nav. Arch.,' vol. 68, p. 131 (1926).

Comparing (14) and (26), we see how the latter approximates to the former when b and c are small compared with a. We have, for instance, a difference which is independent of the velocity in that the factor $(2-\alpha_0)^2(a^2-b^2)^{3/2}$ in (14) is replaced by $4a^3$ in (26); this makes the value of R calculated from (14) greater than that found from (26) in a certain ratio. To give a few numerical examples:—When $a=5b$, $c=b$, the ratio is 1·2; when $a=5b$, $c=\frac{1}{2}b$, it is 1·12; while for $a=10b$, $c=b$, it is about 1·05. Again, comparing the integrals in (14) and (26) the quantity $\alpha=\sqrt{\{(b^2-c^2)/(a^2-b^2)\}}$ is replaced by $\beta=b\sqrt{a^2-b^2}$. From the considerations given in § 5, it appears that this difference would have only a slight effect upon the character of the resistance-velocity curve for a body with proportions like those of a ship.

7. For a ship model with fine ends and the usual ratios of length to beam and draught, experimental results have shown that the theoretical expressions form at least a good first approximation. A more exact solution of the theoretical problem for a surface ship of simple form moving in a frictionless liquid is desirable, but it presents considerable difficulties. As regards comparison with experimental results, such a solution would probably not improve the present position appreciably, on account of the effects of fluid friction in the actual problem. So far as the ship problem is concerned, it seems that the approximate theory might be supplemented by semi-empirical assumptions of a suitable nature, possibly as regards the effective distribution equivalent to a ship under actual conditions.

From the PHILOSOPHICAL MAGAZINE, vol. xi. *l. February* 1931.

The Stability of Motion of Rectilinear Vortices in Ring Formation. By T. H. HAVELOCK, *F.R.S.*

Introduction and Summary.

1. THE stability of the two-dimensional motion of an infinite system of vortices arranged in a single row, or in double rows has been worked out in detail in recent years, but not much attention has been given to cases in

which the number of vortices is finite. The obvious analogous problems arise when the vortices are equally spaced round the circumference of one or more concentric rings; the problems are not perhaps of special importance, but they are of some interest, and, further, one may obtain the infinite straight rows as limiting cases of ring formation.

We examine first the motion of a single ring of vortices, a problem which attracted attention many years ago in connexion with the vortex theory of atoms. Kelvin* worked out the case of three vortices, but failed to obtain a solution for a larger number; it was in this connexion that he drew attention to the now well-known experiments of Mayer with floating magnets. Shortly afterwards the problem was attacked by J. J. Thomson †, and it is usually stated that he proved the configuration to be stable if, and only if, the number of vortices does not exceed six. He, in fact, worked out the small oscillations for the particular cases of three, four, five, six, and seven vortices, obtaining an instability in the last case. It appears that the equations for the general case are capable of a simple explicit solution, and this is given in § 2; a ring of seven vortices is neutral for small displacements, with less than seven it is completely stable, and for more than seven unstable. In § 3 the effect of an assigned velocity field in addition to that of the vortices is examined briefly.

In the next two sections we work out the effect of a concentric circular boundary upon the stability of a single ring, the boundary being either internal or external to the ring. In both cases the stability is diminished, seven or more vortices being unstable whatever the radius of the boundary. For a smaller number there is a limiting ratio of the radius of the ring to the radius of the boundary for stability in each case. For an external boundary the motion is unstable in any case if the radius of the boundary is less than about twice the radius of the ring, and there is a similar result for an internal boundary. The effect of the boundary, estimated in this way, seems larger than might have been anticipated.

In the remaining sections we examine the motion of two concentric rings of vortices, of opposite rotations, the vortices being spaced alternately. A steady state is possible in which the rings rotate and retain their relative positions unaltered, but there are always modes of disturbance which give rise

* Kelvin, Math. and Phys. Papers, iv. p. 135 (1878).
† J. J. Thomson, 'Treatise on Vortex Rings,' p. 94 (1883).

to instability. By suitable choice of the relative strengths of the vortices in the two rings it is possible to limit the instability to only one special mode of disturbance; it is this particular configuration which becomes in the limit the stable Karman vortex street, when we make the radius of a ring and the number of vortices both infinite, keeping their ratio finite.

Single Ring of Vortices.

2. Let there be n equal vortices, each of strength κ, equally spaced round the circumference of a ring of radius a. In steady motion the ring rotates with a certain angular velocity ω. Let the vortices be slightly displaced, and suppose the disturbed positions to be given in polar coordinates by

$$a+r_{s+1}, \quad 2s\pi/n+\omega t+\theta_{s+1}, \quad . \quad . \quad . \quad . \qquad (1)$$

where $s=0, 1, \dots n-1$, and r, θ are small radial and angular displacements from the steady state. Consider the motion of one of the vortices, say that at the point $(a+r_1, \omega t+\theta_1)$; its velocity is due to the other vortices, and the radial component is

$$\frac{\kappa}{2\pi}\sum_{s=1}^{n-1}\frac{(a+r_{s+1})\sin(2s\pi/n+\theta_{s+1}-\theta_1)}{D^2}, \quad . \quad . \quad (2)$$

while the transverse component is

$$-\frac{\kappa}{2\pi}\sum_{s=1}^{n-1}\frac{(a+r_{s+1})\cos(2s\pi/n+\theta_{s+1}-\theta_1)-(a+r_1)}{D^2}, \quad (3)$$

where

$$D^2=(a+r_{s+1})^2+(a+r_1)^2 \\ -2(a+r_{s+1})(a+r_1)\cos(2s\pi/n+\theta_{s+1}-\theta_1).$$

We expand these expressions to the first order terms in r and θ, and so get the equations of motion of the vortex under consideration. After some reduction we obtain

$$\dot{r}_1=-\frac{\kappa}{4\pi a}\sum_1^{n-1}\frac{\theta_{s+1}-\theta_1}{1-C_s},$$

$$(a+r_1)\omega+a\dot{\theta}_1=\frac{\kappa}{4\pi a}\sum_1^{n-1}\left\{1+\frac{C_s}{1-C_s}\frac{r_1}{a}-\frac{1}{1-C_s}\frac{r_{s+1}}{a}\right\}, \quad (4)$$

where $C_s=\cos(2s\pi/n)$.

The steady state is given by $\omega=(n-1)\kappa/4\pi a^2$, and since

$$\sum_1^{n-1}\frac{1}{1-C_s}=\frac{1}{6}(n^2-1),$$

2 S 2

equations (4) give

$$(4\pi a/\kappa)\dot{r}_1 = A\theta_1 - \sum_{1}^{n-1} a_s\theta_{s+1},$$

$$(4\pi a^3/\kappa)\dot{\theta}_1 = Br_1 - \sum_{1}^{n-1} a_s r_{s+1}, \quad . \quad . \quad . \quad . \quad (5)$$

where

$$A = \frac{1}{6}(n^2-1); \quad B = \frac{1}{6}(n-1)(n-11); \quad a_s = 1/(1-C_s).$$

There are similar equations for each vortex, giving altogether a system of $2n$ equations.

The simplest method of treating the equations is to examine a possible simple solution of the form

$$r_{s+1} = \alpha e^{2ks\pi i/n}; \quad \theta_{s+1}\beta e^{2ks\pi i/n}, \quad . \quad . \quad . \quad . \quad (6)$$

where $k=0, 1, 2, \ldots n-1$.

It may be proved that under the conditions stated

$$\sum_{s=1}^{n-1} \frac{e^{2ks\pi i/n}}{1-\cos(2s\pi/n)} = \frac{1}{6}(n^2-1) - k(n-k) \quad . \quad . \quad (7)$$

Hence, from (5), we find that the equations for α, β reduce to

$$(4\pi a/\kappa)\dot{\alpha} = k(n-k)\beta,$$

$$(4\pi a^3/\kappa)\dot{\beta} = \{k(n-k) - 2(n-1)\}\alpha. \quad . \quad . \quad . \quad (8)$$

Finally, taking α and β to be proportional to $e^{\lambda t}$, these give

$$\lambda^2 = \left(\frac{\kappa}{4\pi a^2}\right)^2 k(n-k)\{k(n-k) - 2(n-1)\} \quad . \quad . \quad (9)$$

It follows that in (6), (8), and (9) we have, in general, $2n$ independent solutions of the equations of the system, and that we can build up the complete solution for any arbitrary small initial displacements of the vortices.

An alternative method of solution may be noticed briefly, namely, the method used by previous writers for particular cases; it may be extended to give the general results, though not quite so simply as in (6)–(9). In the $2n$ equations (5) we assume each coordinate to be proportional to $e^{\lambda t}$, and form the determinantal equation for λ. The determinant can be reduced to one of order n in λ^2, and it can be shown that it is of the type known as a circulant, and can be factorized in terms of the nth roots of unity; after some reduction we obtain (9) again, and can deduce the corresponding simple solutions given by (6).

From (9), when $k=0$ we have $\lambda=0$. If we examine this case we find that the displacement consists of a rotation of the ring combined with a small change in its radius; the result is a new steady state with a corresponding small change in the angular velocity. The condition for stability is that λ^2 must be negative for all the other values of k, namely, 1, 2, ... $n-1$. Hence, from (9), the steady state is stable if

$$k(n-k)-2(n-1) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

is negative for all the values of k, and this is the case if it is negative for $k=\frac{1}{2}n$ when n is even, or $\frac{1}{2}(n\pm1)$ when n is odd. It follows at once that the steady state is completely stable when $n<7$. When $n=7$ the expression (10) is zero for $k=3$ or 4; while for $n>7$ there are always some values of k for which λ^2 is positive, and hence the system is unstable.

Whatever the value of n there are always two modes of possible small oscillations, namely, those given by $k=1$ and $k=2$.

When $k=2$ we have

$$\lambda^2=-4\left(\frac{\kappa}{4\pi a^2}\right)^2(n-2), \quad . \quad . \quad . \quad . \quad (11)$$

while for $k=1$

$$\lambda^2=-\left(\frac{\kappa}{4\pi a^2}\right)^2(n-1)^2=-\omega^2. \quad . \quad . \quad . \quad (12)$$

We notice that in the latter case the period of the small oscillation is the same as the period of rotation of the ring in the steady state; this motion was worked out for the particular case of three vortices by Kelvin in the paper already quoted, and it is illustrated in a characteristic manner by the description of a working model to show the motion of the vortices.

The single infinite straight row of vortices may be obtained by making both n and a become infinite, with the ratio $n/2\pi a$ finite and becoming in the limit equal to the distance between consecutive vortices; the usual results then follow from (6) and (9).

Single Ring in assigned Field.

3. We have so far considered the vortices to be moving solely under their mutual actions. Suppose now that there is an assigned velocity field which is maintained independently; for simplicity we suppose the flow to be in circles

round the origin, the angular velocity being $\Omega(r)$, and the transverse fluid velocity at a distance r being $r\Omega$.

Then, referring to equations (4) for the motion of a typical vortex in the ring, the only difference is that we have to add

$$a\Omega(a)+r_1\{\Omega(a)+a\Omega'(a)\}$$

on the right-hand side of the second equation. The angular velocity of the ring in the steady state is now

$$(n-1)\kappa/4\pi a^2+\Omega(a).$$

Following the same procedure, we obtain, instead of (8), the equations

$$(4\pi a/\kappa)\dot{\alpha}=k(n-k)\beta,$$

$$(4\pi a^3/\kappa)\dot{\beta}=\{k(n-k)-2(n-1)+(4\pi a^3/\kappa)\Omega'(a)\}, \quad (13)$$

and hence we have

$$\lambda^2=k(n-k)\{k(n-k)-2(n-1)+(4\pi a^3/\kappa)\Omega'(a)\}, \quad (14)$$

with $k=0, 1, \ldots n-1$.

It follows that the steady state can be stabilized for any value of n, provided $\Omega'(a)$ is negative and sufficiently large.

Two special cases may be noted. First, if the fluid is rotating like a rigid body—that is, if $\Omega(r)$ is constant—the conditions for stability are unaffected. In the second place, suppose there is an assigned vortex fixed at the origin, so that $\Omega(r)=\kappa'/2\pi r^2$; then, if κ' is of the same sign as κ, we can make the steady state stable for any value of n by taking κ' large enough.

Single Ring with Outer Boundary.

4. Suppose the liquid is contained within a circular boundary of radius b, the vortices being in the steady state on a concentric circle of radius a ($<b$). The motion in the liquid is due to the given vortices and their images in the circular boundary.

Taking the steady state first, the radius of the image ring is b^2/a, the strength of each image vortex being $-\kappa$. Writing down the velocity at any vortex in the given ring, the angular velocity in the steady state is given by

$$a\omega=\frac{(n-1)\kappa}{4\pi a}+\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{(b^2/a)C-a}{b^4/a^2+a^2-2b^2C}, \quad (15)$$

where $C=\cos(2s\pi/n)$.

We shall have occasion to use the following summations, which can easily be proved :

$$\sum_{s=1}^{n-1} \frac{1-p^2}{1-2pC+p^2} = \frac{2n}{1-p^n} - \frac{2}{1-p} - (n-1),$$

$$\sum_{s=1}^{n-1} \frac{1-pC}{1-2pC+p^2} = \frac{n}{1-p^n} - \frac{1}{1-p},$$

$$\sum_{s=1}^{n-1} \frac{(1+p^2)C-2p}{(1-2pC+p^2)^2} = \frac{n^2p^{n-1}}{(1-p^n)^2} - \frac{1}{(1-p)^2}, \quad . \quad . \quad (16)$$

with $0 < p < 1$.

Writing $p=a^2/b^2$, we find from (15)

$$a\omega = \frac{\kappa}{4\pi a}\left(\frac{2n}{1-p^n} - n - 1\right). \quad . \quad . \quad . \quad (17)$$

For small displacements from the steady state we have for each vortex κ at a point

$$a + r_{s+1}, \quad \omega t + 2s\pi/n + \theta_{s+1},$$

an image vortex $-\kappa$ at the point

$$\frac{b^2}{a}\left(1 - \frac{r_{s+1}}{a}\right), \quad \omega t + 2s\pi/n + \theta_{s+1}.$$

Considering the motion of the vortex given by $s=0$, we have for the radial velocity the expression (2), together with

$$-\frac{\kappa}{2\pi}\sum_{s=0}^{n-1} \frac{(b^2/a)(1-r_{s+1}/a)\sin\phi}{E^2}, \quad . \quad . \quad (18)$$

and for the transverse velocity we have (3), together with

$$\frac{\kappa}{2\pi}\sum_{s=0}^{n-1} \frac{(b^2/a)(1-r_{s+1}/a)\cos\phi - (a+r_1)}{E^2}, \quad . \quad (19)$$

where

$$\phi = 2s\pi/n + \theta_{s+1} - \theta_1,$$

and

$$E^2 = \frac{b^4}{a^2}\left(1 - \frac{r_{s+1}}{a}\right)^2 + (a+r_1)^2 - 2\frac{b^2}{a}\left(1 - \frac{r_{s+1}}{a}\right)(a+r_1)\cos\phi.$$

The steps in the reduction of the equations of motion need not be reproduced here ; making use of the summations given in (16), and writing

$$p=a^2/b^2\,; \quad S=\sin(2s\pi/n)\,; \quad C=\cos(2s\pi/n), \quad . \quad (20)$$

we obtain eventually the equations

$$\frac{4\pi a^2}{\kappa}\frac{\dot r_1}{a}=\left\{\frac{n^2-1}{6}+\frac{2n^2p^n}{(1-p^n)^2}-\frac{2p}{(1-p)^2}\right\}\theta_1$$

$$-\sum_{s=1}^{n-1}\left\{\frac{1}{1-C}+\frac{2p\{(1+p^2)C-2p\}}{(1-2pC+p^2)^2}\right\}\theta_{s+1}$$

$$-\sum_{s=1}^{n-1}\frac{2p(1-p^2)S}{(1-2pC+p^2)^2}\frac{r_{s+1}}{a}$$

$$\frac{4\pi a^2}{\kappa}\dot\theta_1=\left\{\frac{n^2-1}{6}+2(n-1)+\frac{2n^2p^n}{(1-p^n)^2}\right.$$

$$\left.-\frac{4n}{1-p^n}+\frac{2(2p^2-3p+2)}{(1-p)^2}\right\}\frac{r_1}{a}$$

$$-\sum_{s=1}^{n-1}\left\{\frac{1}{1-C}-\frac{2p\{(1+p^2)C-2p\}}{(1-2pC+p^2)^2}\right\}\frac{r_{s+1}}{a}$$

$$-\sum_{s=1}^{n-1}\frac{2p(1-p^2)S}{(1-2pC+p^2)^2}\theta_{s+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

There are $2n$ equations of this type, and we examine now a possible solution of the form

$$r_{s+1}/a=\alpha e^{2ks\pi i/n}\,;\quad \theta_{s+1}=\beta e^{2ks\pi i/n}, \quad . \quad . \quad . \quad (22)$$

with $k=0, 1, \ldots, n-1$.

On substituting these expressions we obtain two equations in α and β. In simplifying the various coefficients we use the following summations, whose proof need not be given here:—

$$\sum_{s=1}^{n-1}\frac{(1-p^2)\cos(2ks\pi/n)}{1-2pC+p^2}=\frac{n(p^k+p^{n-k})}{1-p^n}-\frac{2}{1-p}+1$$

$$\sum_{1}^{n-1}\frac{(1-pC)\cos(2ks\pi/n)}{1-2pC+p^2}=\frac{n(p^k+p^{n-k})}{2(1-p^n)}-\frac{1}{1-p}$$

$$\sum_{1}^{n-1}\frac{\{(1+p^2)C-2p\}\cos(2ks\pi/n)}{(1-2pC+p^2)^2}=\frac{nk(p^{k-1}-p^{n-k-1})}{2(1-p^n)}$$

$$+\frac{n^2p^{n-1}(p^k-p^{-k})}{2(1-p^n)^2}-\frac{1}{(1-p)^2}, \quad . \quad . \quad (23)$$

valid for $0<p<1$, and $k=1, 2, \ldots, n-1$.

We obtain after some reduction the equations

$$(4\pi a^2/\kappa)\alpha=P\beta-iR\alpha,$$

$$(4\pi a^2/\kappa)\beta=Q\alpha-iR\beta, \quad . \quad . \quad . \quad (24)$$

where

$$P=k(n-k)-\frac{nk(p^k-p^{n-k})}{1-p^n}-\frac{n^2p^{n-k}(1-p^k)^2}{(1-p^n)^2},$$

$$Q=k(n-k)+2(n+1)+\frac{nk(p^k-p^{n-k})}{1-p^n}+\frac{n^2p^{n-k}(1+p^k)^2}{(1-p^n)^2},$$

$$R=\frac{nk(p^k+p^{n-k})}{1-p^n}-\frac{n^2p^{n-k}(1-p^{2k})}{(1-p^n)^2}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

We may check these expressions by deducing the equations for the corresponding disturbance of an infinite double symmetrical row of vortices. If d is the distance between consecutive vortices in each row, and h the distance between the two rows, we have, in the limit,

$$2\pi a/n=d\,;\quad 2k\pi/n=\phi\,;$$
$$p=(1+2\pi h/nd)^{-1}.$$

With these (17) gives the limiting value of the linear velocity of the vortices, namely,

$$\frac{\kappa}{2d}\coth\frac{\pi h}{d};$$

further, the quantities $n^2P/2\pi^2$, $n^2Q/2\pi^2$, and $n^2R/2\pi^2$ become respectively the quantities $A+C$, $A-C$, and B in the notation of Lamb's 'Hydrodynamics,' (5th ed. p. 221).

Returning to equations (24), we take α and β proportional to

$$e^{\kappa\lambda t/4\pi a^2},$$

and obtain

$$\lambda=-iR\pm(PQ)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (26)$$

For complete stability the product PQ must be negative, or zero possibly, for all the values of k. To prove instability it is sufficient to show that PQ is positive for one value at least of k. From the form of the expressions in (25) we see that P and Q are symmetrical in k and $n-k$, and that the critical mode to examine is $k=\frac{1}{2}n$ for n even, or $k=\frac{1}{2}(n\pm1)$ for n odd.

For n even we have

$$P(\tfrac{1}{2}n)=\tfrac{1}{4}n^2-n^2p^{\frac{1}{2}n}/(1+p^{\frac{1}{2}n})^2, \quad . \quad . \quad . \quad (27)$$

which is always positive. Further,

$$Q(\tfrac{1}{2}n)=\tfrac{1}{2}n^2+2(n+1)-\frac{4n}{1-p^n}+\frac{n^2p^{\frac{1}{2}n}}{(1-p^{\frac{1}{2}n})^2}, \quad . \quad (28)$$

and this is positive if

$$(n^2+8n+8)x^3+(3n^2-8n-8)x^2+(3n^2+8n-8)x$$
$$+n^2-8n+8>0, \quad . \quad . \quad (29)$$

where $x=p^{\frac{1}{2}n}=(a/b)^n$.

The left-hand side of (29) is always positive for $n \geqslant 8$. From similar expressions when n is odd we find that there is always a positive value of Q for $n=7$. Hence we conclude that the motion of the ring is unstable when the number of vortices is equal to or greater than seven, whatever the radius of the outer boundary. For $n<7$ we shall see that the motion is stable provided the ratio of the radius of the ring to that of the boundary is less than a certain value in each case. We shall examine the cases briefly, noting that in each case the mode $k=0$ means simply a neutral displacement of the ring.

For $n=2$, $k=1$, we find from the previous expressions that Q is negative if $p<0{\cdot}2137$; and as P is positive, it follows that the circular motion of the two vortices is stable if $a/b<0{\cdot}462$.

For $n=3$, $k=1$ or 2, Q is negative for $p<0{\cdot}322$, and the motion is stable for $a/b<0{\cdot}567$.

Similarly for $n=4$ we find the critical value of a/b to be about $0{\cdot}575$; for $n=5$ it is $0{\cdot}588$, and for $n=6$ it is $0{\cdot}547$. When $n=7$, which is the critical neutral case when there is no boundary, the effect of an outer boundary of any radius is to cause instability.

Single Ring with Inner Boundary.

5. Suppose now that the fluid is bounded internally by a circular barrier ($r=b$), and that a ring of n vortices is rotating in circular motion in a ring of radius a ($>b$). The image of a vortex κ at $r=a$ is a vortex $-\kappa$ at $r=b^2/a$, together with a vortex κ at $r=0$; this combination makes the circulation zero for a circuit enclosing the boundary without including any of the actual vortices

We find the equations of motion of a given vortex, $s=0$ in the previous notation, just as in § 4. The only differences arise (i.) from the additional image vortex $n\kappa$ at the origin, and (ii.) in evaluating the various summations, as b/a is now less than unity instead of a/b. For the steady state we have

$$a\omega=\frac{(n-1)\kappa}{4\pi a}+\frac{n\kappa}{2\pi a}-\frac{\kappa}{2\pi a}\sum_{s=0}^{n-1}\frac{1-qC}{1-2qC+q^2}, \quad . \quad (30)$$

where $q=b^2/a^2$ and $C=\cos(2s\pi/n)$.

This gives

$$a\omega = \frac{\kappa}{4\pi a}\left(3n-1-\frac{2n}{1-q^n}\right). \quad . \quad . \quad . \quad (31)$$

We shall merely state now the results for the general equations of disturbed motion. The equations for $\dot{r}_1$ and $\dot{\theta}_1$ are the same as in (21), with the following alterations:—(i.) write q for p in the coefficients, (ii.) change the sign of the last term in each equation, the coefficients of r_{s+1} and θ_{s+1} respectively, from $-$ to $+$, (iii.) change the coefficient of r_1 in the second equation to

$$\frac{1}{6}(n^2-1)-2(n-1)+\frac{4nq^n}{1-q^n}+\frac{2n^2q^n}{(1-q^n)^2}+\frac{2q}{(1-q)^2}.$$

Taking a simple solution of type (22), and proceeding as in (24), (25), we obtain, instead of (26), the result

$$\lambda = i\mathrm{R}' \pm (\mathrm{P}'\mathrm{Q}')^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where

$$\mathrm{P}' = k(n-k) - \frac{nk(q^k-q^{n-k})}{1-q^n} - \frac{n^2q^{n-k}(1-q^k)^2}{(1-q^n)^2},$$

$$\mathrm{Q}' = k(n-k)-2(n-1)+\frac{4nq^n}{1-q^n}+\frac{n^2q^{n-k}(1-q^k)^2}{(1-q^n)^2} + \frac{(nk(q^k-q^{n-k})}{1-q^n},$$

$$\mathrm{R}' = \frac{nk(q^k+q^{n-k})}{1-q^n} - \frac{n^2q^{n-k}(1-q^{2k})}{(1-q^n)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

As before, it appears that stability depends upon there being values of q less than unity for which Q' is negative for all the values of k. It is easily seen that there is no such value of q when $n \geqslant 7$, and therefore the steady state is unstable when there are seven or more vortices in the ring.

Examining the expressions numerically for smaller values of n, we find that the steady state is stable under the following conditions:—$n=2$, $b/a<0{\cdot}386$; $n=3$, $b/a<0{\cdot}522$; $n=4$, $b/a<0{\cdot}556$; $n=5$, $b/a<0{\cdot}579$; $n=6$, $b/a<0{\cdot}544$.

These values are slightly less than the corresponding limits when the ring is within the circular boundary, but there is little difference in the general conclusions.

Double Alternate Rings.

6. In the previous sections we have been considering in effect a double symmetrical ring, in which the motions of one ring—the image ring—are constrained in accordance with those of the actual ring. We shall leave on one side the general case of a free double symmetrical ring, and proceed to two alternate rings in an unlimited liquid.

Let there be n positive vortices, each of strength κ, equally spaced round a circle of radius a, and n negative vortices of strength κ' equally spaced round a concentric circle of radius b $(>a)$, the arrangement of the vortices being alternate. Thus, if the vortices in the inner ring are given by polar coordinates a, $2s\pi/n$, those of the outer ring are given by b, $2(s+\frac{1}{2})\pi/n$, with $s=0, 1, \ldots, n-1$.

Examine first the possibility of a steady state with the two rings rotating with equal angular velocity, the relative configuration remaining unchanged. The radial velocity of any vortex is zero. The transverse velocity of a vortex in the inner ring is given by

$$\frac{(n-1)\kappa}{4\pi a}+\frac{\kappa'}{2\pi}\sum_{s=0}^{n-1}\frac{bC'-a}{b^2+a^2-2abC'}, \quad . \quad . \quad (34)$$

and in the outer ring by

$$-\frac{(n-1)\kappa'}{4\pi b}-\frac{\kappa}{2\pi}\sum_{s=0}^{n-1}\frac{aC'-b}{a^2+b^2-2abC'}, \quad . \quad . \quad (35)$$

where $C'=\cos\{2(s+\frac{1}{2})\pi/n\}$.

We shall require the following summations, with $p=a/b<1$:

$$\sum_{s=0}^{n-1}\frac{1-pC'}{1-2pC'+p^2}=\frac{n}{1+p^n},$$

$$\sum_{0}^{n-1}\frac{1-p^2}{1-2pC'+p^2}=\frac{n(1-p^n)}{1+p^n},$$

$$\sum_{0}^{n-1}\frac{(1+p^2)C'-2p}{(1-2pC'+p^2)^2}=-\frac{n^2p^{n-1}}{(1+p^n)^2}. \quad . \quad . \quad (36)$$

The condition for equal angular velocity of the two rings then becomes

$$(n-1)\kappa-\frac{2np^n}{1+p^n}\kappa'=\frac{2np^2}{1+p^n}\kappa-(n-1)p^2\kappa'. \quad . \quad (37)$$

It can be seen that for a given ratio of κ' to κ we obtain from this equation a corresponding value of p less than unity,

and hence a possible steady state. Consider now the general equations for the disturbed motion. Let the positions of the vortices in the inner ring be given by polar coordinates

$$a(1+r_{s+1}), \quad 2s\pi/n+\omega t+\theta_{s+1},$$

and those in the outer ring by

$$b(1+\rho_{s+1}), \quad 2(s+\tfrac{1}{2})\pi/n+\omega t+\phi_{s+1}.$$

We form the equations of motion as in the previous sections. We choose a typical vortex, $s=0$, in the inner ring, and to simplify the notation we take the vortex $s=n-1$ in the outer ring. Expanding the components of velocity to first order terms, and reducing the coefficients by means of (36), we obtain the equations for these two vortices:

$$\begin{aligned}
4\pi a^2\dot{r}_1 &= \kappa\left\{\frac{1}{6}(n^2-1)-\frac{2n^2p^n}{(1+p^n)^2}\right\}\theta_1 \\
&\quad -\kappa\sum_1^{n-1}\frac{\theta_{s+1}}{1-C}-\kappa'\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\phi_{s+1} \\
&\quad +\kappa'\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\rho_{s+1}, \\
4\pi a^2\dot{\theta}_1 &= \kappa\left\{\frac{1}{6}(n^2-1)-2(n-1)+\frac{4np^n}{1+p^n}-\frac{2n^2p^n}{(1+p^n)^2}\right\}r_1 \\
&\quad -\kappa\sum_1^{n-1}\frac{r_{s+1}}{1-C}-\kappa'\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\rho_{s+1} \\
&\quad -\kappa'\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\phi_{s+1}, \\
4\pi b^2\dot{\rho}_n &= -\kappa'\left\{\frac{1}{6}(n^2-1)-\frac{2n^2p^n}{(1+p^n)^2}\right\}\phi_n \\
&\quad +\kappa'\sum_1^{n-1}\frac{\phi_{s+1}}{1-C}+\kappa\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}\theta_{s+1} \\
&\quad +\kappa\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}r_{s+1} \\
4\pi b^2\dot{\phi}_n &= -\kappa'\left\{\frac{1}{6}(n^2-1)-2(n-1)+\frac{4n}{1+p^n}-\frac{2n^2p^n}{(1+p^n)^2}\right\}\rho_n \\
&\quad +\kappa'\sum_1^{n-1}\frac{\rho_{s+1}}{1-C}+\kappa\sum_0^{n-1}\frac{2p\{(1+p^2)C'-2p\}}{D^2}r_{s+1} \\
&\quad -\kappa\sum_0^{n-1}\frac{2p(1-p^2)S'}{D^2}\theta_{s+1}, \quad \dots \quad (38)
\end{aligned}$$

where

$$C = \cos(2s\pi/n); \quad C' = \cos\{2(s+\tfrac{1}{2})\pi/n\};$$
$$S' = \sin\{2(s+\tfrac{1}{2})\pi/n\}; \quad D = 1-2pC'+p^2.$$

We now assume a simple solution of the form

$$r_{s+1} = \alpha e^{2ks\pi i/n}; \qquad \theta_{s+1} = \beta e^{2ks\pi i/n};$$
$$\rho_{s+1} = \alpha' e^{2k(s+\frac{1}{2})\pi i/n}; \quad \phi_{s+1} = \beta' e^{2k(s+\frac{1}{2})\pi i/n}; \quad . \quad . \quad (39)$$

and, further, suppose that α, β, α', β' involve the time as a factor

$$e^{\kappa\lambda t/4\pi a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

In simplifying the coefficients we use the following summations, valid for $k=1, 2, \ldots, n-1$, which may be proved without difficulty:

$$\sum_0^{n-1} \frac{(1-p^2)E}{1-2pC'+p^2} = \frac{n(p^k-p^{n-k})}{1+p^n},$$

$$\sum_0^{n-1} \frac{2p\{(1+p^2)C'-2p\}E}{(1-2pC'+p^2)^2} = \frac{n\{kp^k-(n-k)p^{n-k}\}}{1+p^n} - \frac{n^2p^n(p^k-p^{n-k})}{(1+p^n)^2},$$

$$\sum_0^{n-1} \frac{2p(1-p^2)S'E}{(1-2pC'+p^2)^2} = i\left\{\frac{nk(p^k-p^{n-k})}{1+p^n} - \frac{n^2p^n(p^k-p^{-k})}{(1+p^n)^2}\right\}, \quad . \quad (41)$$

where $E = e^{2k(s+\frac{1}{2})\pi i/n}$.

The $4n$ equations of the system now reduce to

$$\lambda\alpha = P_1\beta + Q'\alpha' + R'\beta',$$
$$\lambda\beta = P_2\alpha + R'\alpha' - Q'\beta',$$
$$\lambda\alpha' = P_1'\alpha' + Q\alpha + R\beta,$$
$$\lambda\beta' = P_2'\alpha' + R\alpha - Q\beta, \quad . \quad . \quad . \quad . \quad . \quad (42)$$

where

$$\kappa P_1 = -\frac{2n^2p^n}{(1+p^n)^2}\kappa' + k(n-k)\kappa,$$

$$\kappa P_1' = \frac{2n^2p^{n+2}}{(1+p^n)^2}\kappa - k(n-k)p^2\kappa',$$

$$\kappa P_2 = \{k(n-k) - 2(n-1)\}\kappa + 2\left\{\frac{2np^n}{1+p^n} - \frac{n^2p^n}{(1+p^n)^2}\right\}\kappa',$$

$$\kappa P_2' = -\{k(n-k)-2(n-1)\}p^2\kappa' - 2\left\{\frac{2n}{1+p^n} - \frac{n^2p^n}{(1+p^n)^2}\right\}p^2\kappa,$$

$$Q = ip^2n\frac{p^k\{k-(n-k)p^n\}+p^{n-k}(n-k-kp^n)}{(1+p^n)^2},$$

$$R = p^2n\frac{p^k\{k-(n-k)p^n\}-p^{n-k}(n-k-kp^n)}{(1+p^n)^2},$$

$$Q' = \kappa'Q/\kappa p^2; \quad R' = -\kappa'R/\kappa p^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The equation for λ is

$$\begin{vmatrix} \lambda & -P_1 & -Q' & -R' \\ -P_2 & \lambda & -R' & Q' \\ -Q & -R & \lambda & -P_1' \\ -R & Q & -P_2' & \lambda \end{vmatrix} = 0. \quad (44)$$

Using the relation (37) we see that $P_1-P_2=P_1'-P_2'$, and we find that (44) reduces to a quadratic in λ^2, which can be solved in the form

$$4\lambda^2 = (L^{\frac{1}{2}} \pm M^{\frac{1}{2}})^2 - (P_1-P_2)^2, \quad . \quad . \quad . \quad (45)$$

where

$$L = (P_1+P_2')^2 + 4QQ',$$

$$M = (P_1-P_1')^2 + 4RR'. \quad . \quad . \quad . \quad . \quad (46)$$

The condition for complete stability is that for the n values of k all the values of λ^2 must be real and negative, including possibly zero. From the symmetry of the coefficients (43) in k and $n-k$ it is only necessary to examine the values of k from zero up to $\frac{1}{2}n$ if n is even, or $\frac{1}{2}(n-1)$ if n is odd.

7. We might examine now in detail the case when we take $\kappa'=\kappa$, that is, when the vortices in the two rings are of equal strengths and opposite rotations; we shall state the results without giving the details of the algebraic analysis. It can be shown that when p satisfies the equation (37) the quantity QQ' increases in absolute value from $k=0$ to $k=\frac{1}{2}n$, while the quantity RR' decreases in absolute value as k increases in this range. Further, except at $k=0$ the quantity L of (46) is always negative, and thus the criterion for stability is reduced to M being negative. But when $k=\frac{1}{2}n$ we have $R=R'=0$; and since P_1 is not in general equal to P_1', it follows that M is positive at $k=\frac{1}{2}n$. Hence, if n is even, the system is unstable for the mode $k=\frac{1}{2}n$ at least. It can be seen that in general there are always some unstable modes in the neighbourhood of this mean value of k;

further, there is always an instability associated with the mode $k=0$.

8. It remains to be seen whether we can obtain a greater degree of stability by a suitable choice of the ratio of κ' to κ, that is, with vortices of different strengths in the two rings. To avoid complicating the discussion we shall assume n even. Then the previous discussion suggests that we make the central mode stable, that is, we fix κ' by the condition that $P_1=P_1'$ at $k=\frac{1}{2}n$. From (43) this gives

$$\tfrac{1}{4}\kappa-\frac{2p^n}{(1+p^n)^2}\kappa'=\frac{2p^{n+2}}{(1+p^n)^2}\kappa-\tfrac{1}{4}p^2\kappa'. \quad . \quad . \quad (47)$$

The ratio of κ' to κ and the value of p are now determined by equations (37) and (47).

Without examining the expressions in general a numerical example will show the nature of the results.

Taking $n=10$, the appropriate roots of (37) and (47) are, approximately,

$$p=0{\cdot}8406\ ; \quad \kappa'/\kappa=0{\cdot}897. \quad . \quad . \quad . \quad . \quad (48)$$

The following table shows the values of QQ′ and RR′ and of λ^2 for all the modes, calculated from (43) and (45); the values for $k=6, 7, 8, 9$ are omitted, as they are the same as for $k=4, 3, 2, 1$.

k.	QQ′.	RR′.	λ^2.	
0	0	−411	0	85
1	−61	−350	−318	−142
2	−177	−72	−307	−95
3	−292	−48	−545	−143
4	−369	−5	−475	−316
5	−396	0	−418	−418

We see that the motion is stable in all the possible modes with the exception of $k=0$. Reverting to (40), we find that $\kappa\lambda/4\pi a^2=2\pi\lambda/6{\cdot}3T$ approximately where T is the period of rotation of the rings in the steady state; thus the periods of the small oscillations in the stable modes range from about two-thirds to one-quarter of the period of rotation.

It is easily verified that $\lambda^2=0$ in the mode $k=0$ corresponds to a neutral displacement of the system, consisting of a rotation and dilatation of the rings without alteration of the ratio of their radii. On the other hand, the root $\lambda^2=85$ in this mode gives rise to definite instability.

9. The Karman vortex street may be obtained as a limiting case of the present problem. We make the radius a and the number of vortices n both become infinite, their ratio remaining finite. If the limiting value of $2\pi a/n$ is d, the distance between consecutive vortices in each row, and if h is the distance between the rows, we may put

$$p=(1+2\pi h/nd)^{-1} \; ; \quad . \quad . \quad . \quad . \quad . \quad (49)$$

p approaches unity, while the limiting value of p^n is $e^{-2\pi h/d}$. We see from equations (37) and (47) that the ratio κ'/κ approaches unity, and (47) gives at once, in the limit, the Karman condition

$$\cosh^2(\pi h/d)=2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Further, if from (42) and (45) we write down the expressions for λ^2 when $k=0$, we find that for the positive root, $\kappa\lambda/4\pi a^2$ is of order $n^{-\frac{1}{2}}$; thus, as the limit is approached the instability in this mode merges with the neutral state in the same mode. It is only in this particular limiting case that we obtain a system which is completely stable.

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Ship Waves: the Calculation of Wave Profiles.

By T. H. HAVELOCK, F.R.S.

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1. The surface disturbance produced by a ship is usually analysed into two parts: one is called the local disturbance and the other forms the wave pattern, the supply of energy required for the second part giving rise to the wave resistance of the ship. For a direct comparison between observed and theoretical surface elevation it is necessary to calculate both parts of the disturbance. This has been carried out recently for a certain case by Mr. W. C. S. Wigley,* working at the William Froude Laboratory. The model was of uniform horizontal section and sufficiently deep to be treated as theoretically of infinite draught, while the section consisted of a triangular bow and stern connected by a straight middle body; the surface elevation along the side of the model was observed at various speeds, and compared with the theoretical calculations.

The following paper deals with the calculation of the surface elevation in cases of this type. The theory is developed here from the velocity potential cf a doublet at any given depth below the free surface of the water; this has the advantage of being capable of wide generalisation, and, moreover, the introduction of a small frictional term, which is ultimately made to vanish, keeps the expressions determinate throughout the analysis.

We examine first a uniform distribution of doublets on a vertical line, and then a similar distribution of finite length in the direction of motion; graphs

* W. C. S. Wigley, 'Trans. N.E. Coast Inst. Engineers and Shipbuilders,' vol. 47, p. 153 (1931).

are given of the surface elevation along the line of motion. A similar analysis is given for the distribution corresponding to the model described above, and the connection between the distribution and the model is indicated.

Finally, the results are generalised to give the central surface elevation for a model, of infinite draught, of any sectional form. The general expressions are of simple character and some deductions can be made from their form. In addition, they are suitable for the numerical or graphical calculation of the profile for any required model of this type. A brief analysis of a parabolic model is made to illustrate the general results.

2. Consider a doublet of moment M at a depth f below the surface of water and moving horizontally with constant velocity u. For the present applications we need only the expressions when the axis of the doublet is horizontal and in the direction of motion; further, we take moving axes with Ox in the direction of motion, O in the free surface, Oz vertically upwards, so that the position of the doublet is the point $(0, 0, -f)$. The velocity potential of the fluid motion is given by*

$$\phi = -\frac{i\mathrm{M}}{2\pi}\int_{-\pi}^{\pi}\cos\theta\,d\theta\int_0^\infty e^{-\kappa(z+f)+i\kappa\varpi}\,\kappa\,d\kappa$$

$$+\frac{i\mathrm{M}}{2\pi}\int_{-\pi}^{\pi}\cos\theta\,d\theta\int_0^\infty\frac{\kappa+\kappa_0\sec^2\theta}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}e^{-\kappa(f-z)+i\kappa\varpi}\,\kappa\,d\kappa, \quad (1)$$

where $\varpi = x\cos\theta + y\sin\theta$ and $\kappa_0 = g/u^2$. The real part of (1) is to be taken. The first term expresses the velocity potential of the given doublet in a form valid for $z+f>0$, that is for points above the doublet. In the second term μ is a small positive constant which is ultimately made zero. The surface elevation ζ is given by

$$\frac{\partial\zeta}{\partial t} = -\frac{\partial\phi}{\partial z};\quad z=0. \quad (2)$$

This gives

$$\zeta = \operatorname*{Lim}_{\mu\to 0}\frac{\mathrm{M}}{\pi u}\int_{-\pi}^{\pi}d\theta\int_0^\infty\frac{\kappa^2 e^{-\kappa f+i\kappa\varpi}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa. \quad (3)$$

In this form ζ is finite and continuous, and the expression may be generalised by summation or integration for a distribution of doublets. We shall consider here the distribution to be in the vertical plane $y=0$. If $\mathrm{M}(h,f)$ is the moment per unit area at the point $(h, 0, -f)$ we have

$$\zeta = \frac{1}{\pi u}\int_{-\infty}^{\infty}\int_0^\infty \mathrm{M}(h,f)\,dh\,df\int_{-\pi}^{\pi}d\theta\int_0^\infty\frac{\kappa^2 e^{-\kappa f+i\kappa\varpi'}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa, \quad (4)$$

* 'Proc. Roy. Soc.,' A, vol. 121, p. 518 (1928).

where $\varpi' = (x-h)\cos\theta + y\sin\theta$. We have omitted here the symbol for the limiting value as μ is made to vanish, but that is always to be understood. It is assumed that the integrals are convergent. From a physical point of view it is easily seen that divergent or indeterminate integrals may arise if the distribution contains finite sources or sinks which extend up to the free surface of the water. From the method of obtaining the velocity potential (1), we see that the appropriate form of (4) in such cases will be found by taking the integration with respect to the depth f to extend from a positive quantity d to infinity and then considering the limiting value as d is made to vanish. We may note another form for (4) which is obtained by integrating by parts with respect to h. Provided M is continuous in this variable and is zero at the two limits, we have

$$\zeta = -\frac{i}{\pi u}\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\partial M}{\partial h}\,dh\,df\int_{-\pi}^{\pi}\sec\theta\,d\theta\int_{0}^{\infty}\frac{e^{-\kappa f + i\kappa\varpi'}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\,\kappa\,d\kappa. \quad (5)$$

Further, the normal component of fluid velocity at any point of the vertical plane $y=0$ is equal to

$$2\pi\,\partial M/\partial h. \quad (6)$$

Hence from (5) we may obtain the surface elevation for any assigned distribution of normal fluid velocity over this plane.

3. Consider first a simple line distribution of constant moment M per unit length on the z-axis, extending from the free surface to an infinite depth. Here we shall have to suppose first that the distribution extends up to a depth d below the surface, and then take the limit as d is made small.

Integrating with respect to f, we obtain

$$\zeta = \frac{M}{\pi u}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}\frac{\kappa e^{-\kappa d + i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\,d\kappa. \quad (7)$$

In the integrand we write

$$\frac{\kappa}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} = 1 + \frac{\kappa_0\sec^2\theta}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}, \quad (8)$$

omitting terms which will give no contribution in the limit when μ is made zero. The integrations in θ and κ in (7) corresponding to the first term on the right of (8) give the value $2\pi/\kappa_0(d^2+x^2+y^2)^{\frac{1}{2}}$. Hence, putting $d=0$, the contribution of this part to the surface elevation is $2M/u(x^2+y^2)^{\frac{1}{2}}$. Taking the second part of (8), the corresponding integral in (7) remains convergent

when we put $d = 0$, provided ϖ is not zero. Hence we obtain, for all points other than the origin,

$$\zeta = \frac{2M}{u\,(x^2+y^2)^{\frac{1}{2}}} + \frac{\kappa_0 M}{\pi u}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa. \qquad (9)$$

We shall limit consideration at present to the surface elevation along the line of motion, that is for $y = 0$; we have

$$\zeta = \frac{2M}{u\,|x|} + \frac{4\kappa_0 M}{\pi u}\int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{i\kappa x\cos\theta}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa, \qquad (10)$$

noting that we require the limiting value of the real part as μ is made zero.

The integration in κ may be transformed by regarding κ for the moment as a complex variable and considering a contour integral taken round a suitable path according as x is positive or negative. In this process it is the residue at the pole of the integrand which gives the expression for the waves in the rear of the system. The result, when μ has been made zero, is

$$\int_0^\infty \frac{\cos(\kappa_0 mx\sec\theta)}{1+m}\, dm, \quad \text{for } x > 0;$$

$$2\pi\sin(\kappa_0 x\sec\theta) + \int_0^\infty \frac{\cos(\kappa_0 mx\sec\theta)}{1+m}\, dm, \quad \text{for } x < 0. \qquad (11)$$

For the integration with respect to θ, we require the following results

$$\int_0^{\pi/2} \sec^2\theta\sin(\kappa_0 x\sec\theta)\, d\theta = -\frac{\pi}{2}\,Y_1(\kappa_0 x), \qquad (12)$$

$$\begin{aligned}\int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^\infty \frac{\cos(\kappa_0 mx\sec\theta)}{1+m}\, dm &= -\frac{\pi}{2}\int_0^\infty \frac{J_1(\kappa_0 mx)}{1+m}\, dm \\ &= -\frac{\pi}{2\kappa_0 x} + \frac{\pi}{2\kappa_0 x}\int_0^\infty \frac{J_0(\kappa_0 mx)}{(1+m)^2}\, dm \\ &= -\frac{\pi}{2\kappa_0 x} + \frac{\pi^2}{4}\left\{H_1(\kappa_0 x) - Y_1(\kappa_0 x) - \frac{2}{\pi}\right\}. \qquad (13)\end{aligned}$$

In this J and Y denote Bessel functions, and H is Struve's function, the notation being that of G. N. Watson's "Treatise on Bessel Functions." Collecting these results and putting in (10) we obtain the surface elevation on the line $y = 0$. To avoid any possible ambiguity in signs, we shall find it convenient to write x' for $-x$ and so restrict x and x' to positive values; x is thus distance in front, and x' distance behind the moving system. We obtain

$$\zeta = \frac{\pi\kappa_0 M}{u}\left\{H_1(\kappa_0 x) - Y_1(\kappa_0 x) - \frac{2}{\pi}\right\}; \qquad x > 0$$

$$\zeta = \frac{\pi\kappa_0 M}{u}\left\{H_1(\kappa_0 x') - Y_1(\kappa_0 x') - \frac{2}{\pi}\right\} + \frac{4\pi\kappa_0 M}{u}\,Y_1(\kappa_0 x'); \quad x' > 0. \qquad (14)$$

The quantity $H_1 - Y_1$ is monotonic and decreases to an asymptotic value $2/\pi$. The symmetrical terms in (14) represent the local disturbance, becoming infinite near the origin like x^{-1}. The last term in (14) represents the wave disturbance in the rear. The expressions are easily calculated from tables of the functions, and fig. 1 shows the two parts of the disturbance.

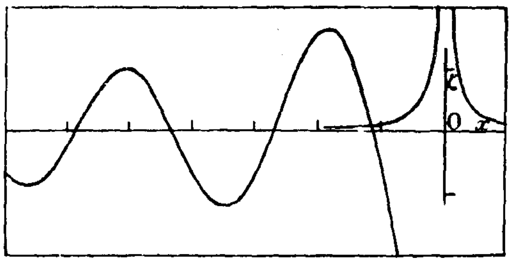

FIG. 1.

It will be seen that there is discontinuity at the origin, but that arises from extending this particular distribution right up to the free surface. If we retain the quantity d used at the beginning of this section, it is easily seen that the discontinuity is associated with the last term of (14); for any finite value of d, this part of the disturbance is zero at the origin.

4. Consider now a uniform distribution over a finite length of the vertical plane $y = 0$, extending over the range $-l < x < l$. This might be deduced from the previous section by integrating with suitable precautions to allow for the discontinuities in those expressions; but we shall use the general formula (4). Suppose in the first place that the distribution extends from a depth d to an infinite depth; then we have

$$\zeta = \frac{M}{\pi u}\int_{-l}^{l} dh \int_{d}^{\infty} df \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{\kappa^2 e^{-\kappa f + i\kappa\varpi'}}{\kappa - \kappa_0 \sec^2\theta + i\mu\sec\theta}\, d\kappa. \tag{15}$$

For the elevation along the line $y = 0$, this gives

$$\zeta = \frac{4iM}{\pi u}\int_{0}^{\pi/2} \sec\theta\, d\theta \int_{0}^{\infty} \frac{e^{-\kappa d}\left\{e^{i\kappa(x-l)\cos\theta} - e^{i\kappa(x+l)\cos\theta}\right\}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa. \tag{16}$$

We may put $d = 0$ in (16). Further, the disturbance separates into equal and opposite disturbances associated with the front and rear of the system, or, as we may call them, into bow and stern systems. Writing q_1 for $x - l$, we have to evaluate the real part of

$$i\int_{0}^{\pi/2} \sec\theta\, d\theta \int_{0}^{\infty} \frac{e^{i\kappa q_1\cos\theta}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa. \tag{17}$$

We transform this as in the previous section, and also make use of the following evaluations

$$\int_0^{\pi/2} \sec\theta \cos(\kappa_0 q_1 \sec\theta)\, d\theta = -\frac{\pi}{2} Y_0(\kappa_0 q_1).$$

$$\int_0^{\pi/2} \sec\theta\, d\theta \int_0^{\infty} \frac{\sin(\kappa_0 m q_1 \sec\theta)}{1+m}\, dm = \frac{\pi}{2}\int_0^{\infty} \frac{J_0(\kappa_0 q_1 m)}{1+m}\, dm$$

$$= \frac{\pi^2}{4}\{H_0(\kappa_0 q_1) - Y_0(\kappa_0 q_1)\}. \qquad (18)$$

Using, as before, q_1 for distance in front of the bow and q_1' for distance behind the bow, we find that the bow system is given by

$$\zeta = \frac{\pi M}{u}\{H_0(\kappa_0 q_1) - Y_0(\kappa_0 q_1)\}\,; \quad q_1 > 0$$

$$\zeta = -\frac{\pi M}{u}\{H_0(\kappa_0 q_1') - Y_0(\kappa_0 q_1')\} - \frac{4\pi M}{u} Y_0(\kappa_0 q_1')\,; \qquad q_1' > 0. \quad (19)$$

There are similar expressions for the stern system with $q_2 = x + l$, all the signs being changed. These results are easily calculated from tables, and curves for the local disturbance and the waves for both bow and stern are shown in fig. 2.

The complete disturbance is the sum of all the curves shown in the figure. The distribution of doublets is equivalent to a vertical line of sources at the bow and a vertical line of sinks at the stern. It may be noticed that the elevation

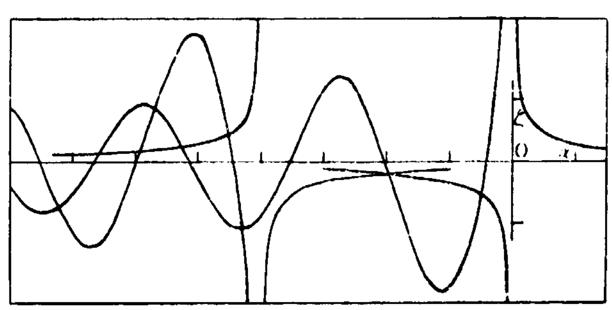

FIG. 2.

becomes logarithmically infinite at bow and stern, and the discontinuities there arise as described in the previous section. The local disturbance is symmetrical fore and aft when taken as a whole, but is anti-symmetrical for bow or stern separately. If the complete disturbance associated with the bow is called a positive system, the stern generates an equal negative system.

5. The system we have just considered may be supposed to correspond to a ship with bluff bow and stern. We may examine the effect of pointing the ends by the following distribution; the moment per unit area

$$
\begin{aligned}
&= \mathrm{M}, \quad \text{for } -a < x < a \\
&= \mathrm{M}\,(l - x)/(l - a), \quad \text{for } a < x < l \\
&= \mathrm{M}\,(l + x)/(l - a), \quad \text{for } -l < x < -a,
\end{aligned} \tag{20}
$$

where M is a constant. The moment is zero outside the range specified in (20).

If we replace M in (20) by $ub/2\pi$, in accordance with (6), we see that if b/l is small the corresponding form of ship is that examined by Wigley in the paper already quoted. Wigley has worked out the surface elevation along the line $y = 0$ from Michell's formulæ, giving suitable interpretations to the indeterminate integrals involved in those formulæ. Here we shall use the general form (5). We may take the distribution to extend right up to the free surface, as it appears that the resulting expressions are finite and continuous throughout.

From (5) and (20), after carrying out the integration with respect to f and h, the surface elevation for $y = 0$ is given by

$$\zeta = \frac{4\mathrm{M}}{\pi u\,(l - a)} \int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^\infty \frac{\mathrm{N}}{\kappa\,(\kappa - \kappa_0 \sec^2\theta + i\mu \sec\theta)}\, d\kappa, \tag{21}$$

where

$$\mathrm{N} = e^{i\kappa\,(x+a)\cos\theta} - e^{i\kappa\,(x+l)\cos\theta} - e^{i\kappa\,(x-l)\cos\theta} + e^{i\kappa\,(x-a)\cos\theta}. \tag{22}$$

We notice from the form of N that the singularity at $\kappa = 0$ in the integral with respect to κ is only apparent. On the other hand, the integral as it stands cannot be separated directly into four parts associated with the points $\pm a$, $\pm l$ respectively; this may, however, be effected by a slight alteration which does not affect the final result for the complete system.

If we write

$$
\begin{aligned}
\zeta(q) &= \frac{4\mathrm{M}}{\pi u\,(l - a)} \int_0^{\pi/2} \sec^2\theta\, d\theta \int_0^\infty \frac{1 - e^{i\kappa q \cos\theta}}{\kappa\,(\kappa - \kappa_0 \sec^2\theta + i\mu \sec\theta)}\, d\kappa \\
&= \frac{4\mathrm{M}}{\pi u\,(l - a)}\, \mathrm{F}(q),
\end{aligned} \tag{23}
$$

then we have

$$\zeta = \zeta(x - l) - \zeta(x - a) - \zeta(x + a) + \zeta(x + l). \tag{24}$$

The integrals in (23) may be transformed in the usual way to separate out the

two parts of the disturbance in each case. We require also the following results

$$\int_0^{\pi/2} \sin(\kappa_0 q \sec\theta)\, d\theta = -\frac{\pi}{2}\int_0^{\kappa_0 q} Y_0(t)\, dt = P_0(\kappa_0 q), \tag{25}$$

where the P functions, which have been used previously in wave analysis, are defined by

$$P_{2n}(p) = (-1)^n \int_0^{\pi/2} \cos^{2n}\theta \sin(p\sec\theta)\, d\theta$$

$$P_{2n+1}(p) = (-1)^{n+1} \int_0^{\pi/2} \cos^{2n+1}\theta \cos(p\sec\theta)\, d\theta. \tag{26}$$

We have also

$$\int_0^{\pi/2} d\theta \int_0^{\infty} \frac{1-\cos(\kappa_0 mq\sec\theta)}{m(1+m)}\, dm$$

$$= \frac{\pi}{2}\int_0^{\kappa_0 q} dt \int_0^{\infty} \frac{J_0(mt)}{1+m}\, dm$$

$$= \frac{\pi^2}{4}\int_0^{\kappa_0 q} \{H_0(t) - Y_0(t)\}\, dt = \frac{\pi}{2} Q_0(\kappa_0 q), \tag{27}$$

using the notation introduced by Wigley for this part of the disturbance.

Retaining q for points in front and q' for points behind the origin of a disturbance, so that $q' = -q$ and q, q' are both positive, we find after collecting these results that

$$F(q) = -\frac{\pi}{2\kappa_0} Q_0(\kappa_0 q), \qquad q > 0$$

$$= -\frac{\pi}{2\kappa_0} Q_0(\kappa_0 q') + \frac{2\pi}{\kappa_0} P_0(\kappa_0 q'), \qquad q' > 0. \tag{28}$$

The complete surface elevation may now be found from (23), (24) and (28). The Q terms represent a local disturbance which is symmetrical fore and aft for the system as a whole, while the P terms give the wave disturbance in the rear of each of the points $\pm a$, $\pm l$.

If M is put equal to $ub/2\pi$, these results will be found to agree with those for the model examined by Wigley in the paper quoted above, and reference may be made to it for a detailed comparison with experimental results.

It should be noted that the method used in (23) and (24) for separating the disturbance into four parts is reflected in the artificial character of the local disturbance associated by (28) with an isolated point $q = 0$; the function Q_0 is zero at its origin and increases indefinitely with distance from it. The

local disturbance decreases with increasing distance when we sum for the system as a whole. The localisation of the disturbance into parts associated with special points is in general no more than a convenient help for purposes of calculation and description.

6. The previous section gives a surface elevation which is finite and continuous throughout, and it is simple to extend the method to cover any form of distribution.

We begin, for simplicity, by considering any limited distribution of which the graph is made up of straight lines.

The general expression (5) gives, for infinite depth of distribution, the elevation along the line of motion as

$$\zeta = -\frac{i}{\pi u}\int_{-\infty}^{\infty}\frac{dM}{dh}\,dh\int_{-\pi}^{\pi}\sec\theta\,d\theta\int_{0}^{\infty}\frac{e^{i\kappa(x-h)\cos\theta}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa. \tag{29}$$

Take the integration with respect to h along two parts of the range meeting at a junction h_{rs}, and we obtain, associated with this junction

$$-\frac{i}{\kappa\cos\theta}\left|\frac{dM}{dh}\right|_r^s e^{-i\kappa h_{rs}\cos\theta}, \tag{30}$$

where the coefficient in straight brackets is the increase in slope of the M, h graph in the positive direction, or $\tan\phi_s - \tan\phi_r$ in terms of the slopes of the adjacent parts of the graph. It should be noted that the positive direction of h, and of x, is taken here in the direction of motion, that is, from stern to bow.

It is clear that for any limited distribution which is zero outside a certain range in h, we have from (29) and (30) the complete surface elevation in the form

$$\zeta = -\frac{1}{\pi u}\int_{-\pi}^{\pi}\sec^2\theta d\,\theta\int_{0}^{\infty}\frac{\Sigma\left|\frac{dM}{dh}\right|_r^s e^{i\kappa(x-h_{rs})\cos\theta}}{\kappa(\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta)}\,d\kappa, \tag{31}$$

where the summation extends to all the junctions, including the bow and stern. Further, the algebraic sum of all the changes of slope is zero; hence we may separate out the calculation for each junction by writing (31) in the form

$$\begin{aligned}\zeta &= \frac{1}{\pi u}\Sigma\left|\frac{dM}{dh}\right|_r^s\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_{0}^{\infty}\frac{1-e^{i\kappa(x-h_{rs})\cos\theta}}{\kappa(\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta)}\,d\kappa\\ &= \frac{4}{\pi u}\Sigma\left|\frac{dM}{dh}\right|_r^s F(x-h_{rs}),\end{aligned} \tag{32}$$

where F is the function specified by (28) for positive and negative values of the argument.

We may now complete the expressions to include a distribution in which there are ranges of continuous change of gradient. It is obvious from the preceding argument that the complete expression is

$$\zeta = \frac{4}{\pi u} \Sigma \left\{ \left| \frac{dM}{dh} \right|_r^s \mathrm{F}(x - h_{rs}) + \int \frac{d^2M}{dh^2} \mathrm{F}(x-h)\, dh \right\}, \tag{33}$$

where the summation covers all points of sudden change of slope and all ranges of continuous variation.

The function F can easily be tabulated and graphed by means of Q_0 and P_0. In summing and integrating in (33) it is to be noticed that the Q_0 terms are symmetrical before and behind each element, while P_0 only exists in the rear of each element. When the distribution M is a sum of integral powers of h, it appears that (33) can be expressed in terms of the P functions defined in (26), for the wave disturbance, together with a similar series of Q functions for the local disturbance. But even if M is not given in simple analytical form, the elevation could be calculated directly from (33) by numerical or graphical methods of integration.

7. We have been discussing the fluid motion due to a given distribution of doublets, the surface elevation we have calculated being one of the stream lines. It would be of interest to trace, if possible, other stream lines so as to exhibit the form of a submerged solid to which the given distribution is equivalent; but the calculations would be lengthy, even in the simplest cases we have considered in the previous sections. For a ship model we have already mentioned the usual approximation for the equivalent distribution of doublets when the ratio of beam to length is small enough. For a model of infinite draught, whose horizontal half-section is given by $y = f(h)$, we have

$$\frac{dM}{dh} = \frac{u}{2\pi} f'(h). \tag{34}$$

Hence (33) gives

$$\zeta = \frac{2}{\pi^2} \Sigma \left\{ \left| f'(h) \right|_r^s \mathrm{F}(x - h_{rs}) + \int f''(h)\, \mathrm{F}(x-h)\, dh \right\}. \tag{35}$$

We note that the magnitude of the contribution due to an angular point on the model is directly proportional to the change of slope that occurs there.

8. We may illustrate the general result by considering briefly a model with

parabolic lines. We take the origin at the bow, and let the form of the half-section for y positive be given by

$$y = b\{1 - (h+l)^2/l^2\}; \qquad -2l < h < 0. \tag{36}$$

The discontinuities of $f'(h)$ at the bow and stern are both positive, and equal to $2b/l$; while $f''(h)$ is constant throughout the range and equal to $-2b/l^2$.

Hence from (28) and (35) we have, from the discontinuity at the bow,

$$\begin{aligned}\zeta_b &= -\frac{2b}{\pi\kappa_0 l}\,Q_0(\kappa_0 x), \qquad x > 0 \\ &= -\frac{2b}{\pi\kappa_0 l}\,Q_0(\kappa_0 x') + \frac{8b}{\pi\kappa_0 l}\,P_0(\kappa_0 x'), \qquad x' > 0.\end{aligned} \tag{37}$$

There is an equal system for the discontinuity at the stern.

Consider now the contribution due to the curved portion and take first the wave terms. For a point behind the stern ($x' > 2l$) we have

$$-\frac{8b}{\pi\kappa_0 l^2}\int_0^{2l} P_0\{\kappa_0(x'-h')\}\,dh'. \tag{38}$$

We have, in a notation already used,

$$\begin{aligned}\int_0^u P_0(u)\,du &= 1 + P_1(u) \\ &= P_0^{-1}(u), \quad \text{say.}\end{aligned} \tag{39}$$

Thus from (38) and (36) the complete wave disturbance at a point behind the stern is given by

$$\zeta_w = \frac{8b}{\pi\kappa_0 l}\left[P_0(\kappa_0 x') + P_0\{\kappa_0(x'-2l)\} - \frac{1}{\kappa_0 l}\{P_0^{-1}(\kappa_0 x') - P_0^{-1}(\kappa_0 \overline{x'-2l})\}\right]. \tag{40}$$

Taking a point between the bow and stern ($0 < x' < 2l$), it is easily verified that (40) gives the wave elevation for all points with the convention that the functions P_0 and P_0^{-1} are to be taken zero for negative values of their arguments. It may be noted that as these functions are zero for zero values of their arguments, the expression is continuous throughout.

Similarly, if we consider the local disturbance and take first a point in front of the bow ($x > 0$), we readily obtain from (28) and (35)

$$\zeta_l = -\frac{2b}{\pi\kappa_0 l}\left[Q_0(\kappa_0 x) + Q_0\{\kappa_0(x+2l)\} + \frac{1}{\kappa_0 l}\{Q_1(\kappa_0 x) - Q_1(\kappa_0\overline{x+2l})\}\right], \tag{41}$$

where

$$Q_1(u) = \int_0^u Q_0(u)\,du. \tag{42}$$

By taking points between the bow and stern and behind the stern, it may be verified that (41) gives this part of the elevation for all points on the understanding that each function Q_0 is symmetrical about the zero of its argument while each function Q_1 is anti-symmetrical, that is $Q_0(-u) = Q_0(u)$ and $Q_1(-u) = -Q_1(u)$.

In (40) and (42) we have the total elevation expressed in terms localised at the bow and stern, and in functions which are easily calculated and tabulated. The quantities have been calculated, without attempting any great degree of accuracy, but sufficiently to show the character of the curves. These are shown in fig. 3 in relation to the length of the model for the velocity given by $\kappa_0 l = gl/u^2 = \pi$.

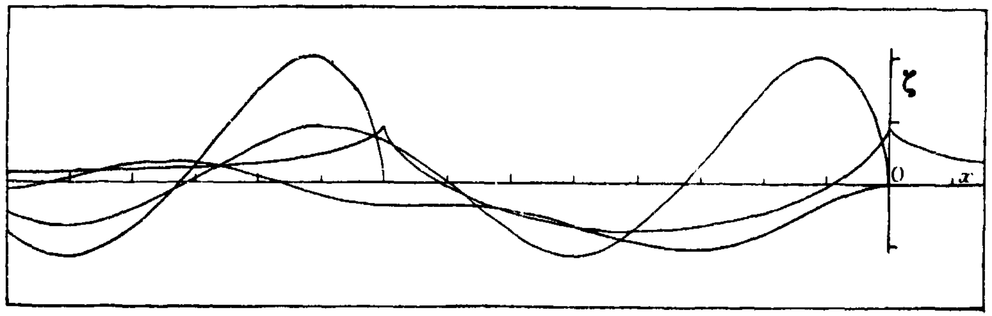

FIG. 3.

The total elevation is the sum of the four curves which are shown in fig. 3. One curve, symmetrical fore and aft, is the complete local disturbance given by the sum of all the Q terms. Then there are two equal curves, one starting at the bow and the other at the stern, for the wave terms due to the discontinuity in slope at the bow and stern. The fourth curve is the total contribution of the curved surface to the wave part of the elevation.

9. Another case of interest, which will only be mentioned here, is an unsymmetrical model whose wave resistance has been discussed previously; its form is given by

$$y = -ah(h+l)^2, \qquad -l < h < 0. \tag{43}$$

Here there is only one discontinuity in $f'(h)$, namely, at the bow, and $f''(h)$ is a linear function of h throughout the range. It will be found that the wave elevation requires the first three terms P_0, P_1, P_2 in the series of P functions, while the local disturbance can be expressed in terms of Q_0, Q_1, Q_2 of a similar series of Q functions.

To return to the general expression (35), it will be seen from the examples that the localisation of the disturbance at special points is largely a matter of

suitable integration. Consider, for instance, the usual form of model which consists of a parallel middle body with a curved entrance extending from the fore-shoulder to the bow and a curved run extending from the aft-shoulder to the stern. In the sense in which the term has been used here, the total elevation can always be separated into parts localised at these four points, the bow and stern and the shoulders. This can readily be expressed analytically by suitable manipulation of (28) and (35); but it is hardly worth while pursuing the general analysis further, as it is simpler to work out the results directly for any particular form of model.

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Ship Waves: their Variation with certain Systematic Changes of Form.

By T. H. HAVELOCK, F.R.S.

(Received February 24, 1932.)

1. The following paper is an examination, by analysis and by curves, of a single definite problem in wave profiles. Consider a ship model, of great draught, in which at some point in the form, at bow, stern or shoulders, for example, there is a sharp corner giving a sudden change of slope of the horizontal lines of the model. What is the effect on the wave profile of replacing this sudden change by a gradual change of slope of the same total amount, but distributed uniformly over any given length of the ship's form? Apart from direct applications, the problem is suggested by certain other considerations. In comparing theoretical and experimental resistance curves, I suggested some years ago* an indirect effect of the friction belt along the sides of the ship in that it may be equivalent to smoothing out the lines of the model, especially towards the stern. From an examination of interference effects with experimental models, it has been estimated that the effective length of the model is roughly 8 per cent. greater than the actual length, and this may probably be ascribed to some such frictional effect. The present paper deals with wave profiles since measurements of surface elevation are now becoming available, though the main results so far are for a simple model with straight lines and sharp corners; such a form simplifies the calculations but no doubt introduces other complications in practice, and a small correction for the smoothing effect of a friction belt would not be likely to account for the remaining differences between calculation and observation. It must be noted, moreover, that there are other approximations in the theory, apart from the neglect of fluid friction, but these need not be discussed here.

For these reasons no attempt has been made to apply the results of the present paper directly to experimental data, but it is hoped that the progressive series of curves will be of interest in showing the changes in profile due to successive changes of form of a definite kind.

2. The general analysis will be quoted from a recent paper,† to which reference may be made for further detail, and the expressions will then be adapted to the particular problem.

* 'Proc. Roy. Soc.,' A, vol. 110, p. 233 (1926).
† 'Proc. Roy. Soc.,' A, vol. 135, p. 1 (1932).

Take O in the free surface, with Ox in the direction of motion and Oz vertically upwards; and let u be the velocity of the model. We consider first a distribution of horizontal doublets in the plane $y = 0$, extending from the free surface down to a great depth, and we take the moment M per unit area to be a function of x only. Further, we suppose that the distribution of M is confined to a finite range in x, is continuous within this range, and is zero at the two limits of the range.

The surface elevation along the median line $y = 0$ is given by

$$\zeta = \frac{4}{\pi u} \Sigma \left\{ \left| \mathrm{M}'(x) \right|_r^s \mathrm{F}(x - x_{rs}) + \int \mathrm{M}''(h) \, \mathrm{F}(x - h) \, dh \right\}, \tag{1}$$

where the summation covers all points of sudden change in the gradient of M, and the integrals extend over the ranges of continuous variation of gradient. The function F is defined for positive and negative values of its argument by

$$\left. \begin{aligned} \mathrm{F}(q) &= -\frac{\pi}{2\kappa_0} \mathrm{Q}_0(\kappa_0 q) \\ \mathrm{F}(-q) &= -\frac{\pi}{2\kappa_0} \mathrm{Q}_0(\kappa_0 q) + \frac{2\pi}{\kappa_0} \mathrm{P}_0(\kappa_0 q), \end{aligned} \right\}, \tag{2}$$

with $q > 0$, and $\kappa_0 = g/u^2$.

We have also, for positive values of p,

$$\left. \begin{aligned} \mathrm{P}_0(p) &= -\frac{\pi}{2} \int_0^p \mathrm{Y}_0(p) \, dp \\ \mathrm{Q}_0(p) &- \frac{\pi}{2} \int_0^p \{\mathrm{H}_0(p) - \mathrm{Y}_0(p)\} \, dp \end{aligned} \right\}, \tag{3}$$

in the usual notation for Struve and Bessel functions.

One of the approximations of the theory lies in the connection between the form of the ship and the equivalent distribution of doublets in the median plane $y = 0$. For a ship model, of infinite draught, whose horizontal half-section is given by $y = f(x)$, the usual approximation amounts to taking

$$\mathrm{M}'(x) = (u/2\pi) f'(x). \tag{4}$$

With this relation, the surface elevation along $y = 0$ is given by

$$\zeta = \frac{2}{\pi^2} \Sigma \left\{ \left| f'(x) \right|_r^s \mathrm{F}(x - x_{rs}) + \int f''(h) \, \mathrm{F}(x - h) \, dh \right\}. \tag{5}$$

Here x, and h, are positive in the direction from stern to bow, x_{rs} being the position of any sharp corner in the form of the model. With this convention

the discontinuities in $f'(x)$ at stern and bow are both positive; at an intermediate sharp corner, say, at a shoulder, the discontinuity would usually be negative. Along the curved lines of the model $f''(h)$ is negative, except for hollow lines where the form is concave outwards and where $f''(h)$ is positive. Thus, knowing the character of the function F, the expression (5) gives a general idea of the contributions of the various parts of the form. These possibilities are illustrated in fig. 1, which represents a half section of a model by a horizontal plane; or, to be more exact, the diagram gives the distribution

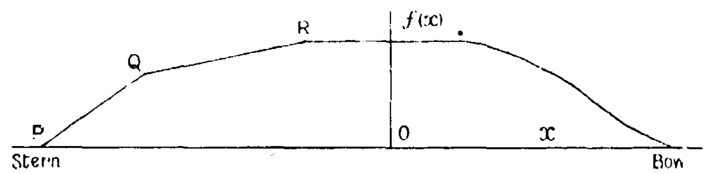

FIG. 1.

of doublets which is approximately equivalent to a model of this form. The figure also indicates the conventions for direction which are adopted throughout this paper.

3. We now isolate one particular feature for examination separately. It should, however, be noted that the function Q defined in (3) increases without limit as its argument becomes greater, though the expression (5) for the model as a whole remains finite everywhere. Therefore there is a certain artificiality, as regards that part of the disturbance, in applying the expressions to an isolated element of the form; but that may be allowed for, and in any case the method gives the differences made by changes in any particular element.

Consider a point on the model, given by $x = x_1$, where the lines of the model are straight lines meeting at a finite angle, for example, P, Q, or R in fig. 1. Let C be the discontinuity in $f'(x)$ at that point; that is, C is the difference of slope of the lines forward and aft of that point. Then, from (2) and (5), the contribution of this element to the surface elevation is

$$\zeta_1 = \frac{2C}{\pi^2} F(x - x_1)$$

$$= (4C/\pi\kappa_0)\{-\tfrac{1}{4}Q_0(\kappa_0 q_1) + P_0(\kappa_0 q'_1)\}, \qquad (6)$$

where $q_1 = x - x_1$ and $q'_1 = x_1 - x$. Further, we may use (6) for all values of x with the convention that P_0 is to be taken zero for negative values of its argument, and that $Q_0(-p) = Q_0(p)$. Now suppose that the same change of slope is carried out uniformly in a given range; that is, suppose the sharp

corner replaced by a parabolic arc extending from $x = x_2$ to $x = x_3$, the point x_1 lying within this range. Considering the effect of this by itself apart from any other changes, we see from (5) that the corresponding contribution to the surface elevation is now

$$\zeta_{23} = \frac{2C}{\pi^2 (x_3 - x_2)} \int_{x_2}^{x_3} F(x - h)\, dh. \tag{7}$$

We shall use the notation

$$Q_1(p) = \int_0^p Q_0(p)\, dp, \tag{8}$$

$$P_0^{-1}(p) = 1 + P_1(p) = \int_0^p P_0(p)\, dp. \tag{9}$$

After evaluating (7) for points in advance of x_3, between x_2 and x_3, and in the rear of x_2, we find that we may express (7) in a single expression for all values of x, namely

$$\zeta_{23} = \frac{4C}{\pi\kappa_0^2 (x_3 - x_2)} \{-\tfrac{1}{4}Q_1(\kappa_0 q_2) + \tfrac{1}{4}Q_1(\kappa_0 q_3) - P_0^{-1}(\kappa_0 q'_2) + P_0^{-1}(\kappa_0 q'_3)\}, \tag{10}$$

with $q_2 = x - x_3 = -q'_2$, $q_3 = x - x_3 = -q'_3$, and with the convention that P_0^{-1} is zero for negative values of its argument, while Q_1 is anti-symmetrical so that

$$Q_1(-p) = -Q_1(p).$$

The expression (6) is, of course, the limiting value of (10) when $x_3 - x_2$ is small and the points x_2 and x_3 ultimately coincide with the point x_1.

Numerical values of the functions may be calculated from their definitions as integrals, or from suitable series; for example, using the expansion of H_0 as a power series, we have

$$Q_0(p) = P_0(p) + \frac{p^2}{2} - \frac{p^4}{1^2 \cdot 3^2 \cdot 4} + \frac{p^6}{1^2 \cdot 3^2 \cdot 5^2 \cdot 6} - \dots \tag{11}$$

$$Q_1(p) = P_0^{-1}(p) + \frac{p^3}{2 \cdot 3} - \frac{p^5}{1^2 \cdot 3^2 \cdot 4 \cdot 5} + \frac{p^7}{1^2 \cdot 3^2 \cdot 5^2 \cdot 6 \cdot 7} - \dots \tag{12}$$

4. The special object in view is a comparison of the relative values of (6) and (10). The quantity C may be either positive or negative, and x_1 may be at any point between x_2 and x_3. But to make the problem definite in the first place, we suppose C negative and take $x_3 = x_1$ and $x_2 < x_1$; thus we are considering a sharp-angled shoulder on the model, such as Q or R in fig. 1, with the smoothing out entirely to the rear of that point. This process, if

carried out on an actual model, would no doubt involve other changes which would have to be considered in a theory capable of taking exact account of actual dimensions; but meantime we may isolate the effect of this particular change.

For convenience we consider separately the effect on the local disturbance and on the wave motion to the rear. Taking the former, we see from (6) and (10) that the difference amounts to replacing $\frac{1}{4} Q_0(\kappa_0 q_1)$ by

$$\frac{1}{4\kappa_0(q_2 - q_1)}\{Q_1(\kappa_0 q_2) - Q_1(\kappa_0 q_1)\}. \tag{13}$$

This can be shown in a form applicable to various velocities and to various ranges of $x_2 - x_1$ by graphing the quantity

$$\frac{1}{4k}\{Q_1(p+k) - Q_1(p)\} \tag{14}$$

on a base p, for several values of k. These curves are shown in fig. 2.

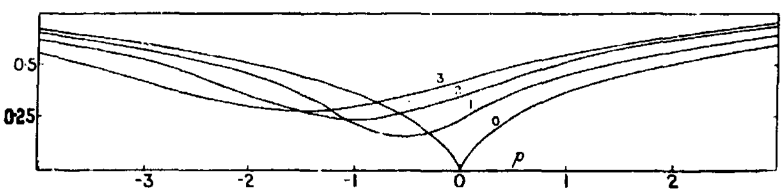

Fig. 2.—Curves of $\{Q(p+k) - Q_1(p)\}/4k$ for different values of k.

In applying these curves to actual distances along the ship model, we note that $p = \kappa_0 x = gx/u^2$, where u is the velocity; and similarly $k = gd/u^2$, where d is the range over which the original sudden change in slope has been distributed. Thus the relative importance of the effects depends upon the ratio gd/u^2, or upon the ratio of d to λ, the wave-length of straight water waves for velocity u. In the diagram, $k = 0$ denotes the curve for the sharp corner; the bow of the model is to the right of the diagram and the stern to the left. Apart from the general smoothing effect, the chief point to notice in these curves is the raising of the profile forward of the point in question and a lowering to the rear of it. This is due to taking the range d entirely to the rear of the original sharp corner. If, on the other hand, the corner is taken at the middle of the range d in each case, by a suitable relative displacement of the curves, it is easily seen that the smoothing of the corner does not make any appreciable difference to the local disturbance except within the range d itself.

Turning now to the wave portion of the surface elevation, the change from (6) to (10) consists in replacing $-\mathrm{P}_0(\kappa_0 q'_1)$ by

$$\frac{1}{\kappa_0 (q'_1 - q'_2)} \{\mathrm{P}_0^{-1}(\kappa_0 q'_2) - \mathrm{P}_0^{-1}(\kappa_0 q'_1)\}. \tag{15}$$

In fig. 3 curves have been drawn for the quantity

$$\{\mathrm{P}_0^{-1}(p-k) - \mathrm{P}_0^{-1}(p)\}/k, \tag{16}$$

on a base p, for several values of k.

There are several points of interest in these curves. Since $k = d/2\pi\lambda$, the relative effect of smoothing out a sharp corner over a given range is less the

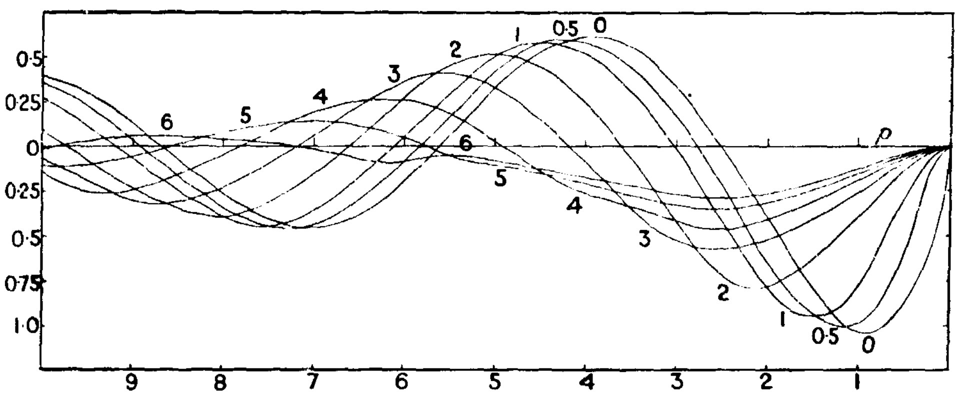

FIG. 3.—Curves of $\{\mathrm{P}_0^{-1}(p-k) - \mathrm{P}_0^{-1}(p)\}/k$ for different values of k.

smaller the ratio of d to λ, as might be expected. In the curves for the smaller values of k, although there is some diminution of amplitude, the more noticeable effect is the displacing of the troughs and crests to the rear, an effect which would increase the apparent interference length of the model. For the larger values of k, from about $k = 2$, there is a pronounced lessening of the amplitudes.

On the convention already described, in calculating these curves from (16) the first term is zero until after $p = k$, and hence within the range k, the curve is simply the value of $-\mathrm{P}_0^{-1}(p)\ .\ /k$. This quantity has a first maximum numerically, at about $p = 2 \cdot 54$, and this may be observed in the curves for $k = 3, 4, 5, 6$. Further, in the curves for the higher values, the effect of later maxima of the same quantity may be noticed; for instance, with $k = 6$ the range of continuous variation of slope is practically equal to the effective wave-length, and so subsidiary interference phenomena of this nature are obtained.

By displacing the curves to right or left we could examine the case when the smoothing of the lines takes place partly in front of the corner; and for a positive discontinuity the curves may be inverted. We may thus obtain, for example, some idea of the effect of smoothing out the lines of a sharp-angled stern, whether actually or by the equivalent effect of a friction belt.

Summary.

An examination, by analysis and by curves, of the changes in wave profile produced by replacing a sudden change of slope in the lines of a model by a continuous variation of the same total amount uniformly distributed over a given length of the model.

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The Theory of Wave Resistance.

By T. H. HAVELOCK, F.R.S.

(Received August 6, 1932.)

Introduction.

1. In the following paper general expressions are obtained for the wave resistance of a continuous distribution of sources and sinks over a surface within the liquid, and also for a similar distribution of normal doublets. These expressions follow directly from results given previously,* and may be applied to give the wave resistance of any solid for which a suitable distribution of sources or doublets over its surface can be found.

The opportunity is taken to give, for comparison, the similar results for a distribution of pressure over the surface of the liquid, using the same notation and the same general method of calculating the wave resistance.

The various results are discussed briefly in relation to the ship problem. Certain interpolation formulæ, of a semi-empirical nature, have been proposed recently in attempting to extend the range of existing expressions for the wave resistance of a ship; these are shown to have their interpretation as particular cases of source distributions of the nature considered here.

Source Distribution.

2. We begin with a simple point source of strength m at a depth f below the free surface of the liquid, and suppose the source to be moving horizontally in the direction Ox with uniform velocity c. Take the origin O in the free surface with Oz vertically upwards, the source being at the point $(0, 0, -f)$ referred to

* 'Proc. Roy. Soc.,' A, vol. 118, p. 24 (1928).

moving axes. Let ζ be the surface elevation, and assume a frictional force in the liquid proportional to velocity, the frictional coefficient being ultimately made zero.

The pressure condition at the free surface is

$$\frac{\partial \phi}{\partial t} - g\zeta + \mu' \phi = \text{constant}, \tag{1}$$

and this gives

$$\frac{\partial^2 \phi}{\partial x^2} + \kappa_0 \frac{\partial \phi}{\partial z} - \mu \frac{\partial \phi}{\partial x} = 0, \tag{2}$$

at $z = 0$, with $\kappa_0 = g/c^2$ and $\mu = \mu'/c$. Assume for the velocity potential

$$\phi = \frac{m}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^{\infty} e^{-\kappa(z+f)+i\kappa\varpi} \, d\kappa + \int_{-\pi}^{\pi} d\theta \int_0^{\infty} F(\kappa, \theta)\, e^{\kappa z + i\kappa\varpi} \, d\kappa, \tag{3}$$

where $\varpi = x\cos\theta + y\sin\theta$, and the real part of the expression is to be taken. The first term in (3) gives the velocity potential of the given source, namely m/r_1, in a form valid for $z + f > 0$. From the surface condition (2) we obtain

$$F(\kappa, \theta) = -\frac{m}{2\pi} \frac{\kappa + \kappa_0 \sec^2\theta + i\mu\sec\theta}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} e^{-\kappa f}. \tag{4}$$

Hence we may write the solution in the form

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} - \frac{\kappa_0 m}{\pi} \int_{-\pi}^{\pi} \sec^2\theta \, d\theta \int_0^{\infty} \frac{e^{-\kappa(f-z)+i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} \, d\kappa, \tag{5}$$

where

$$r_1^2 = x^2 + y^2 + (z+f)^2; \quad r_2^2 = x^2 + y^2 + (z-f)^2.$$

It is to be understood that the limiting value of (5) is taken for $\mu \to 0$.

We may now generalise by integration. We replace x and y by $x - h$ and $y - k$ respectively, and take σ to be the surface density of source at a point $(h, k, -f)$ on a surface S within the liquid. Thus the velocity potential is given by

$$\phi = \int\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sigma\, dS - \frac{\kappa_0}{\pi}\int \sigma dS \int_{-\pi}^{\pi} \sec^2\theta d\theta \int_0^{\infty} \frac{e^{-\kappa(f-z)+i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa, \tag{6}$$

with

$$r_1^2 = (x-h)^2 + (y-k)^2 + (z+f)^2$$
$$r_2^2 = (x-h)^2 + (y-k)^2 + (z-f)^2$$
$$\varpi = (x-h)\cos\theta + (y-k)\sin\theta.$$

It is assumed that the distribution is such that the various integrals are convergent.

3. To calculate the wave resistance R we use the method of the previous paper to which reference has already been made. With the inclusion of the frictional term in the equations of fluid motion, energy is dissipated at a rate equal to $2\mu'$ times the total kinetic energy of the liquid and this must be equal to the product Rc. As μ' is made to approach zero the quantity so calculated approaches a finite limiting value, and its physical interpretation in the limit when there is no fluid friction is the rate at which energy is propagated outwards in the wave motion.

The rate of dissipation of energy is given by

$$-\mu'\rho\int\phi\frac{\partial\phi}{\partial n}\,dS, \tag{7}$$

taken over the boundaries of the liquid. As we require only the limiting value, we have the wave resistance given by

$$R = \operatorname*{Lim}_{\mu\to 0}\mu\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\frac{\partial\phi}{\partial z}\,dx\,dy, \tag{8}$$

taken over the free surface $z = 0$.

Referring to (6), and putting the first two terms in the same integral form as the third, we obtain, at $z = 0$,

$$\phi = -\frac{\kappa_0}{\pi}\int\sigma\,dS\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_0^{\infty}\frac{e^{-\kappa f+i\kappa\varpi}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa, \tag{9}$$

$$\frac{\partial\phi}{\partial z} = -\frac{1}{\pi}\int\sigma\,dS\int_{-\pi}^{\pi}d\theta\int_0^{\infty}\frac{\kappa^2 e^{-\kappa f+i\kappa\varpi}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa, \tag{10}$$

where the real parts are to be taken.

After some reduction, we may write the real part of (9) in the form

$$\begin{aligned}\phi = \int_{-\pi}^{\pi}d\theta\int_0^{\infty}\{&F_1\cos(\kappa x\cos\theta)\cos(\kappa y\sin\theta)\\ &+F_2\sin(\kappa x\cos\theta)\cos(\kappa y\sin\theta)+F_3\cos(\kappa x\cos\theta)\sin(\kappa y\sin\theta)\\ &+F_4\sin(\kappa x\cos\theta)\sin(\kappa y\sin\theta)\}\,\kappa\,d\kappa,\end{aligned} \tag{11}$$

in which

$$
\begin{aligned}
F_1 &= -\{(\kappa - \kappa_0 \sec^2\theta)\, P_e - \mu Q_e \sec\theta\}\, D \\
F_2 &= -\{(\kappa - \kappa_0 \sec^2\theta)\, Q_e + \mu P_e \sec\theta\}\, D \\
F_3 &= -\{(\kappa - \kappa_0 \sec^2\theta)\, Q_0 - \mu P_0 \sec\theta\}\, D \\
F_4 &= -\{(\kappa - \kappa_0 \sec^2\theta)\, P_0 + \mu Q_0 \sec\theta\}\, D \\
D &= \kappa_0 \sec^2\theta / \pi\kappa \{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta\},
\end{aligned}
\tag{12}
$$

and the quantities P, Q are given in terms of the source distribution by

$$
\begin{aligned}
P_e &= \int \sigma e^{-\kappa f} \cos(\kappa h \cos\theta) \cos(\kappa k \sin\theta)\, dS \\
P_0 &= \int \sigma e^{-\kappa f} \sin(\kappa h \cos\theta) \sin(\kappa k \sin\theta)\, dS \\
Q_e &= \int \sigma e^{-\kappa f} \sin(\kappa h \cos\theta) \cos(\kappa k \sin\theta)\, dS \\
Q_0 &= \int \sigma e^{-\kappa f} \cos(\kappa h \cos\theta) \sin(\kappa k \sin\theta)\, dS.
\end{aligned}
\tag{13}
$$

Similarly from (10), $\partial\phi/\partial z$ is obtained in the same integral form as in (11), with quantities G instead of F given by the same expressions as in (12) but with

$$D = \kappa/\pi\{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta\}.$$

The expressions for the surface values of ϕ and $\partial\phi/\partial z$ are now in a form to which we may apply a theorem derived from the Fourier integral theorem in two variables; namely, we have, with the above notation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi \frac{\partial\phi}{\partial z}\, dx\, dy = 4\pi^2 \int_{-\pi}^{\pi} d\theta \int_0^{\infty} (F_1G_1 + F_2G_2 + F_3G_3 + F_4G_4)\, \kappa\, d\kappa. \tag{14}$$

Using (8), this reduces readily to

$$
\begin{aligned}
R &= \operatorname*{Lim}_{\mu \to 0} 4\kappa_0 \rho\mu \int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^{\infty} \frac{\kappa (P_e^2 + P_0^2 + Q_e^2 + Q_0^2)}{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta}\, d\kappa \\
&= 16\pi\kappa_0^2 \rho \int_0^{\frac{1}{2}\pi} (P_e^2 + P_0^2 + Q_e^2 + Q_0^2) \sec^3\theta\, d\theta,
\end{aligned}
\tag{15}
$$

where in (15) the quantities P and Q have the values given by (13) when κ has been replaced by $\kappa_0 \sec^2\theta$.

This result may also be put in the form

$$R = 8\pi\kappa_0^2\rho \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2) \sec^3 \theta \, d\theta, \tag{16}$$

with

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \int \sigma \, e^{\kappa_0 z \sec^2 \theta} \begin{matrix} \cos \\ \sin \end{matrix} \{\kappa_0 (x \cos \theta + y \sin \theta) \sec^2 \theta\} \, dS. \tag{17}$$

In (17), the co-ordinates $(h, k, -f)$ have been replaced by current co-ordinates (x, y, z); since the sources are within the liquid, z is negative over the surface S.

Doublet Distribution.

4. A surface distribution of normal doublets could be obtained by generalising an expression for any two doublets, but it can be deduced directly from (16) and (17). We have simply to regard the surface S in (17) as a double sheet with source densities σ and $-\sigma$ respectively, and then proceed to the limit in the usual manner. The required result is obtained by applying the operator

$$l \frac{\partial}{\partial x} + m \frac{\partial}{\partial y} + n \frac{\partial}{\partial z},$$

to the expressions in (17), (l, m, n) being the direction of the normal to the surface. If M is the doublet moment per unit area, the axes being everywhere normal to the surface S, we obtain, in this way, the wave resistance

$$R = 8\pi\kappa_0^4\rho \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2) \sec^7 \theta \, d\theta, \tag{18}$$

in which

$$\begin{aligned} P &= \int M \, e^{\kappa_0 z \sec^2 \theta} \{-(l \cos \theta + m \sin \theta) \sin (\kappa_0 \varpi \sec^2 \theta) \\ &\qquad\qquad + n \cos (\kappa_0 \varpi \sec^2 \theta)\} \, dS \\ Q &= \int M \, e^{\kappa_0 z \sec^2 \theta} \{(l \cos \theta + m \sin \theta) \cos (\kappa_0 \varpi \sec^2 \theta) \\ &\qquad\qquad + n \sin (\kappa_0 \varpi \sec^2 \theta)\} \, dS, \end{aligned} \tag{19}$$

with $\varpi = x \cos \theta + y \sin \theta$.

These expressions may be put into various alternative forms, and, of course, may be simplified when the surface distribution is symmetrical with respect to the co-ordinate planes. It may be remarked that an expression given previously for the wave resistance of any two finite doublets in given positions may be deduced as a particular case of these results.

Pressure Distribution.

5. The wave resistance for a travelling distribution of pressure applied to the upper surface of the liquid has been worked out by various methods, but not by that used in the previous sections. It is convenient, for comparison, to have the general case set out in the same way and using the same principle for the calculation of the resistance.

We begin by assuming a possible form for the velocity potential and finding the surface pressure to which it corresponds.

We take

$$\phi = \int_{-\pi}^{\pi} \sec\theta\, d\theta \int_0^{\infty} \frac{F(\kappa)\, e^{\kappa z + i\kappa\varpi}}{\kappa - \kappa_0 \sec^2\theta + i\mu \sec\theta}\, \kappa\, d\kappa, \tag{20}$$

with $\varpi = x\cos\theta + y\sin\theta$.

From the kinematical condition at $z = 0$, the surface elevation is given by

$$\zeta = -\frac{i}{c}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^{\infty} \frac{F(\kappa)\, e^{i\kappa\varpi}}{\kappa - \kappa_0 \sec^2\theta + i\mu\sec\theta}\, \kappa\, d\kappa. \tag{21}$$

The pressure at the surface ($z = 0$) is found from

$$\frac{p}{\rho} = -c\frac{\partial\phi}{\partial x} - g\zeta + \mu'\phi. \tag{22}$$

Using (20) and (21), this reduces to

$$p = -ic\rho \int_{-\pi}^{\pi} d\theta \int_0^{\infty} \kappa F(\kappa)\, e^{i\kappa(x\cos\theta + y\sin\theta)}\, d\kappa$$

$$= -2\pi\rho c i \int_0^{\infty} \kappa F(\kappa)\, J_0(\kappa r)\, d\kappa, \tag{23}$$

where $r^2 = x^2 + y^2$. Since we may write

$$p(r) = \int_0^{\infty} J_0(\kappa r)\, \kappa\, d\kappa \int_0^{\infty} p(\alpha)\, J_0(\kappa\alpha)\, \alpha\, d\alpha, \tag{24}$$

we see that

$$\phi = \frac{i}{2\pi c\rho}\int_{-\pi}^{\pi} \sec\theta\, d\theta \int_0^{\infty} \frac{f(\kappa)\, e^{\kappa z + i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, \kappa\, d\kappa, \tag{25}$$

represents the solution for a surface pressure $p(r)$, symmetrical round the moving origin, with

$$f(\kappa) = \int_0^{\infty} p(\alpha)\, J_0(\kappa\alpha)\, \alpha\, d\alpha. \tag{26}$$

To generalise this, we first suppose the pressure concentrated round the origin and of integrated amount P, so that $f(\kappa)$ in (25) is replaced by $P/2\pi$. Then for

any continuous distribution of pressure $p(x, y)$, we obtain by integration

$$\phi = \frac{i}{4\pi^2 c\rho}\int p(h, k)\, dS \int_{-\pi}^{\pi} \sec\theta\, d\theta \int_0^{\infty} \frac{e^{\kappa z + i\kappa\varpi}}{\kappa - \kappa_0 \sec^2\theta + i\mu\sec\theta}\, \kappa\, d\kappa, \qquad (27)$$

where now we have $\varpi = (x - h)\cos\theta + (y - k)\sin\theta$.

6. We obtain the corresponding wave resistance from the rate of dissipation of energy exactly as in the previous sections, and we use the formula (8). The surface values of ϕ and $\partial\phi/\partial z$ are put into the form (11) and the calculation carried out as in (14). From the similarity of the forms for ϕ in the two cases, the result may be written down. We obtain

$$\begin{aligned} R &= \operatorname*{Lim}_{\mu \to 0} \frac{\mu}{4\pi^2 c^2 \rho}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^{\infty} \frac{\kappa^2 (P_e^2 + P_0^2 + Q_e^2 + Q_0^2)}{(\kappa - \kappa_0 \sec^2\theta)^2 + \mu^2 \sec^2\theta}\, d\kappa \\ &= \frac{\kappa_0^2}{\pi c^2 \rho}\int_0^{\frac{1}{2}\pi} (P_e^2 + P_0^2 + Q_e^2 + Q_0^2) \sec^5\theta\, d\theta, \end{aligned} \qquad (28)$$

where the quantities P and Q are as in (13) with f zero and σ replaced by p.

We may also write this in the form

$$R = \frac{\kappa_0^2}{2\pi c^2 \rho}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2) \sec^5\theta\, d\theta, \qquad (29)$$

with

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \int p(x, y) \begin{matrix}\cos \\ \sin\end{matrix} \{\kappa_0 (x\cos\theta + y\sin\theta)\sec^2\theta\}\, dS, \qquad (30)$$

the latter integrations extending over the given surface distribution of pressure.

We may obtain an alternative form by integrating with respect to x in (30); provided the pressure distribution is continuous and is zero at its outer boundaries, we then have

$$R = \frac{1}{2\pi c^2 \rho}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2) \sec^3\theta\, d\theta, \qquad (31)$$

with

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \int \frac{\partial p}{\partial x} \begin{matrix}\cos \\ \sin\end{matrix} \{\kappa_0 (x\cos\theta + y\sin\theta)\sec^2\theta\}\, dS. \qquad (32)$$

We may compare (31) and (32) with the expressions (16) and (17) for a distribution of sources on a surface within the liquid. Suppose we may neglect the depth of this latter surface at every point; then without considering the actual surface elevation, which would require a closer examination, we may say that the wave resistance for the two cases would be the same with the connection between the source density and the pressure distribution given by $4\pi g\rho\sigma = c\,\partial p/\partial x$.

Moving Solid.

7. An obvious application of these results is to the uniform motion of a submerged solid when we replace the solid by a distribution of sources or doublets over its surface; for a first approximation we may take the distribution to be that appropriate to the motion of the solid in an infinite liquid. This will, of course, give the same result as if we had used the system of sources and sinks which is the image of a uniform stream in the solid, or, in fact, any equivalent surface or volume distribution on or within the surface of the solid. Simple forms, such as the sphere or ellipsoid, for which the wave resistance has already been found, have been calculated from the known image system. For instance, the sphere was replaced by a doublet at the centre; it can be verified, after some reduction of integrals, that the expressions (16) and (17) with the proper value of σ over the surface of the sphere, lead to the same result for the wave resistance. In general, the expressions (16) and (17) allow the wave resistance to be calculated for solids for which an image system is not known, but for which the distribution of surface density can be determined by known methods of approximation.

Consider now an open plane distribution of sources and sinks over the vertical zx-plane. In this case the normal fluid velocity at a point on either side is $2\pi\sigma$, where σ is the source density at the point. For a ship of slender form, and small beam, symmetrical about the zx-plane, the normal velocity is taken to be approximately $c\,\partial y/\partial x$ if the surface of the ship is given by an equation $y = f(z, x)$. From (16) and (17), the usual expression for the wave resistance follows:

$$R = \frac{2g^2\rho}{\pi c^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2)\sec^3\theta\, d\theta, \tag{33}$$

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \iint \frac{\partial y}{\partial x} e^{\kappa_0 z \sec^2\theta} \begin{matrix}\cos \\ \sin\end{matrix} (\kappa_0 x \sec\theta)\, dx\, dz, \tag{34}$$

the latter integrations being taken over the vertical longitudinal section.

For the other extreme case, a ship of flat form and small draught, comparison is usually made with a suitable distribution of pressure applied to the surface of the water, with the wave resistance given by, say, (31) and (32).

The similarity between the expressions for the resistance in these two extreme forms has been remarked upon by Weinblum,* and more recently by Hogner.† In an attempt to cover both cases by a single expression, Hogner has proposed

* G. Weinblum, 'Z.A.M.M.,' vol. 10, p. 458 (1930).

† E. Hogner, 'Jahrb. Schiffbautech. Ges.' (1932).

a so-called interpolation formula which, when put into the notation of the present paper, is

$$R = \frac{g^2\rho}{2\pi c^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (P^2 + Q^2)\sec^3\theta\, d\theta, \tag{35}$$

$$\left.\begin{matrix} P \\ Q \end{matrix}\right\} = \iint \frac{\partial z}{\partial x}\, e^{\kappa_0 z \sec^2\theta} \begin{matrix}\cos\\ \sin\end{matrix} \{\kappa_0 (x\cos\theta + y\sin\theta)\sec^2\theta\}\, dx\, dy. \tag{36}$$

In (36) the integrations are taken over the section of the ship by the water surface, and the surface of the ship is given by an equation $z = F(x, y)$. It may be noted that if dS_y and dS_z are the projections of an element of the surface upon the zx-plane and the xy-plane respectively, we have

$$(\partial z/\partial x)\, dS_z = (\partial y/\partial x)\, dS_y.$$

In the limit $y \to 0$, (36) becomes equivalent to (34) under the conditions for a ship of small beam. On the other hand, in the limit $z \to 0$, (36) reduces to the expression (31) for a pressure distribution with the assumption $p = g\rho\zeta$. Without discussing this argument, it may be remarked that (36) is a particular case of the expressions in (16) and (17) for a distribution of sources over the surface of the ship. In the one extreme case, the narrow ship, we take $\sigma = (c/2\pi)\,\partial y/\partial x$, the sources forming in the limit a plane distribution. For the other extreme, the flat ship, a similar approximation would be $\sigma = (c/2\pi)\,\partial z/\partial x$. But it is only in these cases, when the source distribution approximates to a plane, that the normal fluid velocity can be expressed simply in terms of the source density; these expressions do not hold when the distribution is on a curved surface or, in other words, when the finite beam of the ship is taken into account.

It has been remarked that formulæ in use at present are in effect special cases of the general expressions (16) and (17), with simple approximations to the density of the source distribution. If we think of the distribution, appropriate to motion in an infinite liquid, as a suitable first approximation, it might be suggested that this should be used over the curved surface of the ship instead of the present simple expressions over the vertical longitudinal plane. In one sense this would be an improvement, but it is not likely that it would give any better agreement with experimental results; for the more we depart from the simple narrow ship the more necessary it is to take into account the effect of the wave motion upon the distribution of fluid velocity round the ship.

Instead of attempting to assign in advance a distribution of sources or doublets over the surface of the ship, it might be left to be determined, from

suitable integral equations, so that all the conditions of the problem should be satisfied. This, in itself, would not amount to more than a formulation of the general problem in different terms and would not advance its practical solution, unless possibly such a form of statement should lead to improved methods of approximation for the equivalent distribution.

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WAVE PATTERNS AND WAVE RESISTANCE.

By Professor T. H. HAVELOCK, F.R.S.

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INTRODUCTION.

1. It is not my intention to discuss in this paper practical problems of ship resistance, but rather to review briefly certain points in the mathematical theory of ship waves and wave resistance. In doing so, I shall not attempt to give the derivation of formulæ or any mathematical analysis of them; my main object is to give a descriptive or qualitative account of some of the mathematical expressions and to show how in some cases deductions may be drawn from an inspection of them.

The wave pattern made by a ship is familiar both from observation and as a subject of mathematical study, and it is equally fascinating from both points of view. Perhaps the earliest theoretical account is that given by Kelvin in 1887 in his well-known lecture on ship waves to the Institution of Mechanical Engineers. That lecture was based on mathematical work of which a later improved version was published by Kelvin in 1904,* and it is this later work which is usually quoted now in the text-books. The ship, in that work, is idealized to a point disturbance travelling over the water and at the same time sending out groups of waves which combine in such a way as to produce the characteristic pattern of transverse and diverging waves. The early history of this idea of wave groups and group velocity is also of some interest. In a letter written to Stokes in 1873, William Froude describes the motion of a group of waves, how the group as a whole advances with a less velocity than that of the waves composing it, wave crests advancing through the group in its motion and appearing to die away at the front while new ones are formed at the rear; he writes, in his letter from Torquay, "In my long experimental tank or canal here, I have frequent opportunity of noticing this in the propagation of artificially generated waves. I have not, indeed, yet investigated it quantitatively, because my hands are full: but at a later date when experiments on the oscillation of models will be the work in hand, I shall have to establish regular appliances for the generation of waves, and the investigation to which I refer will be comparatively easy." It was in 1876 that Stokes gave the kinematical explanation of group velocity, a more general account being given shortly after by Rayleigh. This was followed in 1877 by Osborne Reynolds' dynamical theory of group velocity, connecting the flow of energy and the rate of work of the fluid pressure in a train of waves; it is this latter point of view which is of fundamental importance in the theory of wave resistance.

Much work has been done since then, both on the detailed structure of wave patterns

* *Edin. Roy. Soc. Proc.*, Vol. XXV, (i), "On Deep Water Two-dimensional Waves produced by any given Initiating Disturbance"; "On the Front and Rear of a Free Procession of Waves in Deep Water"; and "Deep Water Ship Waves."

and on the calculation of wave resistance, and more recently on the comparison of calculated results with experiment; but the fundamental principles remain the same, and it is these which I wish specially to keep in view in the following notes. We begin by considering freely moving wave patterns; that is, not forced waves produced by the motion of a ship, but waves moving freely and steadily over the surface of the water under the action of gravity alone. We imagine the pattern to be produced by the mutual interference of simple plane waves moving freely in all directions, their phases and velocities being suitably adjusted; the elementary properties of the pattern are described from this point of view. Then, considering the waves produced by a ship, we see that these must approximate, at a sufficient distance to the rear of the ship, to such a freely moving pattern; this is illustrated by calculations made for certain ship models. Finally, it is shown how the wave resistance can be obtained from considerations of energy when we know the structure of the wave pattern formed at a great distance in the rear of the ship.

FREE WAVE PATTERNS.

2. The simplest form of free waves on the surface of water consists of simple harmonic waves with straight parallel crests, the procession of waves extending over the whole surface. If the velocity of the waves is c, the wave-length is $2\pi c^2/g$ for deep water; so that if we take an origin O in the surface and take O x in the direction of propagation, the waves might be represented by

$$\zeta = \sin \frac{g}{c^2}(x - c\,t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ζ is the surface elevation, and we have taken the waves to be of unit amplitude.

Suppose now that the waves are travelling in a direction making an angle θ with O x, and that the wave velocity is $c \cos\theta$; then, with O y in the surface and perpendicular to O x, the waves are now represented by

$$\zeta = \sin\left\{\kappa \sec^2\theta\,(x\cos\theta + y\sin\theta - c\,t\cos\theta)\right\} \quad . \quad . \quad . \quad . \quad (2)$$

where we have written $\kappa = g/c^2$.

An equal procession of waves moving in a direction making a negative angle θ with O x is given by

$$\zeta = \sin\left\{\kappa \sec^2\theta\,(x\cos\theta - y\sin\theta - c\,t\cos\theta)\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Superpose these two sets of plane waves, and we have a wave pattern given by the sum of (2) and (3), or

$$\zeta = 2\cos(\kappa\,y\sin\theta\sec^2\theta)\sin\left\{\kappa\,(x - c\,t)\sec\theta\right\} \quad . \quad . \quad . \quad . \quad (4)$$

These have sometimes been called corrugated waves. We may get a rough idea of the result by drawing parallel straight lines to represent the positions of the crests and troughs of the component systems at a given instant: and so we get the picture of a diamond-shaped pattern, covering the whole surface and moving steadily in the direction O x with velocity c.

We now generalize by supposing that we have simple straight-crested waves like (2) travelling forward in all directions included within 90° on either side of O x. Superposing these component plane waves will give a surface elevation

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\left\{\kappa\sec^2\theta\,(x\cos\theta + y\sin\theta - c\,t\cos\theta)\right\} d\theta \quad . \quad . \quad . \quad (5)$$

and this will represent a free wave pattern of some form travelling steadily parallel to Ox with velocity c.

We may again obtain a rough picture of the result by simple graphical methods. Suppose we represent a component plane wave of (5) by parallel straight lines showing the crests and troughs at, say, the instant $t = 0$, in the manner shown in Fig. 1, the full lines representing crests and the broken lines troughs.

Now draw similar lines on the same diagram for a large number of values of θ in the range from $-90°$ to $+90°$. It is instructive to take, for instance, intervals of $10°$ and to

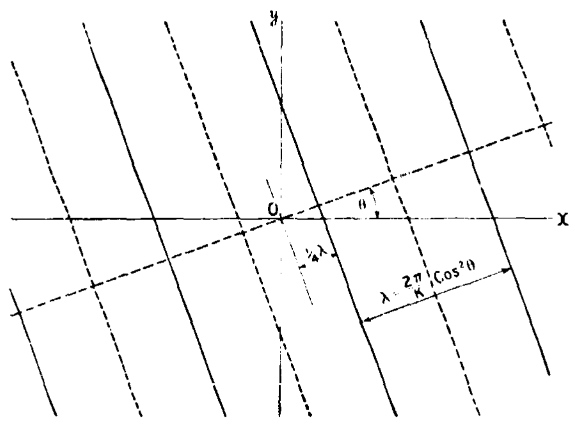

FIG. 1.

draw 19 sets of lines as in Fig. 1. Such a diagram is not given here, as there is too much detail for reproduction on a small scale; but it is interesting to see the picture of a familiar wave pattern emerging from such a diagram. The curves which we see in process of formation are shown in Fig. 2.

These curves are, of course, the envelopes of the lines of constant phase of the component waves, and their mathematical equations are most easily obtained by expressing

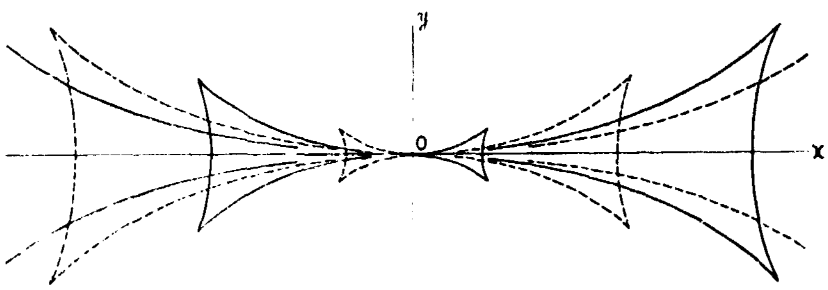

FIG. 2.

that fact. When we look into the formation of the curves we see that they represent places where component crests, or troughs as the case may be, combine together to give prominent features of the pattern; on the other hand, we may say that at points at some distance outside the region covered by these curves the component crests and troughs tend to cancel each other out on the average. We arrive in this way at the picture of a wave pattern of transverse and diverging waves, with a focus point O, and extending in advance of this point as well as to the rear; the whole forms a freely moving pattern travelling forward with steady velocity. It need hardly be said that this description of the pattern represented

by (5) is only a first approximation; detailed mathematical analysis is necessary for a more correct and intimate knowledge of the surface elevation.

Examine more closely one of the curves of Fig. 2, say the portion O A B which is shown in Fig. 3 along with the crest lines of the component plane waves.

We find that the transverse part A B is made up from those plane waves whose direction angles range from zero up to an angle θ_1, which is such that $\cos^2\theta_1 = \frac{2}{3}$, or $\theta_1 = 35^\circ\ 16'$ approximately; the diverging part O A comes from the plane waves whose directions range from θ_1 to 90°. The angle between the crest line O A and the central line O B is $19^\circ\ 28'$, nearly. To complete our picture we require some information about the height of the waves in the pattern defined by the expression (5). All that need be said here is that, following a curve such as B A O, the height is fairly constant over the central portion of the transverse wave, increases in the neighbourhood of a crest point A and then decreases along the diverging wave to zero at the point O.

It may also be noted that the wave-length λ of a component plane wave being $(2\,\pi\, c^2/g)\cos^2\theta$, these wave-lengths range from $2\,\pi\, c^2/g$ to zero.

3. Consider for a moment the difference in these general results if the water, instead of being very deep, is of given finite depth h. The relation between velocity and wave-length for a simple plane wave is different, and, moreover, there can be no plane wave

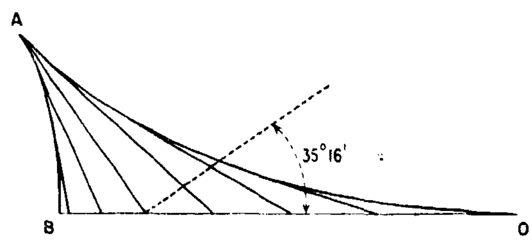

FIG. 3.

whose velocity is greater than $\sqrt{(g\,h)}$. Suppose we build up a pattern like (5) when the velocity c of the pattern is less than this critical value $\sqrt{(g\,h)}$. We could trace the envelope curves in the same way and obtain a wave pattern similar to Fig. 2. The chief difference is that the wave pattern widens out; the angle of the cusp line is greater than the value $19^\circ\ 28'$ for deep water and it increases with the velocity c. In addition, the transverse waves become less curved, the angle θ_1 of Fig. 3 being less than the value 35° for deep water and becoming less as the velocity c is increased.

If the velocity c is made greater than the critical value $\sqrt{(g\,h)}$, we see at once that we must omit a central portion of the integration in (5), because the component plane waves can only begin to exist at such an inclination θ that their wave velocity $c\cos\theta$ is equal to $\sqrt{(g\,h)}$. On working out the wave pattern in more detail, it is found that it consists then of only diverging waves.

4. We return to the expression (5) for deep water. The origin O was taken at a fixed point, but it is more convenient to take a moving origin for the co-ordinates at the focus point of the wave pattern; so in what follows we shall write x instead of $x - c\,t$. Further, for brevity we shall write

$$(x,\ y) = \kappa \sec^2\theta\,(x\cos\theta + y\sin\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We may call the surface elevation given by

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\,(x,\ y)\,d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

a simple sine pattern.

We could also have used a form

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which may be called a simple cosine pattern. The general form of the pattern is the same in both cases, with the necessary changes in wave heights due to the interchange of crests and troughs. It would be of interest to have a more detailed mathematical and numerical analysis of these two simple forms.

In (7) and (8) the amplitudes of the component plane waves are taken to be the same for all directions. We may now proceed to a final generalization by supposing that in each case there is an amplitude factor depending upon the direction of each component: adding the two forms, we arrive at a general expression for a freely moving wave pattern, namely

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \sin(x, y)\, d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{F}(\theta) \cos(x, y)\, d\theta \quad . \quad . \quad . \quad . \quad (9)$$

It is true that the amplitude factors may alter considerably our picture of the pattern, especially if they have pronounced maxima or minima; however, we shall see that most cases which have been calculated for ship models can be reduced to terms like (9) with simple amplitude factors.

Ship Waves.

5. We have been dealing so far with a free wave pattern: that is, we have supposed the system to be completely in existence at some instant, and then afterwards it moves freely and steadily forward.

Consider now the disturbance produced, in a frictionless liquid, by a moving ship or by a disturbing pressure system moving steadily forward. At some distance in advance of the ship there can be no appreciable disturbance, as we suppose it moving forward into still water. In the immediate neighbourhood of the ship the disturbance will be of a complicated character. But as we go further and further to the rear, the surface disturbance must approximate more and more to some freely moving wave pattern following on with the same speed as the ship.

For instance, if a long cylindrical log is moved with steady velocity c at right angles to its length, the disturbance at a great distance in the rear must approximate to a simple plane wave of velocity c, whose wave-length is therefore $2\pi c^2/g$. It could be expressed by (1), taking some suitable point as the origin O, and including some definite amplitude factor; this amplitude factor would be the important thing left to be determined from the form of the cross-section of the cylinder and its velocity. Similarly for an ordinary ship form, the waves at a great distance in the rear must approximate to some freely moving wave pattern such as we have been considering; and for some suitable origin O, in or near the ship, they must therefore be expressible in the form (9), with amplitude factors $f(\theta)$ and $F(\theta)$ depending upon the form of the ship and the speed. Without going into the details of calculating these expressions we shall now examine a few cases in order to illustrate the types of wave pattern which occur in such problems.

Point Disturbance and Sphere.

6. On account of its historical interest we may mention first the travelling point

disturbance examined by Kelvin in the paper to which reference has already been made. In this case the waves in the rear approximate to the form

$$\zeta = A \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 \theta \sin (x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where A is a constant, and we use the notation specified in section 4.

We may describe this as a sine pattern with an amplitude factor $\sec^2 \theta$, which varies from unity at $\theta = 0°$ to infinity at $\theta = 90°$. We have seen, in section 2, that the transverse waves of the pattern come from the range 0° to 35° approximately, while the diverging waves come from the rest of the range 35° to 90°, taking one side of the central line O x. Thus we should expect the diverging waves in this case (10) to be increased in magnitude

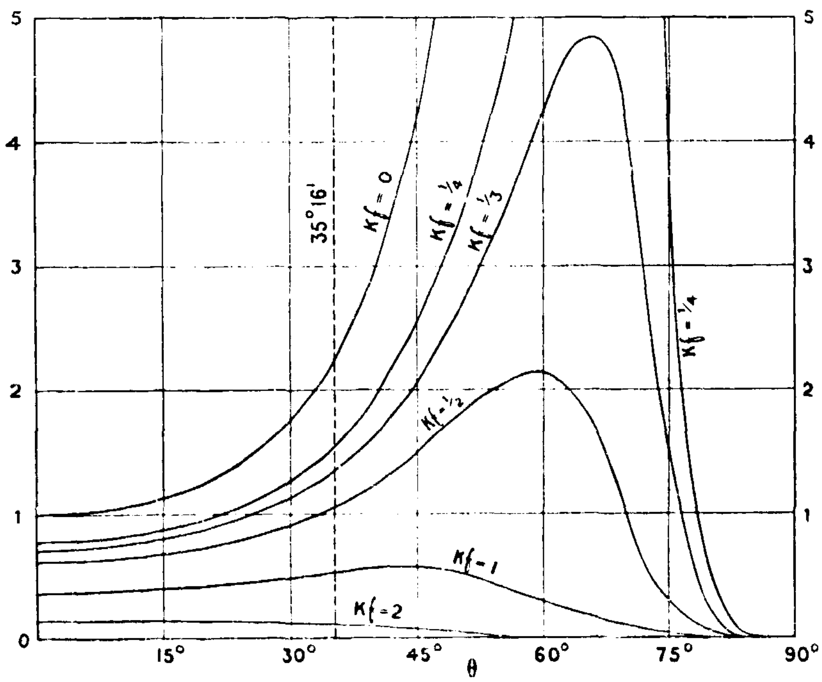

FIG. 4.—GRAPHS OF $\mathrm{SEC}^4 \theta\, e^{-\kappa f \sec^2 \theta}$ FOR DIFFERENT VALUES OF κf.

compared with those for the simple sine pattern (7); these features are unduly prominent in the Kelvin pattern in comparison with those made by an ordinary ship model. Incidentally we may note that the factor $\sec^2 \theta$ causes the expression (10) to have an infinite value at the focus point.

An interesting contrast is for a small sphere, of radius a, submerged in the water with its centre at a depth f and moving with velocity c. The expression is now

$$\zeta = 2\kappa^2 a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^4 \theta\, e^{-\kappa f \sec^2 \theta} \sin (x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the focus point O being vertically above the centre of the sphere.

In Fig. 4 are shown curves of the amplitude factor $\sec^4 \theta \exp. (-kf \sec^2 \theta)$ for different values of κf, that is, of gf/c^2.

From these curves we get at once some idea of the relative importance of the transverse

and diverging waves for different depths or for different speeds. We see that the effect of increasing depth, at the same speed, is to diminish relatively the diverging waves.

But these are perhaps details of purely theoretical interest, and we turn now to some cases of ship models.

Models of Great Draught.

7. We consider first a model of great draught, of uniform horizontal cross-section throughout and with parabolic lines; this is a model which has been investigated by Mr. W. C. S. Wigley, working at the William Froude Laboratory. Fig. 5 shows the horizontal section.

Taking the origin O at the mid-point, the equation of the curve A C B is $y = b(1 - x^2/l^2)$,

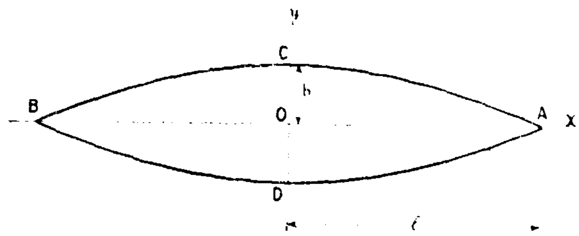

Fig. 5.

the beam being $2b$ and the length $2l$. It can be shown that, on the usual mathematical theory, the waves in the rear of the model approximate to

$$\zeta = -\frac{8b}{\pi\kappa^2 l^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\kappa l\cos(\kappa l\sec\theta) - \cos\theta\sin(\kappa l\sec\theta)\right\}\sin(x,y)\,d\theta \quad . \quad (12)$$

This might be regarded as a sine pattern with a somewhat complicated amplitude factor; but fortunately we can dissect it into simpler components, for it is identically equal to

$$\zeta = \frac{4b}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{c^2}{gl}\sin(x-l,y) + \frac{c^2}{gl}\sin(x+l,y)\right.$$

$$\left. + \frac{c^4}{g^2l^2}\cos\theta\cos(x-l,y) - \frac{c^4}{g^2l^2}\cos\theta\cos(x+l,y)\right\}d\theta \quad (13)$$

Here the pattern is seen to be the combined result of superposing four simple patterns, two focussed at the bow and two at the stern. The first two are simple sine patterns, with constant amplitude factors at a given speed; they may, in fact, be attributed directly to the finite angle of the model at the bow A and at the stern B respectively. The other two terms in (13) are cosine patterns, with an amplitude factor $\cos\theta$ in each case; although one is focussed at the bow and the other at the stern, it is more appropriate to regard these two terms together as representing the resultant effect of the curved sides A C B and A D B of the model.

A matter of great interest is the mutual interference of these four patterns according to the speed, the extent to which it is possible to make the crests of one pattern coincide with the troughs of another and the speeds at which maximum effects of this kind occur; however, these points are better considered in connection with the corresponding wave resistance.

Notice first the magnitudes of the terms in (13). The bow and stern systems are factored by $c^2/g\,l$, while the effect of the curved sides has the factor $c^4/g^2\,l^2$. Hence at low speeds the bow and stern provide the greater part of the wave system, but as the speed increases their relative importance becomes less. Then we have the effect of the amplitude factor $\cos\theta$ in the last two terms of (13). Remembering the distinction between transverse waves and diverging waves in a simple pattern, and that $\cos\theta$ diminishes from unity at 0° to about 0·8 at 35° and then to zero at 90°, we may describe the result in general terms: the effect of the gradual change of slope along the curved sides of the model compared with the finite angle at bow and stern is to diminish the relative importance of the diverging waves. This point is amplified further in the following model.

8. In this model one end, say the stern, is drawn out to a fine point. The model is again of great draught and is of uniform horizontal section throughout. Fig. 6 shows the form of the horizontal section.

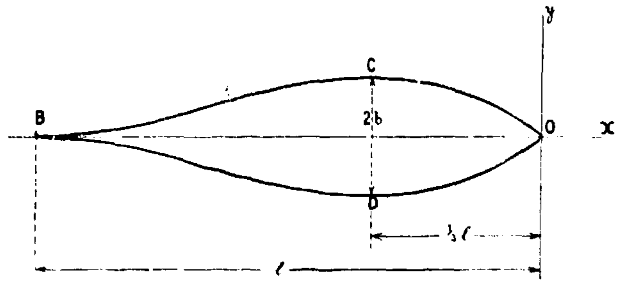

FIG. 6.

Taking the origin O at the bow, as in the diagram, the equation of the curved side O C B is

$$y = -\frac{27\,b}{4\,l^3}\,x\,(x+l)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

The maximum beam $2\,b$ occurs at one-third of the length from the bow.

The wave pattern in the rear is given, in our abbreviated notation, by

$$\zeta = \frac{27\,b}{2\,\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\{\frac{c^2}{g\,l}\sin(x,y) - 4\left(\frac{c^2}{g\,l}\right)^2\cos\theta\cos(x,y) - 6\left(\frac{c^2}{g\,l}\right)^3\cos^2\theta\sin(x,y)\right.$$
$$\left. - 2\left(\frac{c^2}{g\,l}\right)^2\cos\theta\cos(x+l,y) + 6\left(\frac{c^2}{g\,l}\right)^3\cos^2\theta\sin(x+l,y)\right\}d\,\theta \qquad (15)$$

Here we have five simple patterns, the first three focussed at the bow and the last two at the stern. The first term in (15) is the simple sine pattern due to the finite angle of the model at the bow; we notice there is no similar term for the stern because the angle has been smoothed away completely at the stern. The last four terms of (15) taken together represent the resultant effect of the curved sides O C B and O D B of the model. The general inferences are the same as for the previous model; but we notice that we have now, in (15), patterns with an amplitude factor $\cos^2\theta$, and for such the relative importance of the diverging waves is still further diminished.

EFFECT OF DRAUGHT.

9. Another point about which we may make some broad deductions from the formulæ for the wave patterns is the effect of the draught of the model. In the previous cases

we have supposed this to be very large, or theoretically infinite. Let us suppose now that the model is of uniform horizontal section down to a depth d below the surface and is then cut off by a horizontal plane. For our present descriptive purpose, we may make some simplifying assumptions in deducing the formulæ for the wave system, but these need not be investigated here; it is sufficient to state the general result. The effect of making the model of draught d, instead of infinite draught, is simply to introduce into each of the terms for the component patterns, in say (13) or (15), an additional amplitude factor, namely

$$1 - e^{-\kappa d \sec^2 \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Since $\kappa d = g d/c^2$, the value of this factor depends upon the speed. Fig. 7 shows curves of this quantity (16), for different values of κd, for the half range of values of θ from 0° to 90°.

From inspection of this diagram we see at once that, for a given speed, if the draught is diminished the transverse waves of the pattern become less important. We may put

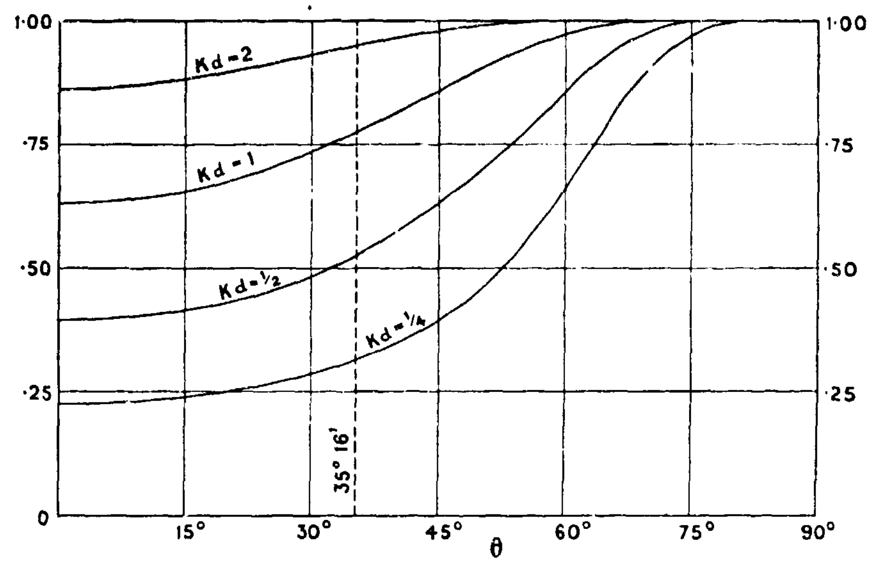

FIG. 7.—GRAPHS OF $1 - e^{-\kappa d \sec^2 \theta}$ FOR DIFFERENT VALUES OF κd.

alongside this a remark drawn from observation; for instance, in Taylor's *Speed and Power of Ships* there is the statement: "Narrow deep ships have wave patterns whose transverse features are more strongly accentuated than those of broad shallow ships."

This might, of course, be anticipated without any mathematical expressions. For the effect of a plane wave on the surface is only appreciable down to a depth of, say, half its wave-length. But of the component plane waves which combine to make the pattern, that which is travelling in the same direction as the ship has the greatest wave-length, and in inclined directions the wave-length is proportional to $\cos^2 \theta$ and diminishes to zero at 90°. Thus as we diminish the draught, for a given speed, the first components to be affected are those of longest wave-length and those are the components which provide the transverse waves of the pattern. However, the mathematical expressions enable us to obtain at least a rough quantitative estimate of the effect.

WAVE RESISTANCE.

10. We turn now to the calculation of wave resistance, and for this purpose it is essential to have a knowledge of the wave patterns we have been considering. Throughout

all this work we are assuming the liquid to be frictionless; or, rather, we suppose that frictional resistance and the effects of viscosity have been treated separately and so eliminated from the wave problem in order to make it more amenable to calculation. It is true that the most direct idea of wave resistance is to regard it as what it is in fact, namely, the combined backward resultant of the fluid pressures taken over the hull of the ship; but this is by no means the simplest method for purposes of calculation.

On the other hand, by a direct application of the method of energy and work, we shall see that we only need to know the wave pattern at a great distance in the rear of the ship.

Denote by S the position of the ship at any instant, by A and B two infinite vertical planes in given fixed positions at right angles to the direction of motion of the ship, the plane A being in advance of the ship and the plane B to the rear.

Consider the rate of increase of the energy of the fluid in the region between the surface of the ship and these two planes, and consider also the forces operating at the boundaries of this portion of fluid. The fluid possesses kinetic energy due to its motion and potential energy arising from alterations in the surface elevation. Calculate the rate at which total energy, kinetic and potential, is flowing into the region in question across the plane B and call this E (B). A similar calculation would give E (A) for the rate at which total energy is flowing out of this region across the plane A. At any point of the plane B let p be the fluid pressure and u the component fluid velocity inwards at right angles to this plane. The fluid to the left of B is doing work on the fluid to the right at a rate pu per unit area at each point of the plane; summing up for the whole plane, we call W (B) the rate at which work is being done on the fluid in question across the plane B. Similarly, $-$ W (A), calculated in the same way for the plane A, is the rate of work across that plane upon the fluid between the two planes. Finally, if R is the resultant resistance to the motion of the ship and c its velocity, the ship is doing work on the fluid at a rate R c. Hence, equating the total rate of work upon this portion of fluid to the rate of increase of its total energy, we deduce a general expression for R,

$$\mathrm{R}\,c = \mathrm{E}\,(\mathrm{B}) - \mathrm{W}\,(\mathrm{B}) - \{\mathrm{E}\,(\mathrm{A}) - \mathrm{W}\,(\mathrm{A})\} \quad . \quad . \quad . \quad . \quad (17)$$

This holds for any two fixed planes, one in advance of the ship and the other to the rear. If we take plane A further and further in advance, the quantities E (A) and W (A) approximate to zero, since the ship is advancing into still water. And if we take B further and further to the rear, the disturbance approximates to a free wave pattern such as we have considered in the previous sections and we can calculate the quantities E and W for any plane of that free wave pattern. Thus we have finally

$$\mathrm{R}\,c = \mathrm{E} - \mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where E and W are calculated from the free wave pattern to which the disturbance approximates at a great distance in the rear of the ship.

11. This method is familiar in its application to plane waves with straight parallel crests. It is probable that the first calculations of wave resistance were those made in this way for plane waves, the argument being usually expressed in terms of group velocity. For simple harmonic waves of height h the average total energy is $\frac{1}{2} g \rho h^2$ per unit area of surface thus the quantity E of (18) is $\frac{1}{2} g \rho h^2 c$ per unit length parallel to the crests. The quantity W is exactly one-half of this amount; or, as it is usually expressed, the group velocity is one-half the wave velocity. Hence from (18) we ha R $= \frac{1}{4} g \rho h^2$, where R is the wave resistance per unit length of the cylindrical body to whose motion the waves are due.

It is rather curious that this method has not been used for obtaining the wave resistance from the wave pattern produced by ordinary ship forms. The formulæ in use at present have been developed by other methods. In some cases they have been found from the

resultant fluid pressure on the ship. Another method is to introduce an artificial kind of fluid resistance, calculate the rate of dissipation of energy, and so ultimately arrive at expressions for the wave resistance. All these methods must lead to the same results if carried out correctly; but perhaps the most natural method is that outlined above and embodied in the general expression (18).

It has been shown recently that the necessary calculations can readily be extended to wave patterns of the general type which occur in ship waves.* The results may be given here, without going into the detailed analysis.

Suppose first that we have a free wave pattern given by

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta) \sin(x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and suppose that the amplitude factor $f(\theta)$ is an even function of θ, so that (19) is equivalent to

$$\zeta = 2\int_{0}^{\frac{\pi}{2}} f(\theta) \sin(\kappa\, x \sec\theta) \cos(\kappa\, y \sin\theta \sec^2\theta)\, d\theta \quad . \quad . \quad . \quad (20)$$

We can write down the velocity potential of the fluid motion for the wave form (19) and so obtain the pressure and velocity at any point of the fluid. The quantities E and W of (18) can then be calculated, with suitable limitations on the function $f(\theta)$ which amount to ensuring that E and W are in fact finite and calculable. Under these conditions it is found that E − W for the pattern (19) is given by a remarkably simple expression, namely,

$$\mathrm{E} - \mathrm{W} = \pi \rho c^3 \int_0^{\frac{\pi}{2}} \{f(\theta)\}^2 \cos^3\theta\, d\theta \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Hence the wave resistance of a body moving with velocity c and leaving in its rear a pattern (19) would be given by

$$\mathrm{R} = \pi \rho c^2 \int_0^{\frac{\pi}{2}} \{f(\theta)\}^2 \cos^3\theta\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

12. Suppose, for illustration, that the amplitude factor is independent of θ and that we have

$$\zeta = h \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

a simple sine pattern, with h possibly a function of the velocity c. This is certainly a hypothetical case: (23) is like the first term of (13) or (15), so presumably the sort of body which would produce this wave pattern would be the bow of a ship of great draught, but without any sides or stern. However, without inquiring any further into that, if the wave pattern is (23), then from (22) the corresponding wave resistance would be

$$\mathrm{R} = \pi \rho c^2 h^2 \int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta = \tfrac{2}{3}\pi \rho c^2 h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

* *Proc. Roy. Soc. A.*, 144, p. 514, 1934.

We might even carry this calculation a step further and divide the integration into two parts: (i) from $\theta = 0$ to $\theta = 35° 16'$, (ii) from $\theta = 35° 16'$ to $\theta = 90°$; and we might associate the first part of R so calculated with the transverse waves of the pattern, and the second part with the diverging waves. On that basis we easily find that for (23) the transverse waves account for about 77 per cent. of the wave resistance, and the diverging waves for the remaining 23 per cent.

The formula for R given in (22) was for a sine pattern (19), but the same expression holds for a similar cosine pattern. For instance, to compare with (23) we may take the case

$$\zeta = h \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cos(x, y)\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24a)$$

which is like a term of (13) or (15) giving the effect of the curved sides of the model. For this pattern the corresponding wave resistance is

$$R = \pi \rho c^2 h^2 \int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta = \tfrac{8}{15}\pi \rho c^2 h^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If we make a similar division into transverse waves and diverging waves we find that the former now account for a greater proportion of the total resistance, about 86 per cent. However, this is, no doubt, carrying the dissection too far; the wave pattern as a whole should be treated as a single system.

13. As an example of (22) we may consider the model with parabolic lines for which the wave pattern was given in the expression (12). We have at once the wave resistance given by

$$R = \frac{64\, b^2 \rho c^2}{\pi \kappa^4 l^4} \int_0^{\frac{\pi}{2}} \left\{\kappa l \cos(\kappa l \sec\theta) - \cos\theta \sin(\kappa l \sec\theta)\right\}^2 \cos^3\theta\, d\theta \quad . \quad (26)$$

On expanding this expression we have

$$R = \frac{32\, b^2 \rho c^2}{\pi \kappa^4 l^4} \int_0^{\frac{\pi}{2}} \{\kappa^2 l^2 + \cos^2\theta + \kappa^2 l^2 \cos(2\kappa l \sec\theta)$$
$$- 2\kappa l \cos\theta \sin(2\kappa l \sec\theta) + \cos^2\theta \cos(2\kappa l \sec\theta)\} \cos^3\theta\, d\theta \quad (27)$$

And this leads to

$$R = \frac{32 \rho b^2 c^2}{\pi} \left[\frac{2}{3}\left(\frac{c^2}{g l}\right)^2 + \frac{8}{15}\left(\frac{c^2}{g l}\right)^4 + \left(\frac{c^2}{g l}\right)^2 \int_0^{\frac{\pi}{2}} \cos^3\theta \cos(2\kappa l \sec\theta)\, d\theta\right.$$
$$\left. - 2\left(\frac{c^2}{g l}\right)^3 \int_0^{\frac{\pi}{2}} \cos^4\theta \sin(2\kappa l \sec\theta)\, d\theta - \left(\frac{c^2}{g l}\right)^4 \int_0^{\frac{\pi}{2}} \cos^5\theta \cos(2\kappa l \sec\theta)\, d\theta \right] \quad (28)$$

The result has been put into this form for direct comparison with the expression for the waves given in (13), where they are analysed into four simple patterns, one for the bow, one for the stern, and two for the combined effects of the curved sides of the model. From this, and the calculations of the previous section, we can now identify the origin of each of the terms in the expression (28). The first term is the resistance due to the bow and stern patterns as if each existed alone, while the second term is similarly due to the curved sides calculated separately. The last three terms of (28) have been left in the form

of integrals: these integrals have been tabulated for numerical work, but we are only considering here some general inferences. These three terms represent the mutual interference of the four simple patterns contained in (13), and it is obvious from the power of the factor $(c^2/g\,l)$ whence they arise. The first of these represents the interference of bow and stern patterns, the second the interference of bow or stern with entrance or run, and the last term the mutual interference of the two patterns from the curved sides or, as one may say, the interference of entrance and run. It is these last three terms in (28) which have oscillating values, and so give rise to the well-known humps and hollows on the curve of wave resistance.

14. We have seen that the wave pattern left behind by a ship can in general be put into the form given in (9); we have described this as sine and cosine patterns with known amplitude factors. The calculation of the quantity E — W can readily be extended to this general form and we obtain then the wave resistance for any general case.

We first put (9) into the equivalent form

$$\zeta = \int_0^{\frac{\pi}{2}} (F_1 \sin A \cos B + F_2 \cos A \sin B + F_3 \cos A \cos B + F_4 \sin A \sin B)\, d\theta \quad (29)$$

where $A = \kappa\, x \sec\theta$, $B = \kappa\, y \sin\theta \sec^2\theta$, and the F's are functions of θ, in general, and of the form of the ship and its speed. The calculation of R follows as in the simpler case of (19), and leads to the general result

$$R = \tfrac{1}{4}\,\pi\,\rho\,c^2 \int_0^{\frac{\pi}{2}} (F_1^2 + F_2^2 + F_3^2 + F_4^2) \cos^3\theta\, d\theta \quad . \quad . \quad . \quad (30)$$

The determination of the functions F is, of course, another matter. Approximate methods in use at present amount to replacing the ship by some equivalent distribution of sources and sinks; the functions F then usually appear as integrals taken over the surface of the ship, or over its longitudinal section for a first approximation. One of the outstanding problems of ship wave resistance is the improvement of methods for determining these functions; the line of attack open at present would seem to be by further steps of mathematical and numerical approximation, assisted and corrected by comparison with experimental results.

The object of the present paper was to recall some of the elementary properties of wave patterns and their production by the mutual interference of simple plane waves, to illustrate these by examples from ship models, and further to emphasize the direct connection between the wave pattern and the wave resistance.

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The Calculation of Wave Resistance.

By T. H. Havelock, F.R.S.

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1. The wave resistance of a body moving in a frictionless liquid has been calculated by various methods. In a few cases it has been found directly as the resultant of the fluid pressures on the surface of the body. Another method, which has been more generally useful, involves the introduction of a certain type of fluid friction into the equations of motion. The wave resistance is then found by calculating the rate of dissipation of energy and taking the limiting value when the frictional coefficient is made vanishingly small. This method has certain important analytical advantages, nevertheless it is highly artificial. A third method, dealing directly with a frictionless liquid, consists in examining the flow of energy in the wave motion; this has hitherto been used only for two-dimensional problems when the wave motion consists of simple waves with straight parallel crests, the usual theory of group velocity being directly applicable.

In the following note this method is extended to three-dimensional fluid motion. Although no new special results are obtained so far as expressions for wave resistance are concerned, it seemed of sufficient interest to obtain them by this direct method, namely, by considering the flow of energy and the rate of work across planes far in advance and far in the rear of the moving body.

These quantities are examined first for a free wave pattern of simple type. Then a general expression is given for wave resistance in terms of the velocity potential of the free wave pattern to which the disturbance approximates at a great distance in the rear, and this is applied to a general form of wave pattern and to some special cases. Finally, a similar examination is made of a certain problem when the water is of finite depth.

2. With the origin O in the free surface of deep water, and Oz vertically upwards, the surface condition is

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \zeta}{\partial t} = 0, \qquad z = 0, \tag{1}$$

where ϕ is the velocity potential and ζ the surface elevation. For a wave

pattern advancing steadily with velocity c in the direction Ox, we may write (1) in the form

$$\frac{\partial^2 \phi}{\partial x^2} + \kappa_0 \frac{\partial \phi}{\partial z} = 0, \qquad z = 0, \tag{2}$$

with $\kappa_0 = g/c^2$.

A simple plane wave advancing in a direction making an angle θ with Ox is given by

$$\left.\begin{aligned} \zeta &= a \sin\{\kappa_0 \sec^2\theta\,(x\cos\theta + y\sin\theta - ct\cos\theta)\} \\ \phi &= ac\cos\theta\, e^{\kappa_0 z \sec^2\theta} \cos\{\kappa_0 \sec^2\theta\,(x\cos\theta + y\sin\theta - ct\cos\theta)\} \end{aligned}\right\} . \tag{3}$$

We may generalize this to obtain a free wave pattern made up of plane waves advancing in all directions, so that the pattern itself moves steadily with velocity c in the direction Ox; we have then

$$\zeta = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\theta) \sin\{\kappa_0 \sec^2\theta\,(x\cos\theta + y\sin\theta - ct\cos\theta)\}\, d\theta. \tag{4}$$

We shall suppose in the first place that the pattern is symmetrical with respect to Ox, so that we have

$$\left.\begin{aligned} \zeta &= 2\int_0^{\frac{1}{2}\pi} f(\theta)\sin(\kappa_0 x' \sec\theta)\cos(\kappa_0 y\sin\theta\sec^2\theta)\, d\theta \\ \phi &= 2c\int_0^{\frac{1}{2}\pi} f(\theta)\, e^{\kappa_0 z\sec^2\theta}\cos(\kappa_0 x'\sec\theta)\cos(\kappa_0 y\sin\theta\sec^2\theta)\cos\theta\, d\theta \end{aligned}\right\} , \tag{5}$$

with $x' = x - ct$.

Consider a fixed vertical plane $x =$ constant. The rate of flow of total energy across this plane is given by

$$\tfrac{1}{2}\rho c \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left\{ \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right\} dy + \tfrac{1}{2} g\rho c \int_{-\infty}^{\infty} \zeta^2\, dy. \tag{6}$$

The variable part of the fluid pressure being $\rho\partial\phi/\partial t$, or $-\rho c\partial\phi/\partial x$, the rate at which work is being done across the same plane is

$$\rho c \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left(\frac{\partial \phi}{\partial x}\right)^2 dy. \tag{7}$$

We shall assume that the wave pattern is such that these quantities are finite and determinate.

To evaluate these expressions with the values (5) for ϕ and ζ we use the following theorem:

If

$$\left.\begin{aligned} F_1(y) &= \int_0^\infty (A_1 \cos yu + B_1 \sin yu)\, du, \\ F_2(y) &= \int_0^\infty (A_2 \cos yu + B_2 \sin yu)\, du, \end{aligned}\right\} \tag{8}$$

A, B, being functions of u, then

$$\int_{-\infty}^{\infty} F_1(y) F_2(y)\, dy = \pi \int_0^\infty (A_1A_2 + B_1B_2)\, du, \tag{9}$$

assuming that the integrals are convergent.

To take one of the integrals in (6) as an example, we have

$$\frac{\partial\phi}{\partial x} = -2\kappa_0 c \int_0^{\frac{1}{2}\pi} f(\theta)\, e^{\kappa_0 z \sec^2\theta} \sin(\kappa_0 x' \sec\theta) \cos(\kappa_0 y \sin\theta \sec^2\theta)\, d\theta. \tag{10}$$

To put this into the form (8), we write $u = \kappa_0 \sin\theta \sec^2\theta$, then carry out the process (9) and finally replace the variable u in terms of θ; it is clear that we shall have to introduce into the integral in the final form a factor $d\theta/du$; that is, a factor $\cos^3\theta/\kappa_0(1 + \sin^2\theta)$. Thus we have

$$\begin{aligned} &\int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left(\frac{\partial\phi}{\partial x}\right)^2 dy \\ &\quad = 4\pi\kappa_0 c^2 \int_{-\infty}^{0} dz \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 e^{2\kappa_0 z \sec^2\theta} \sin^2(\kappa_0 x' \sec\theta)\, \frac{\cos^3\theta\, d\theta}{1 + \sin^2\theta} \\ &\quad = 2\pi c^2 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \sin^2(\kappa_0 x' \sec\theta)\, \frac{\cos^5\theta\, d\theta}{1 + \sin^2\theta}. \end{aligned} \tag{11}$$

From (6) we find in this way that the rate of flow of total energy across the vertical plane is

$$\begin{aligned} \pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \{(3 - \sin^2\theta) \sin^2(\kappa_0 x' \sec\theta) \\ + (1 + \sin^2\theta) \cos^2(\kappa_0 x' \sec\theta)\} \frac{\cos^3\theta\, d\theta}{1 + \sin^2\theta}, \end{aligned} \tag{12}$$

and that the rate at which work is being done across this plane is

$$2\pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \sin^2(\kappa_0 x' \sec\theta)\, \frac{\cos^5\theta\, d\theta}{1 + \sin^2\theta}. \tag{13}$$

It is the difference of these two quantities that is significant for our purpose; it is, as would be expected, independent of the time and of the position of the

plane. Subtracting (13) from (12), we find that the rate at which energy is being propagated less the rate of work reduces to the simple expression

$$\pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \cos^3\theta \, d\theta. \tag{14}$$

It may be noted that if we take mean values of (12) and (13) we have as the mean rate of flow of energy

$$2\pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \frac{\cos^3\theta}{1+\sin^2\theta} \, d\theta, \tag{15}$$

and as the mean rate of work

$$\pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \frac{\cos^5\theta}{1+\sin^2\theta} \, d\theta. \tag{16}$$

The connection indicated in (15) and (16) is a generalization of the well-known result for simple plane waves that the mean rate of work is half the mean rate of flow of energy.

3. Consider now the forced wave pattern produced by a body moving through the liquid, or by a localized pressure disturbance. The complete surface elevation may be separated into a local disturbance and a wave pattern. In a frictionless liquid a possible solution is one in which the wave pattern extends to an infinite distance in advance of the body as well as in the rear. The determinate practical solution is that for which the wave pattern vanishes at a great distance in advance, and we may suppose this obtained by superposing over the whole surface a suitable free wave pattern. In that case, considering the flow of energy and rate of work across two fixed vertical planes, one far in advance and the other far in the rear, we see that (14) is equal to Rc, where R is the wave resistance. Hence we have

$$R = \pi\rho c^3 \int_0^{\frac{1}{2}\pi} \{f(\theta)\}^2 \cos^3\theta \, d\theta, \tag{17}$$

when the wave pattern at a great distance to the rear approximates to the form (4).

For example, the forced wave pattern produced by a submerged sphere, or more precisely by a horizontal doublet of moment M at depth f, approximates at a great distance behind the disturbance, to the free wave pattern

$$\zeta = \frac{4\kappa_0^2 M}{c} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sec^4\theta \, e^{-\kappa_0 f \sec^2\theta} \sin\{\kappa_0 (x'\cos\theta + y\sin\theta)\sec^2\theta\} \, d\theta. \tag{18}$$

Hence, from (17), the wave resistance is

$$R = 16\pi\rho\kappa_0{}^4M^2 \int_0^{\frac{1}{2}\pi} \sec^5\theta\, e^{-2\kappa_0 f \sec^2\theta}\, d\theta, \tag{19}$$

which is the known result for this case.

4. Before generalizing these results we may put (6) and (7) into an explicit form for the wave resistance.

The kinetic energy of the liquid in a strip between two parallel vertical planes at a distance δx apart is

$$\tfrac{1}{2}\rho\,\delta x \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dy. \tag{23}$$

Transform (23) into the equivalent form of a surface integral over the boundaries of this portion of fluid, assuming the wave pattern to be such that the various integrals are convergent. Thus we obtain the rate of flow of kinetic energy across a vertical plane as

$$\tfrac{1}{2}\rho c \int_{-\infty}^{\infty} \left(\phi \frac{\partial\phi}{\partial z}\right)_{z=0} dy + \tfrac{1}{2}\rho c \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left\{\phi \frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial\phi}{\partial x}\right)^2\right\} dy. \tag{24}$$

Further, we may transform the other terms in (6) and (7) by using the surface condition (2) together with $g\zeta = -c\partial\phi/\partial x$ at $z = 0$.

Finally, equating the difference between (6) and (7) to Rc, we obtain for the wave resistance

$$R = \frac{\rho}{2\kappa_0} \int_{-\infty}^{\infty} \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 - \phi \frac{\partial^2\phi}{\partial x^2}\right\}_{z=0} dy - \tfrac{1}{2}\rho \int_{-\infty}^{0} dz \int_{-\infty}^{\infty} \left\{\left(\frac{\partial\phi}{\partial x}\right)^2 - \phi \frac{\partial^2\phi}{\partial x^2}\right\} dy. \tag{25}$$

In this expression ϕ is the velocity potential of the free wave pattern to which the disturbance approximates at a great distance in the rear. Considering the disturbance produced by a body of any form, it appears that this free wave pattern must be expressible, in general, in the form

$$\begin{aligned}\zeta = &\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\theta) \sin\{\kappa_0 \sec^2\theta\,(x'\cos\theta + y\sin\theta)\} d\theta \\ &+ \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} F(\theta) \cos\{\kappa_0 \sec^2\theta\,(x'\cos\theta + y\sin\theta)\} d\theta,\end{aligned} \tag{26}$$

that is, in the form

$$\zeta = \int_0^{\frac{1}{2}\pi} (P_1 \sin A \cos B + P_2 \cos A \sin B + P_3 \cos A \cos B + P_4 \sin A \sin B)\, d\theta, \tag{27}$$

where $A = \kappa_0 x' \sec\theta$, $B = \kappa_0 y \sin\theta \sec^2\theta$.

The corresponding velocity potential is

$$\phi = c\int_0^{\frac{1}{2}\pi} (\mathrm{P}_1 \cos \mathrm{A} \cos \mathrm{B} - \mathrm{P}_2 \sin \mathrm{A} \sin \mathrm{B} - \mathrm{P}_3 \sin \mathrm{A} \cos \mathrm{B} + \mathrm{P}_4 \cos \mathrm{A} \sin \mathrm{B})\, e^{\kappa_0 z \sec^2\theta} \cos\theta\, d\theta. \qquad (28)$$

With this value of ϕ in (25), we use (8) and (9) to evaluate the integrations with respect to y as in § 2 ; and we obtain readily the general result

$$\mathrm{R} = \tfrac{1}{4}\pi\rho c^2 \int_0^{\frac{1}{2}\pi} (\mathrm{P}_1^2 + \mathrm{P}_2^2 + \mathrm{P}_3^2 + \mathrm{P}_4^2) \cos^3\theta\, d\theta. \qquad (29)$$

The actual calculation of the quantities P for a body of given form is, of course, another problem. Methods in use at present amount to replacing the body by some approximately equivalent system of sources and sinks ; the functions P then appear, in general, in the form of integrals taken over the surface of the body. We need not consider these here as the expressions for R given above lead to the same results as those obtained previously by different methods.

5. It is of interest to examine a similar problem when the water is of finite depth h. It is clear from the derivation of (25) that we may use it in this case also, taking the lower limit of integration with respect to z to be $-h$ instead of $-\infty$.

For the simple symmetrical type of free wave pattern given by (4), the corresponding velocity potential is

$$\phi = 2c\int_0^{\frac{1}{2}\pi} (\theta)\, \frac{\cosh \kappa (z + h)}{\sinh \kappa h} \cos(\kappa x' \cos\theta) \cos(\kappa y \sin\theta) \cos\theta\, d\theta, \qquad (30)$$

the relation between κ and θ being

$$\kappa - \kappa_0 \sec^2\theta \tanh \kappa h = 0. \qquad (31)$$

We shall assume first $\kappa_0 h > 1$, that is $c^2 < gh$, so that (31) as an equation for κ has one real root for each value of θ in the range of integration. In evaluating (25) we carry out the integrations with respect to y by means of (8) and (9). For this we have to change from an integration in θ to one in a variable u given by

$$u = \kappa \sin\theta, \qquad (32)$$

together with (31). The corresponding factor $d\theta/du$ has now the value

$$\frac{\cos^3\theta\,(\coth \kappa h - \kappa h \operatorname{cosech}^2 \kappa h)}{\kappa_0 (1 + \sin^2\theta - \kappa_0 h \operatorname{sech}^2 \kappa h)}. \qquad (33)$$

We have, for example,

$$\int_{-h}^{0} dz \int_{-\infty}^{\infty} \left(\frac{\partial\phi}{\partial x}\right)^2 dy$$

$$= 4\pi c^2 \int_{-h}^{0} dz \int_{0}^{\frac{1}{2}\pi} \{f(\theta)\}^2 \frac{\cosh^2 \kappa (z+h)(\coth \kappa h - \kappa h \operatorname{cosech}^2 \kappa h)}{\kappa_0 \sinh^2 \kappa h (1 + \sin^2\theta - \kappa_0 h \operatorname{sech}^2 \kappa h)}$$

$$\times \sin^2(\kappa x' \cos\theta)\, \kappa^2 \cos^7\theta \, d\theta$$

$$= 2\pi c^2 \int_{0}^{\frac{1}{2}\pi} \{f(\theta)\}^2 \frac{\coth^2 \kappa h - \kappa^2 h^2 \operatorname{cosech}^4 \kappa h}{\kappa_0 (1 + \sin^2\theta - \kappa_0 h \operatorname{sech}^2 \kappa h)} \sin^2(\kappa x' \cos\theta)\, \kappa \cos^7\theta \, d\theta. \tag{34}$$

Evaluating the remaining terms in (25), we obtain after a little reduction the result

$$R = \pi\rho c^2 \int_{0}^{\frac{1}{2}\pi} \{f(\theta)\}^2 (\coth \kappa h - \kappa h \operatorname{cosech}^2 \kappa h) \cos^3\theta \, d\theta, \tag{35}$$

with κ given in terms of θ by (31).

This may be compared with (17) for the similar wave pattern in deep water.

For a horizontal doublet M at depth f in water of depth h, an expression for the complete surface elevation can be derived from results given previously.* We have

$$\zeta = \frac{M}{\pi c} \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{\cosh \kappa (h-f)\, e^{i\kappa (x'\cos\theta + y \sin\theta)}}{\cosh \kappa h\,(\kappa - \kappa_0 \sec^2\theta \tanh \kappa h + i\mu \sec\theta)} \kappa^2 \, d\kappa, \tag{36}$$

where we take the limiting value of the real part for $\mu \to 0$.

From this we may easily deduce the free wave pattern to which the disturbance approximates at a great distance in the rear. It is given by

$$\zeta = \frac{4\kappa_0{}^2 M}{c} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cosh \kappa (h-f) \tanh^2 \kappa h \sec^4\theta}{\cosh \kappa h\,(1 - \kappa_0 h \sec^2\theta \operatorname{sech}^2 \kappa h)} \sin\{\kappa (x'\cos\theta + y\sin\theta)\}\, d\theta. \tag{37}$$

From (35) this gives

$$R = 16\pi\rho\kappa_0 M^2 \int_{0}^{\frac{1}{2}\pi} \frac{\kappa^3 \cos\theta \cosh^2 \kappa (h-f)}{\cosh^2 \kappa h\,(1 - \kappa_0 h \sec^2\theta \operatorname{sech}^2 \kappa h)} d\theta. \tag{38}$$

It will be found that this agrees with the result obtained by a different method in the paper just quoted, when the previous expression is corrected for an obvious slip; in formula (37) of that paper 32 should be replaced by 16 and $\tanh \kappa h\,(1 + \tanh \kappa h)$ by $(1 + \tanh \kappa h)^2$.

*'Proc. Roy. Soc.,' A., vol. 118, p. 33 (1928). [This paper is No. 22 of this collection and the error mentioned above has been corrected.—Editor.]

When $\kappa_0 h < 1$, that is $c^2 > gh$, the equation (31) for κ has a real root only for a more limited range of values of θ, the lower limit being $\theta_0 = \cos^{-1}\sqrt{(\kappa_0 h)}$ instead of zero. It is readily seen that the expression for R will be as in (38) with θ_0 as the lower limit of the integral.

Summary.

An examination is made of the transfer of energy in a free wave pattern, and expressions for wave resistance are deduced. These are applied to certain cases both for deep water and for water of finite depth.

Ship Waves: The Relative Efficiency of Bow and Stern

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1. It seems fairly certain that one of the main causes of differences between theoretical and experimental results is the neglect of fluid friction in the calculation of ship waves, and further that the influence of fluid friction may be regarded chiefly as one which makes the rear portion of the ship less effective in generating waves than the front portion. The process may be pictured, possibly, in terms of a friction belt or boundary layer whose more important effect is equivalent to smoothing the lines of the model towards the rear. Some calculations were made from this point of view in a previous paper,* the purpose then being to show how such an asymmetry, fore and aft, reduced the magnitude of interference effects between bow and stern waves. We may also describe the frictional effect as a diminution in the effective relative velocity of the model and the surrounding water as we pass from bow to stern. This is not very satisfactory from a theoretical point of view; but, on the other hand, it leads to a comparatively simple modification of expressions for the waves produced by the model. From a formal point of view, we may regard the modification as an empirical introduction of a reducing factor to allow for decrease in efficiency of the elements of the ship's surface as we pass from bow to stern.

There are now available experimental results, for wave profiles as well as for wave resistance, which make it possible to attempt such a comparison. The following work is limited to a few simple cases, and the assumptions are made in as simple a form as possible for the purpose of the calculations; these deal with the wave profile and wave resistance of a model of symmetrical form, and also with the difference between motion bow first and motion stern first for a simple asymmetrical model.

2. Take the origin O in the undisturbed free surface of the water, with Ox horizontal and Oz vertically upwards; and let the origin O be moving with uniform velocity c in the direction Ox. We suppose that there is a given distribution of sources and sinks over the zx-plane, or, alternatively, that the normal fluid velocity is given over this plane; let it be F (h, f) at

* 'Proc. Roy. Soc.,' A, vol. 110, p. 233 (1926).

the point $(h, 0, -f)$. Then the surface elevation ζ due to this travelling distribution is given by

$$\zeta = -\frac{i}{2\pi^2 c}\int_{-\infty}^{\infty}\int_{0}^{\infty} F(h,f)\,dh\,df \int_{-\pi}^{\pi} \sec\theta\,d\theta \int_{0}^{\infty} \frac{\kappa e^{-\kappa f + i\kappa\varpi}\,d\kappa}{\kappa - \kappa_0 \sec^2\theta + i\mu\sec\theta}, \quad (1)$$

where $\varpi = (x - h)\cos\theta + y\sin\theta$, and the limiting value is to be taken as the positive quantity μ tends to zero.

If the form of the ship is given by y as a function of h and f, the usual approximation is to take $F(h, f)$ as equal to $c\,\partial y/\partial h$. We modify this now by supposing that the effective value of c in this expression for $F(h, f)$ diminishes from bow to stern; we introduce what may be called a reducing factor $f(h)$, so that we shall use in (1)

$$F(h, f) = cf(h)\frac{\partial y}{\partial h}. \quad (2)$$

We have assumed that the reducing factor is independent of the depth. It will, no doubt, depend upon the velocity and form of the model, and in particular upon the value of the Reynolds number; but, meantime, we shall neglect any such considerations. It may even be that, in certain circumstances, the factor should allow for an increase of apparent efficiency near the bow of the model. However, it appears from such experimental evidence as is available that the wave profile near the bow agrees fairly well, for simple models, with calculations made without any allowance for frictional effects; so that the chief effect of the latter appears to be a reduction in efficiency over the rear portion of the model. In view of these considerations, and also to lighten the numerical calculations, very simple expressions have been used in the following work. Calculations are made for two cases, and in both we assume the reducing factor to be constant and less than unity over the rear portion; in one case the factor is taken as constant and equal to unity over the front portion, while in the other, to avoid possible discontinuities, it is assumed to diminish uniformly from the bow to the value which it has for the rear portion.

We shall consider only models of great draught and of uniform horizontal section; for such, (1) and (2) give for the surface elevation

$$\zeta = -\frac{i}{2\pi^2}\int_{-\infty}^{\infty} f(h)\frac{\partial y}{\partial h}\,dh \int_{-\pi}^{\pi} \sec\theta\,d\theta \int_{0}^{\infty} \frac{e^{i\kappa\varpi}\,d\kappa}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}. \quad (3)$$

3. We consider a model of length $2l$ and beam $2b$, and of symmetrical parabolic lines given by

$$y = b(1 - h^2/l^2). \quad (4)$$

The reduction factor $f(h)$ is to mean a diminution of effective velocity from the value c at the bow to a smaller value βc at the stern. In order to allow the calculations to be made in terms of known functions, we shall suppose the diminution to take place uniformly over the front half of the model; thus we assume

$$\begin{aligned} f(h) &= \beta + (1-\beta)\,h/l, \qquad 0 < h < l \\ &= \beta, \qquad -l < h < 0. \end{aligned} \tag{5}$$

Using (5) and (4) in (3) and carrying out the integration with respect to h, we obtain

$$\zeta = \frac{b}{\pi^2 l^3}\int_{-\pi}^{\pi}\sec^2\theta\,d\theta\int_0^{\infty}\frac{A\,d\kappa}{\kappa^3(\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta)}, \tag{6}$$

where

$$\begin{aligned} A = \{2(1-\beta)\sec^2\theta + (2-\beta)\,i\kappa l\sec\theta - \kappa^2 l^2\}\,e^{i\kappa[(x-l)\cos\theta+y\sin\theta]} \\ - 2(1-\beta)\sec^2\theta\,e^{i\kappa(x\cos\theta+y\sin\theta)} \\ - (i\beta\kappa l\sec\theta + \beta\kappa^2 l^2)\,e^{i\kappa[(x+l)\cos\theta+y\sin\theta]}. \end{aligned} \tag{7}$$

This expression gives finite and continuous values for the surface elevation. It is convenient, for purposes of calculation, to separate it into finite and continuous expressions associated respectively with the bow ($x = l$), amidships ($x = 0$), and the stern ($x = -l$). Further, for points on the central line $y = 0$, we can express these in terms of known functions.

Writing

$$G(q) = i\int_{-\pi}^{\pi}\sec\theta\,d\theta\int_0^{\infty}\frac{e^{i\kappa q\cos\theta}\,d\kappa}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}, \tag{8}$$

$$G_0(q) = \int_0^q G(q)\,dq, \qquad G_1(q) = \int_0^q G_0(q)\,dq, \tag{9}$$

and so on, it can readily be shown that (6) gives, for the wave profile along $y = 0$,

$$\begin{aligned} \zeta = -\frac{b}{\pi^2 l^3}\{l^2 G_0(x-l) + (2-\beta)\,l\,G_1(x-l) + 2(1-\beta)\,G_2(x-l) \\ - 2(1-\beta)\,G_2(x) + \beta l^2 G_0(x+l) - \beta l\,G_1(x+l)\}. \end{aligned} \tag{10}$$

In the limit, when we take μ zero, we have*

$$\begin{aligned} G(q) &= \pi^2\{H_0(\kappa_0 q) - Y_0(\kappa_0 q)\}, \quad q > 0 \\ &= -\pi^2\{H_0(\kappa_0 q) - Y_0(\kappa_0 q)\} - 4\pi^2 Y_0(\kappa_0 q), \quad q < 0. \end{aligned} \tag{11}$$

* 'Proc. Roy. Soc.,' A, vol. 135, p. 5 (1932).

In the notation used in previous work, we have

$$\left.\begin{aligned} Q_0(u) &= \frac{\pi}{2}\int_0^u \{H_0(u) - Y_0(u)\}\,du, \\ Q_1(u) &= \int_0^u Q_0(u)\,du, \qquad Q_2(u) = \int_0^u Q_1(u)\,du, \\ P_0(u) &= -\frac{\pi}{2}\int_0^u Y_0(u)\,du, \\ P_0^{-1}(u) &= \int_0^u P_0(u)\,du = 1 + P_1(u), \\ P_0^{-2}(u) &= \int_0^u P_0^{-1}(u)\,du = u + P_2(u). \end{aligned}\right\} \tag{12}$$

Summing up these results, we obtain finally for the wave profile

$$\zeta = -\frac{2b}{\pi}\left\{\frac{1}{p}F_0(\kappa_0 q_1) + \frac{2-\beta}{p^2}F_1(\kappa_0 q_1) + \frac{2(1-\beta)}{p^3}F_2(\kappa_0 q_1) - \frac{2(1-\beta)}{p^3}F_2(\kappa_0 q_2) + \frac{\beta}{p}F_0(\kappa_0 q_3) - \frac{\beta}{p^2}F_1(\kappa_0 q_3)\right\}, \tag{13}$$

with $\kappa_0 = g/c^2$, $p = \kappa_0 l$, $q_1 = x - l$, $q_2 = x$, $q_3 = x + l$. Also we have

$$\left.\begin{aligned} F_0(u) &= Q_0(u), \qquad u > 0 \\ &= Q_0(-u) - 4P_0(-u), \qquad u < 0, \\ F_1(u) &= Q_1(u), \qquad u > 0 \\ &= -Q_1(-u) + 4P_0^{-1}(-u), \qquad u < 0, \\ F_2(u) &= Q_2(u), \qquad u > 0 \\ &= Q_2(-u) - 4P_0^{-2}(-u), \qquad u < 0. \end{aligned}\right\} \tag{14}$$

Using tables and graphs of the various P and Q functions, the wave profile can now be found, for any speed, for any assigned value of β. We have chosen the value $\beta = 0\cdot 6$, and calculations have been made for a sufficient number of values of x to give the wave profile for two different speeds; the speeds are those for which $\kappa_0 l = 6$ and $\kappa_0 l = 3$, or for $c/\sqrt{(gl)}$ equal to $0\cdot 408$ and $0\cdot 577$ respectively. The wave profile has also been calculated at these speeds for the value $\beta = 1$, that is for the usual theory without any allowance for frictional effects. The four curves are shown in fig. 1, the full curves being for $\beta = 1$ and the dotted curves for $\beta = 0\cdot 6$.

These curves may be compared with some given recently by Wigley* in a comparison of experimental and calculated wave profiles.

* 'Proc. Roy. Soc.,' A, vol. 144, p. 144 (1934).

In fig. 2 of that paper, the full curves are calculated from the usual theory, that is, for $\beta = 1$ in the notation of the present paper; while the dotted curves are observed values. It may be concluded that the value $\beta = 0 \cdot 6$ is of the right order of magnitude to bring the calculated values into better agreement with observed values, at least for medium values of $c/\sqrt{(gl)}$. It should be noted that Wigley's model is slightly different from that of the present calculation, in that it has a certain amount of parallel middle body inserted between the parabolic ends.

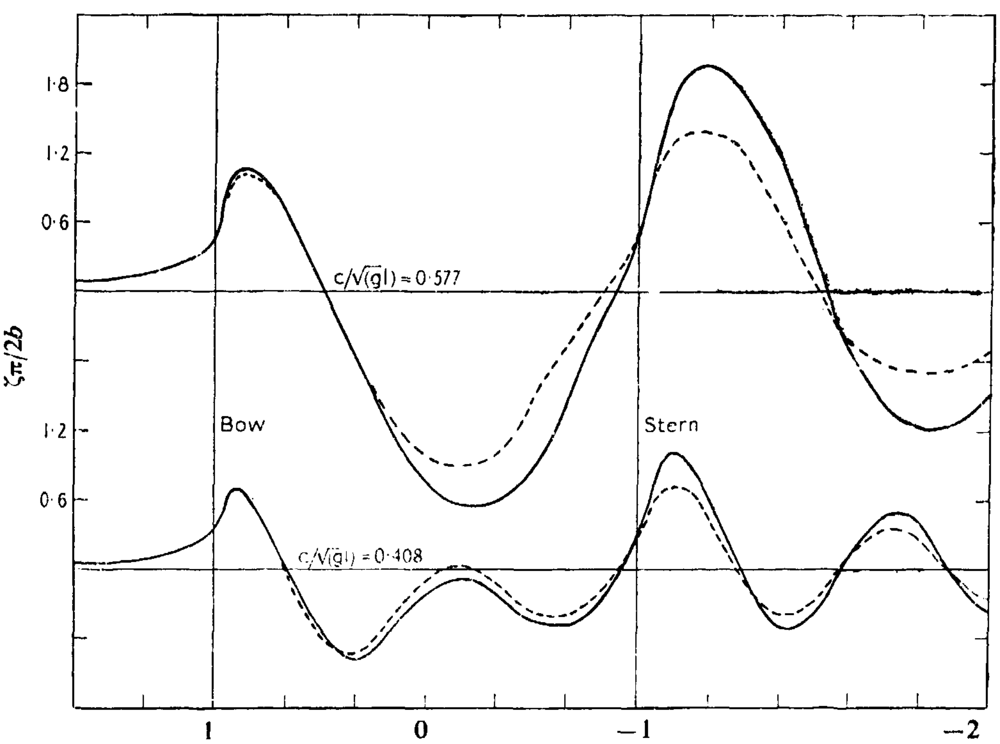

FIG. 1

For that, and other obvious reasons, it is not worth while attempting any closer comparison of the results meantime.

4. It is of interest to examine the corresponding change in the wave resistance for this model. It can easily be deduced from (6) that the wave pattern at a great distance to the rear of the model approximates to

$$\zeta = -\frac{4b}{\pi p^3}\int_{-\pi/2}^{\pi/2}\{p^2 \sin(x-l, y) - (2-\beta)\, p \cos\theta \cos(x-l, y)$$
$$- 2(1-\beta)\cos^2\theta \sin(x-l, y) + 2(1-\beta)\cos^2\theta \sin(x, y)$$
$$+ \beta p^2 \sin(x+l, y) + \beta p \cos\theta \cos(x+l, y)\}\, d\theta, \qquad (15)$$

where we have used the abbreviation

$$(q, y) = \kappa_0 \sec^2 \theta \, (q \cos \theta + y \sin \theta).$$

In (15) the wave pattern is analysed into simple constituents associated with the bow, amidships, and the stern; putting the expression into the form

$$\zeta = \int_0^{\pi/2} \{A_1 \sin (\kappa_0 x \sec \theta) + A_2 \cos (\kappa_0 x \sec \theta)\} \cos (\kappa_0 y \sin \theta \sec^2 \theta) \, d\theta, \tag{16}$$

the wave resistance is given by*

$$R = \tfrac{1}{4} \rho \pi c^2 \int_0^{\pi/2} (A_1^2 + A_2^2) \cos^3 \theta \, d\theta. \tag{17}$$

Carrying out the reduction we obtain

$$R = \frac{16\rho b^2 c^2}{\pi} \left\{ \frac{2(1+\beta^2)}{3p^2} + \frac{16\beta^2}{15p^4} + \frac{128(1-\beta)^2}{35p^6} + \frac{2\beta}{p^2} P_3(2p) + \frac{2\beta(\beta-3)}{p^3} P_4(2p) - \frac{2\beta(3\beta-4)}{p^4} P_5(2p) - \frac{4\beta(1-\beta)}{p^5} P_6(2p) - \frac{4(1-\beta^2)}{p^4} P_5(p) + \frac{8(1-\beta)}{p^5} P_6(p) - \frac{8(1-\beta)^2}{p^6} P_7(p) \right\}. \tag{18}$$

In terms of P functions which have been tabulated this becomes, for the particular case $\beta = 0 \cdot 6$,

$$R = \frac{16\rho b^2 c^2}{\pi} \left\{ \frac{2 \cdot 72}{3p^2} + \frac{0 \cdot 384}{p^4} + \frac{20 \cdot 48}{35p^6} + \left(\frac{1 \cdot 2}{p^2} - \frac{0 \cdot 32}{p^4} \right) P_3(2p) - \left(\frac{2 \cdot 88}{p^3} - \frac{0 \cdot 8}{p^5} \right) P_4(2p) + \frac{2 \cdot 32}{p^4} P_5(2p) + \frac{3 \cdot 52}{7p^4} P_3(p) - \frac{18 \cdot 88}{7p^5} P_4(p) - \left(\frac{14 \cdot 4}{7p^4} - \frac{7 \cdot 68}{7p^6} \right) P_5(p) \right\}. \tag{19}$$

This is to be compared with the value for the same model without any reducing factor, that is, with (18) when $\beta = 1$, or

$$R = \frac{16\rho b^2 c^2}{\pi} \left\{ \frac{4}{3p^2} + \frac{16}{15p^4} + \frac{2}{p^2} P_3(2p) - \frac{4}{p^3} P_4(2p) + \frac{2}{p^4} P_5(2p) \right\}. \tag{20}$$

The curves are given in fig. 2, and show the variation of R/c^2 with the quantity $c/\sqrt{(gl)}$; in addition to the smaller value of the resistance from (19) compared with (20), tnere is also a relative decrease in interference effects.

* 'Proc. Roy. Soc.,' A, vol. 144, p. 519 (1934).

5. The wave resistance of a ship model in a frictionless liquid is the same whether it is moving bow first or stern first, even when the model is not symmetrical fore and aft. If, however, we introduce a reducing factor to represent the effect of fluid friction, it is clear that we shall obtain a difference between the two cases, and it is also easy to foresee the general character of the result. Suppose that the bow is finer than the stern, and assume that the reducing factor is the same whether going ahead or astern. Then it is obvious that the resistance will be less when going bow first than when going stern first; and further, that interference effects between bow and stern waves will be relatively more marked in the former case than in the latter.

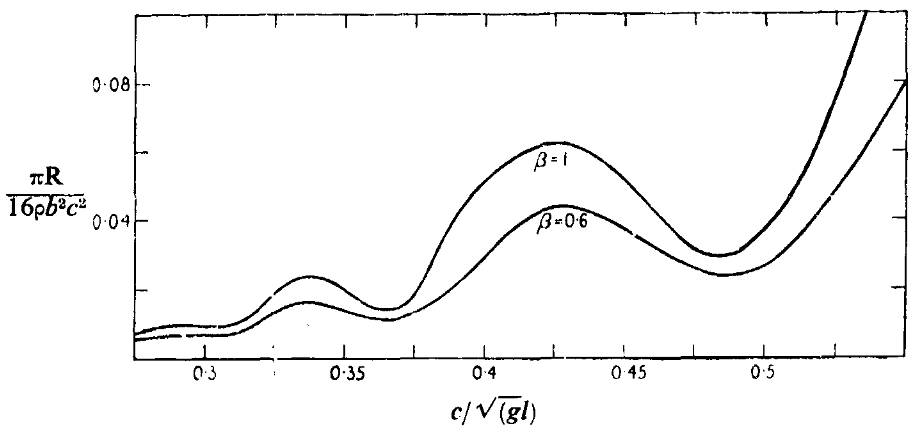

FIG. 2

We shall now work out a particular case, a model of great draught with parabolic ends and with some parallel middle body. The lines of the horizontal section are given by

$$\begin{aligned} y &= b\,(1 - h^2/l^2), & 0 < h < l \\ &= b, & -\tfrac{1}{2}l < h < 0 \\ &= -\frac{4b}{l^2}(h^2 + lh), & -l < h < -\tfrac{1}{2}l. \end{aligned} \tag{21}$$

In this model the change of gradient at the stern is twice that at the bow.

In order to simplify the calculations, we shall assume that the reducing factor is constant and equal to unity over the front half of the model, and has a constant value β over the rear half; there will be only a small difference between the results so obtained and those with a more natural form of reducing factor, because in any case the middle portion of this

model does not contribute much to the wave-making. We shall not examine the wave profile in this case. For the wave resistance we have

$$\mathrm{R} = (4\rho/\pi) \int_0^{\pi/2} (\mathrm{A}^2 + \mathrm{B}^2) \cos\theta \, d\theta, \tag{22}$$

where

$$\begin{aligned} \mathrm{A} - i\mathrm{B} = &- \frac{4\beta bc}{l^2} \int_{-l}^{-\frac{1}{2}l} (2h + l)\, e^{-i\kappa_0 h \sec\theta}\, dh \\ &- \frac{2bc}{l} \int_0^l h e^{-i\kappa_0 h \sec\theta}\, dh \end{aligned} \tag{23}$$

This leads to the result

$$\begin{aligned} \mathrm{R} = \frac{16\rho b^2 c^2}{\pi} \Big\{ &\frac{2(1 + 4\beta^2)}{3p^2} + \frac{16(1 + 16\beta^2)}{15p^4} + \frac{4\beta}{p^2} \mathrm{P}_3(2p) \\ &- \frac{12\beta}{p^3} \mathrm{P}_4(2p) + \frac{8\beta}{p^4} \mathrm{P}_5(2p) + \frac{8\beta}{p^3} \mathrm{P}_4(\tfrac{3}{2}p) - \frac{8\beta}{p^4} \mathrm{P}_5(\tfrac{3}{2}p) \\ &- \frac{2(1 - 2\beta)}{p^3} \mathrm{P}_4(p) + \frac{2(1 - 4\beta)}{p^4} \mathrm{P}_5(p) - \frac{16\beta^2}{p^3} \mathrm{P}_4(\tfrac{1}{2}p) \\ &+ \frac{8\beta(1 + 4\beta)}{p^4} \mathrm{P}_5(\tfrac{1}{2}p) \Big\}. \end{aligned} \tag{24}$$

This expression may be written as

$$\mathrm{R} = \mathrm{R}_0 + \beta \mathrm{R}_1 + \beta^2 \mathrm{R}_2. \tag{25}$$

The form (25), with β a positive quantity less than unity, applies to the model when going bow first. It is easily seen that the corresponding result for motion stern first, assuming the same reduction factor β, is

$$\mathrm{R} = \beta^2 \mathrm{R}_0 + \beta \mathrm{R}_1 + \mathrm{R}_2. \tag{26}$$

Numerical calculations have been made from these expressions for $\beta = 0 \cdot 6$, and from these curves have been drawn showing the variation of R/c^2 with speed, on a base of $c/\sqrt{(gl)}$; these are given in fig. 3.

The curve A in fig. 3 is for motion bow first, the curve B for motion stern first. The curve C is for (24) with $\beta = 1$, that is, it is the resistance curve for motion in either direction when no allowance is made for frictional effects. There are few experimental data available for comparison; but in any case it should be noted that, apart from other simplifying assumptions, the preceding calculations are for a model of very great draught. However, reference should be made to some experimental curves given by Wigley;* in fig. 3 of his paper there are three resistance

* 'Trans. Inst. Nav. Arch.,' vol. 72, p. 216 (1930).

curves which correspond to curves A, B and C of fig. 3 below, and the mutual relations of the three curves in the two cases have much in common.

6. In the preceding work, the reducing factor has been given specially simple forms in order that the calculations might be made in terms of functions which have already been tabulated. However, for the wave resistance of a model of ordinary form and draught, the calculations are usually made by numerical and graphical methods for the particular case; the introduction of a reducing factor of suitable form would not

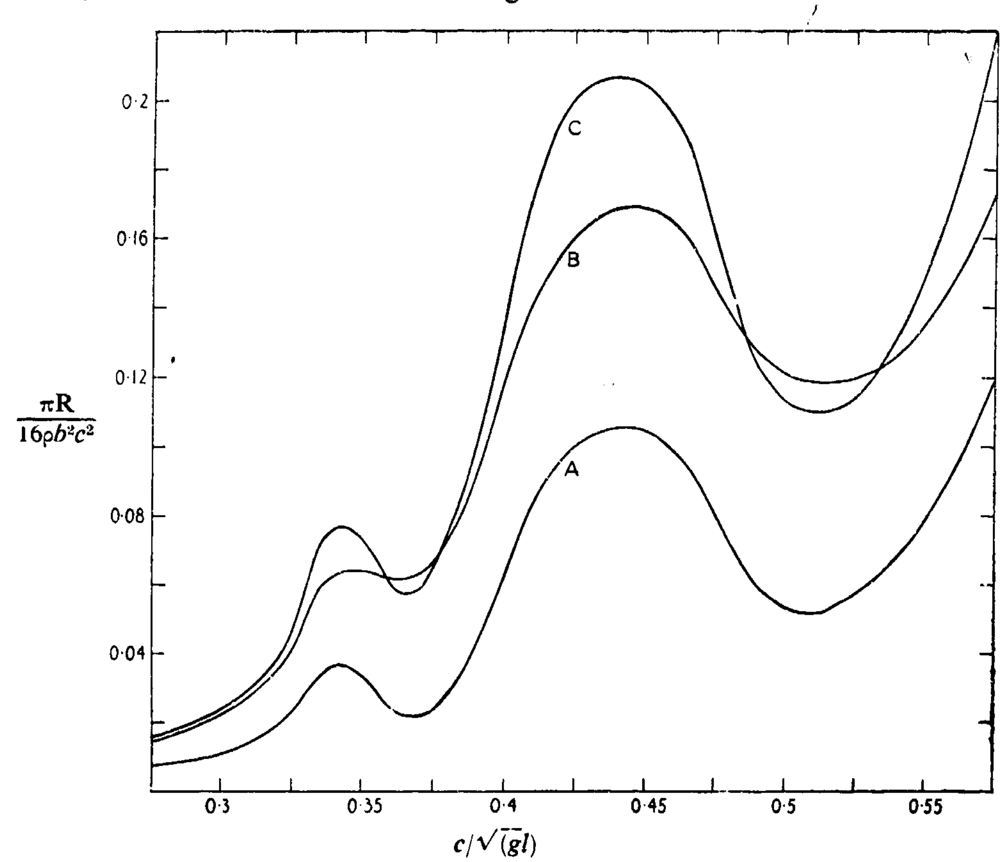

FIG. 3

add any great complication. The usefulness of such a factor would depend largely upon whether it proved to be sufficiently independent of speed and of variation of form of the model.

SUMMARY

The main effect of fluid friction in regard to the production of waves by a ship may be described as a decreasing efficiency of elements of the ship's surface with increasing distance from the bow. A reducing factor,

of a semi-empirical nature, is introduced into the theory of ship waves to represent this effect. With certain assumptions, calculations are made for the wave profile for a simple model, and curves are also given; these are compared with available experimental data. It appears that, as a a rough estimate for such forms at moderate speeds, the efficiency of the stern is of the order of 60% of that of the bow. Curves are also drawn to show the corresponding change in the wave resistance. The introduction of the reducing factor leads to different wave resistances for a model going ahead and going astern, when the model is not symmetrical fore and aft; this is illustrated by calculations and curves for a particular case.

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Wave Resistance: the Mutual Action of Two Bodies

By T. H. HAVELOCK, F.R.S.

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1—Methods of calculating wave resistance which depend upon energy considerations are appropriate for a single body or a single system for which we require the total resistance. There are, however, certain problems in which there are two or more bodies and we wish to calculate the resistance of each separately, or more generally the resultant force on each body in any required direction. For instance, the effect of the walls of a tank upon the resistance of a model might be calculated from the resistance of one model among a series of models abreast of each other. Another problem is suggested by experiments made by Barrillon.* Two or more models were towed in various relative positions and the resistances measured separately; the results for a model in the waves produced by other models in advance of it were considered to show interference effects due to both the transverse and the diverging waves from the leading models. Without attempting to deal with these actual problems at present, the following paper contains a method of calculating wave resistance which seems suitable for the purpose. It depends upon obtaining the force on a body as the resultant of certain forces on the sources and sinks to which it is equivalent hydrodynamically. A general discussion is given first and then a simple case is worked out in some detail ; this may be described as two equal small spheres at the same depth, first with one directly behind the other, then with the two abreast of each other, and finally in any given relative positions.

2—Consider a solid body held at rest in a liquid in steady irrotational motion. We shall suppose the motion to be due to a uniform stream together with given sources and sinks in the region outside the body, and we suppose the effect of the body to be equivalent to a certain distribution of sources and sinks within the surface of the body; the latter may be called the internal sources. It is known that the resultant forces and couples on the body may be calculated from forces on the internal sources due to attractions or repulsions between the external and internal sources taken in pairs; the fictitious force between two sources m, m' is $4\pi\rho mm'/r^2$ and is an attraction when m and m' are of like sign. Another way of expressing this theorem is that if m is a typical internal source, the

* 'C.R. Acad. Sci. Paris,' vol. 182, p. 46 (1926).

force on it may be taken as the vector $-4\pi\rho m q$, where q is the resultant fluid velocity at that point due to all the other sources, in which the remaining internal sources may be included as their actions and reactions do not affect the final result.

It is true that for a solid of given form an important and difficult part of the problem is the complete determination of the internal sources so as to satisfy all the required conditions. However, assuming this has been done, we can proceed to calculate the resultant forces. Further, in certain problems results of some value may be obtained by using distributions of internal sources which satisfy the conditions approximately.

3—Take the origin O in the free surface of deep water which is streaming with uniform velocity c in the negative direction of Ox, and take Oz vertically upwards. Let there be a source of strength m in the fluid at the point $(0, 0, -f)$. The velocity potential of the fluid motion is given* by

$$\phi = cx + \frac{m}{r_1} - \frac{m}{r_2} - \frac{\kappa_0 m}{\pi}\int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{-\kappa(f-z)+i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa, \quad (1)$$

where the limit is to be taken as the positive quantity μ tends to zero, and

$$r_1^2 = x^2 + y^2 + (z+f)^2; \qquad r_2^2 = x^2 + y^2 + (z-f)^2;$$
$$\varpi = x\cos\theta + y\sin\theta; \qquad \kappa_0 = g/c^2.$$

The second term on the right of (1) is the given source, the third term represents an equal sink at the image point above the free surface, while the last term could be interpreted as a certain continuous distribution of sources lying in the plane $z - f = 0$. The expression (1) may be generalized by summation and integration for the velocity potential of any given distribution of sources in the liquid. We shall assume that this distribution is such that it represents a solid body in the stream, the total source strength being therefore zero.

4—Consider in the first place a continuous distribution over a finite part of the vertical plane $y = 0$, the surface density of source strength being σ at a point $(h, 0, -f)$. The velocity potential is

$$\phi = cx + \iint\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sigma\, dh\, df$$
$$- \frac{\kappa_0}{\pi}\iint \sigma\, dh\, df \int_{-\pi}^{\pi} \sec^2\theta\, d\theta \int_0^\infty \frac{e^{-\kappa(f-z)+i\kappa\varpi}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa, \quad (2)$$

* Havelock, 'Proc. Roy. Soc.,' A, vol. 138, p. 340 (1932).

where

$$r_1^2 = (x-h)^2 + y^2 + (z+f)^2$$
$$r_2^2 = (x-h)^2 + y^2 + (z-f)^2$$
$$\varpi = (x-h)\cos\theta + y\sin\theta.$$

Using the theorem given in § 2, we may write down the total wave resistance for the body which is represented by the given distribution. It is given by

$$\mathrm{R} = 4\pi\rho \iint \sigma(h', f')\, u\, dh'\, df', \tag{3}$$

taken over the distribution, u being the x-component of fluid velocity at the point $(h', 0, -f')$.

Consider the contribution of the various terms in (2) to the value of u in (3). We may omit the uniform stream c since the total source strength is zero, and also any contribution from the internal sources. Further, it is easily seen that there is no total horizontal force on the internal sources due to the image system represented in (2) by the term involving r_2. Thus the only part of u which gives any integrated effect from (3) comes from the x-derivative of the last term in (2). Thus we obtain the expression

$$\mathrm{R} = 4\kappa_0\rho i \iint \sigma'\, dh'\, df' \iint \sigma\, dh\, df \int_{-\pi}^{\pi} \sec\theta\, d\theta \times \int_0^{8} \frac{e^{-\kappa(f+f')+i\kappa(h'-h)\cos\theta}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\,\kappa\, d\kappa. \tag{4}$$

The integrations in θ and κ may be written as

$$2\int_0^{\pi/2} \sec\theta\, d\theta \int_0^{\infty} \left\{ \frac{e^{i\kappa(h'-h)\cos\theta}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta} - \frac{e^{-i\kappa(h'-h)\cos\theta}}{\kappa - \kappa_0\sec^2\theta - i\mu\sec\theta} \right\} e^{-\kappa(f+f')}\,\kappa\, d\kappa. \tag{5}$$

Regarding κ as a complex variable we may transform the integrals by taking as contour an appropriate quadrant bounded by the positive half of the real axis and the positive or negative half of the imaginary axis according to the sign of $h' - h$. Reducing the expressions and finally putting μ zero, the integral with respect to κ in (5) is equivalent to

$$-2i\int_0^{\infty} \frac{\kappa_0\sec^2\theta\sin m(f+f') - m\cos m(f+f')}{m^2 + \kappa_0^2\sec^4\theta}\, e^{-m(h'-h)\cos\theta}\, m\, dm,$$
$$\text{for } h' - h > 0, \tag{6}$$

and

$$2i\int_0^\infty \frac{\kappa_0 \sec^2\theta \sin m(f+f') - m\cos m(f+f')}{m^2+\kappa_0^2\sec^4\theta} e^{m(h'-h)\cos\theta} m\,dm$$
$$- 4\pi\kappa_0\, i\sec^2\theta e^{-\kappa_0(f+f')\sec^2\theta}\cos\{\kappa_0(h'-h)\sec\theta\},$$
$$\text{for} \quad h'-h<0, \quad (7)$$

the function defined by (6) and (7) being continuous at $h'-h=0$. Writing $-i\mathrm{F}(h', f', h, f, \theta)$ for this function, we have

$$\mathrm{R} = 8\kappa_0\rho\iint \sigma'\,dh'\,df'\iint \sigma\,dh\,df\int_0^{\pi/2}\mathrm{F}(h', f', h, f, \theta)\sec\theta\,d\theta. \quad (8)$$

It is obvious from (6) and (7) that the part of F from the integrals in m will give zero result when integrated twice over the distribution; and we are left with

$$\mathrm{R} = 32\pi\kappa_0^2\rho\iint \sigma'\,dh'\,df'\iint \sigma\,dh\,df$$
$$\times\int_0^{\pi/2}\sec^3\theta e^{-\kappa_0(f+f')\sec^2\theta}\cos\{\kappa_0(h'-h)\sec\theta\}\,d\theta, \quad (9)$$

with $h'-h<0$, the integrand being zero for $h'-h>0$.

This is the wave resistance expressed in a form which brings out more clearly than the usual forms the fact that the solution we require is one in which the regular waves trail aft from each element of the distribution.

It is easily seen that the limitation $h'-h<0$ in (9) is equivalent to taking one-half the result of the repeated integration over the distribution without this limitation. Hence we obtain the result

$$\mathrm{R} = 16\pi\kappa_0^2\rho\int_0^{\pi/2}(\mathrm{P}^2+\mathrm{Q}^2)\sec^3\theta\,d\theta,$$

with

$$\mathrm{P}+i\mathrm{Q} = \iint \sigma e^{i\kappa_0 h\sec\theta - \kappa_0 f\sec^2\theta}\,dh\,df. \quad (10)$$

This agrees with the general result obtained from energy considerations in the paper already quoted, where the distribution was not necessarily confined to the plane $y=0$. There is no difficulty in extending the present method to more general cases, but that is left over until occasion arises for applying the results to some particular problem.

5—To proceed to the case of two bodies, it is only necessary to suppose that the distribution of sources is divisible into two parts, each contained

within a distinct closed surface. For convenience, we shall limit the discussion to a distribution in the vertical plane $y = 0$. We suppose that the total distribution σ of the previous section can be divided into two distributions σ_1, σ_2, each representing a solid body and one being aft of the other. The resistance for either body is given by the same general expression (3), the integration being taken over the corresponding partial distribution. For instance, for the resistance R_1 of the body σ_1, the velocity u at any element of σ_1 will be that due to the rest of σ_1 and to σ_2, and the integration is to be taken over σ_1. The velocity potential is given by (2) with $\sigma = \sigma_1 + \sigma_2$. It is convenient to regard (2) as made up from the following parts: the uniform stream c, the given distributions σ_1 and σ_2, distributions $-\sigma_1$ and $-\sigma_2$ over image positions above the free surface, and finally a part represented by a certain integral taken over the distribution $\sigma_1 + \sigma_2$.

Consider the contributions of these parts to the value of R_1. The uniform stream gives no resultant effect as we suppose the integrated source strength of σ_1 to be zero. We have now a resultant force from the mutual actions between σ_1 and σ_2, given numerically by

$$4\pi\rho \iint \sigma_1\, dh_1\, df_1 \iint \sigma_2\, dh_2\, df_2 \frac{h_1 - h_2}{\{(h_1 - h_2)^2 + (f_1 - f_2)^2\}^{3/2}}, \tag{11}$$

the sign depending upon whether σ_1 is in advance of σ_2 or to the rear of it. It may be noted in passing that this corresponds to the apparent repulsion between two bodies, one behind the other, in a uniform stream of infinite extent. There is also a similar resultant due to the actions between $-\sigma_2$ and σ_1, given numerically by

$$4\pi\rho \iint \sigma_1\, dh_1\, df_1 \iint \sigma_2\, dh_2\, df_2 \frac{h_1 - h_2}{\{(h_1 - h_2)^2 + (f_1 + f_2)^2\}^{3/2}}. \tag{12}$$

Finally we have the part due to the last term in (2) for the velocity potential, and this will be given in the notation of (8) by

$$8\kappa_0\rho \iint \sigma'_1\, dh'_1\, df'_1 \iint (\sigma_1\, dh_1\, df_1 + \sigma_2\, dh_2\, df_2) \int_0^{\pi/2} \mathrm{F} \sec\theta\, d\theta, \tag{13}$$

where F is given by (6) and (7).

The terms in F represented by the integrals in m will give a resultant effect different from zero when summed over the partial distribution σ_1, arising from the part due to σ_2 when summed over σ_1. From the term in (7) representing the regular waves, the part due to σ_1 when summed over σ_1 will give the wave resistance of σ_1 as if existing alone; the part due to

σ_2 will give no effect over σ_1 if σ_2 is aft of σ_1, but the full interference effect of the two systems will be added to R_1 if σ_1 is aft of σ_2.

Summing up this general discussion, we see that the total resistance of each system consists of various parts: the resistance of each as if existing alone, mutual actions between the two systems which are equal and opposite and may be classed as due to local disturbances, and wave interference acting on that system which is to the rear of the other. It may be noted again that in this analysis we are assuming the source distributions to be given. It has been shown how the various terms in the resistances can be calculated when the two systems are in one and the same vertical plane. A similar analysis could be made for more general cases; but we shall consider in some detail a simple distribution consisting of two isolated doublets.

6—Suppose that there are two equal horizontal doublets A, B each of moment M in the liquid at the points $(0, 0, -f)$ and $(-l, 0, -f)$ respectively; thus A is directly in advance of B. If the points are sufficiently far apart, the corresponding bodies would be, approximately, spheres each of radius b given by $M = \frac{1}{2}b^3c$. However, all we shall assume meantime is that the doublets are far enough apart to represent two distinct bodies, one enclosing each doublet, whatever their actual shapes may be.

The velocity potential is given by

$$\phi = cx + \phi_A + \phi_B, \tag{14}$$

where

$$\phi_A = \frac{Mx}{r_1^3} - \frac{Mx}{r_2^3} + \frac{i\kappa_0 M}{\pi}\int_{-\pi}^{\pi} \sec\theta\, d\theta \int_0^\infty \frac{e^{-\kappa(f-z)+i\kappa(x\cos\theta+y\sin\theta)}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\,\kappa\, d\kappa, \tag{15}$$

and ϕ_B is a similar expression with $x + l$ instead of x, the notation being the same as in (1).

The form which (3) takes for an isolated doublet is

$$R = -4\pi\rho M\frac{\partial^2\phi}{\partial x^2}, \tag{16}$$

where in $\partial^2\phi/\partial x^2$ we must omit the term in ϕ due to the doublet at the point in question. Thus we may calculate the resistances R_A and R_B separately. In the process we have to evaluate the expression

$$-\lim_{\mu\to 0} i\int_{-\pi}^{\pi} \cos\theta\, d\theta \int_0^\infty \frac{e^{-2\kappa f+i\kappa x\cos\theta}}{\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta}\,\kappa^3\, d\kappa. \tag{17}$$

By the method already described, this is transformed into

$$4\int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\infty}\frac{\kappa_0\sec^2\theta\sin 2mf - m\cos 2mf}{m^2+\kappa_0^2\sec^4\theta}\,e^{-mx\cos\theta}\,m^3\,dm, \qquad \text{for } x>0, \quad (18)$$

$$-4\int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\infty}\frac{\kappa_0\sec^2\theta\sin 2mf - m\cos 2mf}{m^2+\kappa_0^2\sec^4\theta}\,e^{mx\cos\theta}\,m^3\,dm$$
$$-8\pi\kappa_0^3\int_0^{\pi/2}\sec^5\theta e^{-2\kappa_0 f\sec^2\theta}\cos(\kappa_0 x\sec\theta)\,d\theta, \qquad \text{for } x<0, \quad (19)$$

the two expressions having the same limiting value as x tends to zero.

Writing R_0 for the resistance of either doublet if existing alone, and using these expressions in (16), we get at once the known result

$$R_0 = 16\pi\rho\kappa_0^4 M^2\int_0^{\pi/2}\sec^5\theta e^{-2\kappa_0 f\sec^2\theta}\,d\theta. \qquad (20)$$

Considering R_A, the contribution from the doublet M at B and the image doublet $-M$ is easily found to be $-R'$ where

$$R' = 24\pi\rho M^2\left\{\frac{1}{l^4}-\frac{l(l^2-6f^2)}{(l^2+4f^2)^{7/2}}\right\}. \qquad (21)$$

It may be noted that if we put $M=\frac{1}{2}b^3c$, the first term in (21) gives $6\pi\rho b^6 c^2/l^4$, which is the usual approximate value of the repulsion between two equal spheres moving in the line of their centres in an infinite liquid. Finally, for R_A, there is the term which comes from (18) and (16); we denote this by $-R''$, with

$$R'' = 16\kappa_0\rho M^2\int_0^{\pi/2}\cos\theta\,d\theta\int_0^{\infty}\frac{\kappa_0\sec^2\theta\sin 2mf - m\cos 2mf}{m^2+\kappa_0^2\sec^4\theta}\,e^{-ml\cos\theta}\,m^3\,dm. \qquad (22)$$

If we calculate R_B, remembering that A is in advance of B, the forces R' and R'' are reversed, and we have in addition the effect of the second term in (19). We obtain finally

$$R_A = R_0 - R' - R'' \qquad (23)$$

$$R_B = R_0 + R' + R'' + 32\pi\rho\kappa_0^4 M^2\int_0^{\pi/2}\sec^5\theta e^{-2\kappa_0 f\sec^2\theta}\cos(\kappa_0 l\sec\theta)\,d\theta. \qquad (24)$$

The sum of R_A and R_B is the result which would be given by energy methods for the two parts regarded as one system. In (23) and (24) we have the separate resistances with the wave-interference part assigned

to the rear system. In addition we have the terms R′ and R″, which may be regarded as a local action and reaction, their magnitudes diminishing rapidly with increasing distance. It may be noted that with M proportional to the velocity c, R′ increases as the square of the velocity; this may be associated with the fact that, although the regular wave system diminishes to zero ultimately with increasing velocity, there is a permanent local surface elevation.

7—Suppose now that the two doublets are abreast of each other at a distance $2k$ apart, that is, suppose equal doublets A and B at the points $(0, 0, -f)$ and $(0, 2k, -f)$ respectively. The velocity potential is

$$\phi = cx + \phi_A + \phi_B, \tag{25}$$

with ϕ_A given by (15), and ϕ_B by a similar expression with $y - 2k$ instead of y.

We have

$$R_A = -4\pi\rho M\left(\frac{\partial^2\phi_A}{\partial x^2} + \frac{\partial^2\phi_B}{\partial x^2}\right), \tag{26}$$

evaluated at the point A and omitting from ϕ_A the term representing the doublet at A.

It is clear from the symmetry of the arrangement, that the local terms give no effect; reducing the remaining terms we obtain the result

$$R_A = R_0 + 16\pi\rho\kappa_0{}^4M^2\int_0^{\pi/2}\sec^5\theta e^{-2\kappa_0 f\sec^2\theta}\cos(2\kappa_0 k\sin\theta\sec^2\theta)\,d\theta, \tag{27}$$

with R_0 given by (20).

Taking $M = \frac{1}{2}b^3c$, we may regard this as the resistance of a small sphere at depth f in a stream and at a distance k from a vertical wall parallel to the stream; it is of some interest to estimate the influence of the wall upon the resistance. R_0 has been expressed previously in terms of Bessel functions; it is given by (using the notation of Watson's Treatise on Bessel Functions)

$$R_0 = \frac{\pi\rho g^4b^6}{c^6}e^{-\alpha}\left\{K_0(\alpha) + \left(1 + \frac{1}{2\alpha}\right)K_1(\alpha)\right\}, \tag{28}$$

with $\alpha = \kappa_0 f = gf/c^2$.

The integral in (27) is equal to $\frac{1}{4}\partial^2X/\partial\alpha^2$, where, with $\beta = \kappa_0 k$,

$$X = \int_0^{\pi/2}\sec\theta e^{-2\alpha\sec^2\theta}\cos(2\beta\sin\theta\sec^2\theta)\,d\theta$$

$$= \int_0^{\infty}e^{-2\alpha\cosh^2 u}\cos(\beta\sinh 2u)\,du = \tfrac{1}{2}e^{-\alpha}K_0(\sqrt{\alpha^2+\beta^2}). \tag{29}$$

Using these results in (27), we obtain

$$R = \frac{\pi\rho g^4 b^6}{c^6} e^{-\alpha}\left[K_0(\alpha) + \left(1 + \frac{1}{2\alpha}\right)K_1(\alpha) + \frac{2\alpha^2 + \beta^2}{2(\alpha^2 + \beta^2)} K_0(\sqrt{\alpha^2 + \beta^2}) + \left\{\frac{\alpha}{(\alpha^2 + \beta^2)^{1/2}} + \frac{\frac{1}{2}(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^{3/2}}\right\} K_1(\sqrt{\alpha^2 + \beta^2})\right], \tag{30}$$

with $\alpha = gf/c^2$, $\beta = gk/c^2$.

Values have been calculated from this, using tables of Bessel functions, and graphs are shown in fig. 1.

The ordinates are values of $Rf^3/\pi g\rho b^6$, while the abscissae are those of $c/\sqrt{(gf)}$. The curves are for different values of the ratio k/f; the curve

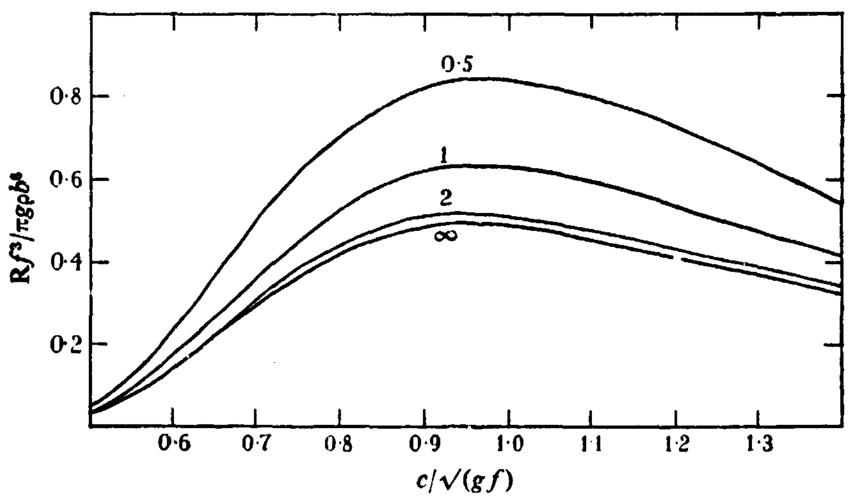

FIG. 1.

for $k/f = \infty$ is the graph of R_0, the resistance in an infinite stream, and the other curves show the increase of resistance with increasing nearness of the wall.

8—It may be remarked that the present method of calculating the forces gives not only the wave resistance, which is the resultant force in the line of motion, but can also be used to give the resultant force in any direction; for instance, in (3) it is only necessary to replace u by the component velocity in the required direction.

For the problem treated in § 7, the force on A in the direction towards the wall is given by

$$Y = 4\pi\rho M\left(\frac{\partial^2\phi_A}{\partial x\,\partial y} + \frac{\partial^2\phi_B}{\partial x\,\partial y}\right). \tag{31}$$

Carrying out the calculation as before, we find

$$Y = \tfrac{3}{4}\pi\rho M\left\{\frac{1}{k^4} - \frac{k}{(h^2+f^2)^{5/2}}\right\}$$
$$-16\kappa_0\rho M^2\int_0^{\pi/2}\sin\theta\, d\theta\int_0^{\infty}\frac{\kappa_0\sec^2\theta\sin 2mf - m\cos 2mf}{m^2+\kappa_0^2\sec^4\theta}e^{-2mk\sin\theta}m^3\, dm$$
$$-16\pi\rho\kappa_0^4M^2\int_0^{\pi/2}\sin\theta\sec^6\theta e^{-2\kappa_0 f\sec^2\theta}\cos(2\kappa_0 k\sin\theta\sec^2\theta)\, d\theta. \quad (32)$$

Here again with $M = \frac{1}{2}b^3c$, the first term is the usual approximation for the attraction between two spheres moving abreast in an infinite liquid at a distance $2k$ apart.

9—When one sphere is directly behind the other, the oscillating part of its resistance is due to the transverse waves in the pattern made by the leading sphere. When the two spheres are abreast of each other, there are no similar oscillating terms. We shall now consider the more general case of any relative positions, when in suitable circumstances we can distinguish between interference due to transverse waves and diverging waves.

With the same notation, we take the doublets A and B to be at the points $(0, 0, -f)$ and $(-l, k, -f)$ respectively; thus, with l and k positive, B is a distance l to the rear of A and a distance k to one side. The velocity potential is

$$\phi = cx + \phi_A + \phi_B, \quad (33)$$

with ϕ_A given by (15), and ϕ_B a similar expression with $x + l$ instead of x and $y - k$ instead of y. Each resistance is given by the expression in (26), evaluated at A or B in the manner already explained, and the calculation of the various terms follows the same lines.

For the term corresponding to R′ in (21) we now obtain

$$R' = 12\pi\rho M^2 l\left\{\frac{2l^2-3k^2}{(l^2+k^2)^{7/2}} - \frac{2l^2-3k^2-12f^2}{(l^2+k^2+4f^2)^{7/2}}\right\}. \quad (34)$$

The remaining terms are more complicated than in the previous simpler cases; for their contribution to R_A we have to evaluate an expression

$$i\int_{-\pi}^{\pi}\cos\theta\, d\theta\int_0^{\infty}\frac{e^{i\kappa(l\cos\theta - k\sin\theta)}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}e^{-2\kappa f}\kappa^3\, d\kappa. \quad (35)$$

We first reduce the integration in θ to the range 0 to $\frac{1}{2}\pi$. Then the various integrals in κ are transformed by contour integration, the form of the results depending upon the sign of $l\cos\theta - k\sin\theta$; this involves

dividing the integration in θ into the ranges 0 to α and α to $\frac{1}{2}\pi$, where $\tan\alpha = l/k$. Reducing the various expressions we find that the part corresponding to R″ in (22) is now given by

$$R'' = 8\kappa_0\rho M^2 \int_{-\alpha}^{\pi-\alpha} \cos\theta \, d\theta \int_0^\infty \frac{\kappa_0 \sec^2\theta \sin 2mf - m\cos 2mf}{m^2 + \kappa_0^2 \sec^4\theta} \times e^{-m(l\cos\theta + k\sin\theta)} m^3 \, dm. \quad (36)$$

The remaining terms give contributions to both R_A and R_B. It is found that the complete results for the two resistances can be put into the form

$$R_A = R_0 - R' - R'' + 16\pi\rho\kappa_0^4 M^2 \int_{-\pi/2}^{-\alpha} \sec^5\theta e^{-2\kappa_0 f \sec^2\theta} \cos\{\kappa_0 \sec^2\theta (l\cos\theta + k\sin\theta)\} \, d\theta, \quad (37)$$

$$R_B = R_0 + R' + R'' + 16\pi\rho\kappa_0^4 M^2 \int_{-\alpha}^{\pi/2} \sec^5\theta e^{-2\kappa_0 f \sec^2\theta} \cos\{\kappa_0 \sec^2\theta (l\cos\theta + k\sin\theta)\} \, d\theta, \quad (38)$$

where R_0 is given by (20), R′ by (34), R″ by (36), and $\tan\alpha = l/k$.

The previous results for A and B in line, and for A and B abreast, are particular cases of these expressions with $\alpha = \frac{1}{2}\pi$ and $\alpha = 0$ respectively.

The sum of (37) and (38) could have been obtained from expressions given previously for the total resistance of A and B considered as one system. Perhaps the most interesting difference between R_A and R_B, compared with simpler cases, occurs in the last terms in (37) and (38). It might appear that both A and B experience effects of wave-interference, in the usual meaning of that term, although A is in advance of B. However, this is not so, and this can be seen most easily if we suppose $\kappa_0\sqrt{(l^2 + k^2)}$ to be large and apply the Kelvin method of approximation to the integrals in question. According to this, the important parts of the integral come from narrow ranges of θ in the neighbourhood of the stationary values of $l\sec\theta + k\sec\theta\tan\theta$, that is, near values of θ given by

$$\tan\theta = -\tfrac{1}{4}\tan\alpha \pm \tfrac{1}{4}\sqrt{(\tan^2\alpha - 8)}. \quad (39)$$

Such values only exist if $\tan^2\alpha > 8$; moreover, even if they do exist, they do not contribute to the value of the integral unless the values of θ given by (39) lie within the range of integration. It is easily seen that they do not come within the range for the integral in (37); hence the resistance of the leading sphere does not exhibit any characteristic interference effects.

On the other hand, there are such effects for the other sphere if $\tan^2 \alpha > 8$, that is, if $\frac{\pi}{2} - \alpha < 19° 28'$ approximately. Thus the interference effects occur if this sphere lies within the wave pattern left by the leading sphere; and the two prominent terms in the evaluation of the integral correspond respectively to the transverse waves and the diverging waves of the pattern.

SUMMARY

A new method is given for calculating wave resistance directly from the source distribution equivalent to the body producing the waves. The method can be applied to two source systems representing two distinct bodies in any relative positions, giving the resistance of each separately. It can also be used to obtain the resultant force in any direction, or the resultant couples.

Results are obtained for a simple case representing two small spheres in various relative positions. With the two spheres in the line of motion, the resistances differ by certain forces of action and reaction and also by the wave-interference effects, which are assigned entirely to the following sphere.

Taking the two spheres abreast, the results are interpreted as showing the effect of a vertical wall upon the resistance of a sphere; the expressions are given in terms of Bessel functions and curves show the magnitude of the influence of the wall for various distances and velocities. An expression is also given for the force towards the wall.

Finally, with the spheres in any relative positions, it is shown that effects of wave interference occur when the following sphere lies within the wave pattern produced by the leading sphere, and arise from both the transverse waves and the diverging waves.

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The Forces on a Circular Cylinder Submerged in a Uniform Stream

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1—Although many investigations have been made on the wave resistance of submerged bodies, no case has been solved completely in the sense of taking fully into account the condition of zero normal velocity at the surface of the body. The simplest case is that of the two-dimensional motion produced by a long circular cylinder, with its axis horizontal and perpendicular to the stream, submerged at a certain depth below the upper free surface. This problem was propounded many years ago by Kelvin, and it was solved later, as regards a first approximation, by Lamb; in that solution the cylinder was replaced by a doublet, and the effect of the disturbance at the surface of the cylinder was neglected. Applying the method of images, I examined a second approximation,† and also by the same method obtained a first approximation for the vertical force on the cylinder.‡

Although the problem is not in itself of practical importance, it seems of sufficient interest to obtain a more complete analytical solution, and this is given in the present paper. The solution contains an infinite series, whose coefficients are given by an infinite set of linear equations; expansions are given for the coefficients in terms of a certain parameter, and corresponding expressions obtained for both the wave resistance and the vertical force. Numerical calculations have been made from these for various velocities and for different ratios of the radius of the cylinder to the depth of its axis. These confirm the general impression that the first approximation is a good one over a considerable range. The effect of the complete expressions appears in an increase in the wave resistance at lower velocities and a slight decrease at high velocities; this may be described as due largely to a shifting of the maximum of resistance towards the lower velocities, an effect which might have been anticipated.

The similar three-dimensional problems of the submerged sphere, or spheroid, are of more practical interest, as the first approximations which I have given for these cases have had certain applications in ship resis-

† 'Proc. Roy. Soc.,' A, vol. 115, p. 268 (1926).
‡ 'Proc. Roy. Soc.,' A, vol. 122, p. 387 (1928).

tance; the corresponding extension of the solutions would require more complicated analysis than for the two-dimensional case, but it seems probable that the general deductions on the range of applicability of the approximate formulae would be of a similar character.

2—Consider the two-dimensional fluid motion due to a fixed circular cylinder, of radius a, placed in a uniform stream of great depth, the axis of the cylinder being at a depth f below the undisturbed surface of the stream. Take the origin at the centre of the circular section, with Ox horizontal and Oy vertically upwards, and suppose the stream to be of velocity c in the negative direction of Ox. We write the velocity potential of the motion as

$$\phi = cx + \phi_0. \tag{1}$$

To obtain a solution which gives regular waves to the rear of the cylinder, we adopt the hypothesis of a frictional force proportional to the deviation of the fluid velocity from the uniform flow c, thus introducing a coefficient μ' which is made zero after the various analytical calculations have been effected. The pressure is then given by†

$$\frac{p}{\rho} = \text{const} - gy + \mu'\phi_0 - \tfrac{1}{2}q^2. \tag{2}$$

If η is the surface elevation and we make the usual approximation for small surface disturbances, we have

$$-c\frac{\partial\eta}{\partial x} = -\frac{\partial\phi}{\partial y}; \qquad y = f. \tag{3}$$

Hence, from (2), the condition to be satisfied at the free surface is

$$\frac{\partial^2\phi_0}{\partial x^2} + \kappa_0\frac{\partial\phi_0}{\partial y} - \mu\frac{\partial\phi_0}{\partial x} = 0; \qquad y = f, \tag{4}$$

where $\kappa_0 = g/c^2$, and $\mu = \mu'/c$.

We may regard ϕ_0 as made up of two parts, one part having singularities within the circle $r = a$, and the other having singularities in the region of the plane for which $y > f$. The first part is the potential of a system of sources and sinks, of total strength zero, within the circle, and can clearly be expressed by the real part of a series

$$\sum_1^\infty \mathrm{A}_n z^{-n}. \tag{5}$$

† Lamb, "Hydrodynamics," 6th ed., p. 399.

where $z = x + iy$, and the coefficients are complex. Now we have

$$z^{-n} = \frac{(-i)^n}{(n-1)!}\int_0^\infty \kappa^{n-1} e^{i\kappa z}\, d\kappa, \qquad y > 0. \tag{6}$$

Hence, in order to satisfy (4), we write ϕ_0 in the form

$$\phi_0 = \int_0^\infty \mathrm{F}(\kappa)\, e^{i\kappa x - \kappa y}\, d\kappa + \int_0^\infty \mathrm{G}(\kappa)\, e^{i\kappa x + \kappa(y-2f)}\, d\kappa, \tag{7}$$

where the real part is to be taken, and

$$\mathrm{F}(\kappa) = \sum_1^\infty (-i)^n \mathrm{A}_n \kappa^{n-1}/(n-1)!. \tag{8}$$

Putting (7) in (4), we obtain

$$\mathrm{G}(\kappa) = -\frac{\kappa + \kappa_0 + i\mu}{\kappa - \kappa_0 + i\mu}\mathrm{F}(\kappa). \tag{9}$$

With this value in (7) the surface condition is satisfied. Further, we may change the sign of i throughout the second term of (7), and we obtain

$$\phi_0 = \int_0^\infty \mathrm{F}(\kappa)\, e^{i\kappa x}\, d\kappa^{-\kappa y} - \int_0^\infty \frac{\kappa + \kappa_0 - i\mu}{\kappa - \kappa_0 - i\mu}\mathrm{F}^*(\kappa)\, e^{-i\kappa x + \kappa(y-2f)}\, d\kappa, \tag{10}$$

where the real part is to be taken, and the asterisk denotes the conjugate complex quantity. It may be noted that this method of satisfying the condition at the free surface is quite general, and independent of the form of the submerged body.

It is convenient for the present problem to alter the notation slightly from (8), and we write

$$\left.\begin{aligned} \mathrm{F}(\kappa) &= -ica^2 f(\kappa) \\ f(\kappa) &= b_0 + b_1(\kappa a) + \frac{b_2}{2!}(\kappa a)^2 + \frac{b_3}{3!}(\kappa a)^3 + \dots \end{aligned}\right\}. \tag{11}$$

Further, the expression (10) is a function of the complex variable z; hence we have for the complex potential function w, or $\phi + i\psi$,

$$w = cz - ica^2\int_0^\infty f(\kappa)\, e^{i\kappa z}\, d\kappa - ica^2\int_0^\infty \frac{\kappa + \kappa_0 - i\mu}{\kappa - \kappa_0 - i\mu} f^*(\kappa)\, e^{-i\kappa z - 2\kappa f}\, d\kappa, \tag{12}$$

this being in a form valid in the liquid in the region $y > 0$, it also being noted that ultimately μ is to be made zero.

3—We have now to determine the function $f(\kappa)$ so as to satisfy the condition $\partial\phi/\partial r = 0$ for $r = a$. For this we turn the second term in

(12) back to the form (5); it gives, with the form (11) for $f(\kappa)$, the series

$$\sum_1^\infty ca^2 (ia)^{n-1} b_{n-1} z^{-n}. \tag{13}$$

Further, the last term in (12) represents the potential of image sources and sinks in the region of the plane for which $y > 2f$, and hence it can be expanded in the neighbourhood of the circle $|z| = a$ in a series of ascending powers of z. Thus we obtain w in the form

$$w = \text{const} + cz + \sum_1^\infty ca^2 (ia)^{n-1} b_{n-1} z^{-n} + \sum_1^\infty \mathrm{B}_n z^n,$$

$$\mathrm{B}_n = ca^2 \frac{(-i)^{n+1}}{n!} \int_0^\infty \frac{\kappa + \kappa_0 - i\mu}{\kappa - \kappa_0 - i\mu} \kappa^n e^{-2\kappa f} f^*(\kappa)\, d\kappa. \tag{14}$$

With the potential in the form

$$w = \text{const} + \sum_1^\infty (\mathrm{C}_n z^n + \mathrm{D}_n z^{-n}), \tag{15}$$

the condition of zero normal velocity on the circle $|z| = a$ is satisfied, provided

$$\mathrm{D}_n = a^{2n} \mathrm{C}^*_n. \tag{16}$$

Hence, from (14) we obtain the equations

$$b_0 = 1 - a^2 \int_0^\infty \frac{\kappa + \kappa_0 + i\mu}{\kappa - \kappa_0 + i\mu} \kappa f(\kappa) e^{-2\kappa f} d\kappa,$$

$$b_{n-1} = -\frac{a^{n+1}}{n!} \int_0^\infty \frac{\kappa + \kappa_0 + i\mu}{\kappa - \kappa_0 + i\mu} \kappa^n f(\kappa) e^{-2\kappa f} d\kappa. \tag{17}$$

These relations, with (11), may be expressed in the form of an integral equation satisfied by the function $f(\kappa a)$; it is easily found to be

$$v^{\frac{1}{2}} f(v) = v^{\frac{1}{2}} - \int_0^\infty \frac{u + \gamma + i\mu}{u - \gamma + i\mu} e^{-2fu/a} u^{\frac{1}{2}} f(u) \, \mathrm{I}_1 (2\sqrt{uv})\, du, \tag{18}$$

where $\gamma = \kappa_0 a$, I_1 is the modified Bessel function, and the limit of the integral is to be taken as the positive quantity, μ approaches zero.

For purposes of calculation, we use (17) as a set of linear equations for the coefficients $b_0, b_1, \ldots$. We write

$$q_n = \lim_{\mu \to 0} \int_0^\infty \frac{u + 1 + i\mu}{u - 1 + i\mu} e^{-2\kappa_0 f u} u^n \, du. \tag{19}$$

Substituting the power series (11) for $f(\kappa)$ on the right of (17), we obtain the infinite set of equations

$$\left.\begin{aligned}
&b_0(1+q_1\gamma^2)+q_2\gamma^3 b_1+\frac{q_3\gamma^4}{2\,!}b_2+\frac{q_4\gamma^5}{3\,!}b_3+\dots &&=1\\
&\frac{q_2\gamma^3}{2\,!}b_0+\left(1+\frac{q_3\gamma^4}{2\,!}\right)b_1+\frac{q_4\gamma^5}{2\,!\,2\,!}b_2+\frac{q_5\gamma^6}{2\,!\,3\,!}b_3+\dots &&=0\\
&\frac{q_3\gamma^4}{3\,!}b_0+\frac{q_4\gamma^5}{3\,!}b_1+\left(1+\frac{q_5\gamma^6}{2\,!\,3\,!}\right)b_2+\frac{q_6\gamma^7}{3\,!\,3\,!}b_3+\dots &&=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots &&=0
\end{aligned}\right\} . \quad (20)$$

From the integral expression for q_n given in (19), and also the fact that $a/f<1$, it can readily be shown that the infinite determinant formed by the coefficients of $b_0, b_1, \dots,$ on the left of (20) is convergent.

Evaluating the expression (19) and putting

$$q_n = r_n - is, \qquad (21)$$

we find

$$r_n = \frac{n\,!}{\alpha^{n+1}} + 2\left\{\frac{(n-1)\,!}{\alpha^n} + \frac{(n-2)\,!}{\alpha^{n-1}} + \dots + \frac{1}{\alpha} - e^{-\alpha}\,li\,(e^{\alpha})\right\},$$

$$s = 2\pi e^{-\alpha}, \qquad (22)$$

where $\alpha = 2\kappa_0 f$, and li denotes the logarithmic integral.

For any given values of $a, f,$ and c, we have in (20) a set of equations for the b's with complex numerical coefficients.

Although expansions in terms of other parameters may be more suitable for special ranges, it is convenient to assume that the coefficients b can be expanded in power series of the quantity γ, that is $\kappa_0 a$. These expansions will be of the form

$$\left.\begin{aligned}
b_0 &= 1 + b_{02}\gamma^2 + b_{04}\gamma^4 + b_{06}\gamma^6 + \dots\\
b_1 &= b_{13}\gamma^3 + b_{15}\gamma^5 + b_{17}\gamma^7 + \dots\\
b_2 &= b_{24}\gamma^4 + b_{26}\gamma^6 + b_{28}\gamma^8 + \dots\\
\dots &= \dots\dots\dots\dots\dots\dots
\end{aligned}\right\} . \quad (23)$$

Substituting in (21) and collecting the various powers of γ, the new coefficients may be found to any required stage. For the calculations which follow, it was found sufficient to obtain the results:

$$\begin{aligned}
b_{02} &= -q_1\\
b_{04} &= q_1^2\\
b_{06} &= \tfrac{1}{3}q_2^2 - q_1^3\\
b_{08} &= q_1^4 - q_1 q_2^2 + \tfrac{1}{12}q_3^2
\end{aligned}$$

$$
\begin{aligned}
b_{010} &= -q_1^5 + \tfrac{3}{2} q_1^2 q_2^2 - \tfrac{1}{6} q_1 q_3^2 - \tfrac{1}{4} q_2^2 q_3 + \tfrac{1}{144} q_4 \\
b_{13} &= -\tfrac{1}{2} q_2 \\
b_{15} &= \tfrac{1}{2} q_1 q_2 \\
b_{17} &= \tfrac{1}{4} q_2 q_3 - \tfrac{1}{2} q_1^2 q_2 \\
b_{19} &= -\tfrac{1}{4} q_2^3 + \tfrac{1}{2} q_1^3 q_2 - \tfrac{1}{4} q_1 q_2 q_3 + \tfrac{1}{24} q_3 q_4 \\
b_{111} &= -\tfrac{1}{6} q_2 q_3^2 + \tfrac{1}{4} q_1^2 q_2 q_3 - \tfrac{1}{2} q_1^4 q_2 + \tfrac{1}{2} q_1 q_2^3 - \tfrac{1}{24} q_1 q_3 q_4 + \tfrac{1}{288} q_4 q_5 \\
b_{24} &= -\tfrac{1}{6} q_3 \\
b_{26} &= \tfrac{1}{6} q_1 q_3 \\
b_{28} &= -\tfrac{1}{6} q_1^2 q_3 + \tfrac{1}{12} q_2 q_4 \\
b_{35} &= -\tfrac{1}{24} q_4 \\
b_{37} &= \tfrac{1}{24} q_1 q_4 \\
b_{46} &= -\tfrac{1}{120} q_5 .
\end{aligned}
\tag{24}
$$

4—Consider now the forces acting on the cylinder per unit length. The pressure is given by

$$p/\rho = \text{const} - gy - \tfrac{1}{2}q^2. \tag{25}$$

The term in gy gives the usual buoyancy, equal to the weight of displaced liquid, as part of the vertical force on the cylinder. Apart from this term, let X, Y be the resultant horizontal and vertical forces on the cylinder in the positive directions of Ox, Oy. Then, by the Blasius formula, we have

$$X - iY = \tfrac{1}{2}\rho i \oint \left(\frac{dw}{dz}\right)^2 dz, \tag{26}$$

taken round the circle $|z| = a$.

We note that $-$X will be the force known as the wave resistance, while Y is the addition to the upward force of buoyancy arising from the fluid motion. The value of the integral in (26) is $2\pi i$ times the residue of the integrand; with w given in the form (15), and, using (16), this gives

$$X - iY = 2\pi\rho \sum_1^\infty \frac{n(n+1)}{a^{2n+2}} D_n D^*_{n+1}. \tag{27}$$

Using (14), we have the result

$$X - iY = -2\pi\rho c^2 a i \{1.2 b_0 b^*_2 + 2.3 b_1 b^*_2 + \ldots + n(n+1) b_{n-1} b^*_n + \ldots\}. \tag{28}$$

This may be expanded in powers of γ, that is of $\kappa_0 a$, by substituting from (23) and (24), the results given there being sufficient to include the term

in γ^{11}. Using the notation of (21), and separating out the real and imaginary parts, we obtain, after some reduction,

$$\begin{aligned} -\mathrm{X} = 4\pi^2\rho c^2 a\,(\kappa_0 a)^3\, e^{-2k_0 f}\,[1 &- 2r_1\,(\kappa_0 a)^2 - (r_2 - 3r_1^2 + s^2)(\kappa_0 a)^4 \\ &- (4r_1^3 - r_2^2 - 2r_1 r_2 + \tfrac{1}{6}r_4 - 4r_1 s^2 + s^2)\,(\kappa_0 a)^6 \\ &+ \{5r_1^4 - 3r_1 r_2^2 - 3r_1^2 r_2 + \tfrac{1}{4}r_2^2 + \tfrac{1}{6}r_3^2 + \tfrac{1}{3}r_1 r_3 + \tfrac{1}{2}r_2 r_3 - \tfrac{1}{72}r_4 \\ &- (10r_1^2 - 3r_1 - 3r_2 + \tfrac{5}{12})\,s^2 + s^4\}\,(\kappa_0 a)^8 + \ldots], \end{aligned} \tag{29}$$

$$\begin{aligned} \mathrm{Y} = 4\pi\rho c^2 a\,(\kappa_0 a)^3\,[&-\tfrac{1}{2}r_2 + r_1 r_2\,(\kappa_0 a)^2 + \tfrac{1}{2}(r_2 r_3 - 3r_1^2 r_2 + r_2 s^2)\,(\kappa_0 a)^4 \\ &+ (2r_1^3 r_2 - \tfrac{1}{2}r_2^3 - r_1 r_2 r_3 + \tfrac{1}{12}r_3 r_4 + \tfrac{1}{2}r_2 s^2 - 2r_1 r_2 s^2)\,(k_0 a)^6 \\ &+ \{\tfrac{3}{2}r_1 r_2^3 - \tfrac{5}{2}r_1^4 r_2 - \tfrac{1}{3}r_2 r_3^2 + \tfrac{3}{2}r_1^2 r_2 r_3 - \tfrac{1}{8}r_2^2 r_4 - \tfrac{1}{6}r_1 r_3 r_4 + \tfrac{1}{288}r_4 r_5 \\ &+ (5r_1^2 r_2 - \tfrac{1}{2}r_1 r_2 - \tfrac{1}{2}r_2 r_3 - \tfrac{11}{24}r_2 + \tfrac{1}{4}r_3 - \tfrac{1}{8}r_4 + \tfrac{1}{24})\,s^2 \\ &\qquad - r_2 s^4\}\,(\kappa_0 a)^8 + \ldots], \end{aligned} \tag{30}$$

with r_n, s given by (22).

The first term in (29) is the expression for the wave resistance of a circular cylinder which was obtained by Lamb. The first term in (30) is, after putting in the value of r_2 from (22), the first approximation for the vertical force which I obtained by the method of images in the paper already quoted.

5—It is of interest to obtain the wave resistance, which should be equal to $-\mathrm{X}$, from considerations of energy applied to the regular waves behind the cylinder. The current function ψ is given by the imaginary part of the expression (12). Putting $y = f + \eta$, we obtain at once the complete expression for the surface elevation as

$$\eta = ia^2 \int_0^\infty f(\kappa)\, e^{i\kappa x - \kappa f}\, d\kappa + ia^2 \int_0^\infty \frac{\kappa + \kappa_0 - i\mu}{\kappa - \kappa_0 - i\mu}\, f^*(\kappa)\, e^{-i\kappa x - \kappa f}\, d\kappa, \tag{31}$$

where the imaginary part is to be taken, $f(\kappa)$ is given by (11), and μ is to be made zero ultimately. This expression separates into two parts, a local disturbance η_1 which decreases with increasing distance from the cylinder, and a system of regular waves η_2 to the rear, that is, for negative values of x. The latter part is found, by methods familiar in these problems, to be given by

$$\eta_2 = -\,4\pi\kappa_0 a^2 f^*(\kappa_0)\, e^{-i\kappa_0 x - \kappa_0 f}, \tag{32}$$

the imaginary part to be taken.

If h is the amplitude of the regular waves at a great distance behind the cylinder, the wave resistance R is given by

$$\mathrm{R} = \tfrac{1}{4}g\rho h^2. \tag{33}$$

Hence from (32) we have

$$R = 4\pi^2 g\rho\kappa_0^2 a^4 f(\kappa_0) f^*(\kappa_0)\, e^{-2\kappa_0 f}. \tag{34}$$

With

$$f(\kappa_0) = b_0 + b_1(\kappa_0 a) + \frac{b_2}{2!}(\kappa_0 a)^2 + \dots,$$

and with the equations (20), it could presumably be shown that (34) is the same as the real part, with sign changed, of the expression (28). However, it has been used here simply to verify the previous expansion; substituting from (23) and (24) we obtain from (34) the same result as is given in (29).

6—We may now examine the expressions (29) and (30) numerically. It is easily seen that if the ratio a/f is small, the first term in each case gives a close approximation at all velocities. Further, the ratio of the second term to the first in (29) and in (30) is $-2r_1\kappa_0^2 a^2$, that is

$$-\frac{1}{2}\frac{a^2}{f^2}\{1 + 2\alpha - 2\alpha^2 e^{-\alpha}\, li\,(e^{\alpha})\}, \tag{35}$$

with $\alpha = 2\kappa_0 f = 2gf/c^2$.

The quantity in brackets in (35) approaches the value -1 as c becomes zero and the value $+1$ as c becomes infinite. It has a maximum negative value of $-2\cdot 57$ at $\alpha = 4\cdot 5$ approximately, and a maximum positive value of $1\cdot 9$ at about $\alpha = 0\cdot 6$. Hence the effect of the second approximation in (29) is to increase the wave resistance at low velocities and to give a rather smaller value at high speeds.

Taking $a/f = \frac{1}{4}$, as a moderate value of this ratio, and calculating the resistance from (29), it is found that the value does not differ by more than about 9% of the value of the first approximation at any velocity. As an example of the numerical values in this case, for $\alpha = 6$, that is for $c = 0\cdot 58\sqrt{(gf)}$, the following are the values of the successive terms in the expansion in square brackets in (29):

$$1 + 0\cdot 0746 + 0\cdot 0134 + 0\cdot 0015 + 0\cdot 0001.$$

Another case which has been worked out in some detail is $a/f = \frac{1}{2}$, this being definitely outside the range of the first approximation for the most part. Numerical values were calculated for both X and Y for $\alpha = 8$, 6, 5, 4, 3, $2\cdot 5$, 2, and 1. On account of slower convergence of the series at the higher values of α, an estimate was made of the next term beyond those shown in (29) and (30). The results are shown in fig. 1.

The curves R and Y are the wave resistance and vertical force calculated from (29) and (30); R_1, Y_1 are the curves given by the first approximations,

that is by the first term in (29) or (30). The unit of force in each case is $\pi g \rho a^2$, that is the weight of liquid displaced by the cylinder per unit length. It should be noted also that, in addition to the vertical force Y, there is the usual hydrostatic buoyancy. The curves for the wave resistance show clearly the increased values at lower velocities and also the displacement of the position of maximum resistance, the latter occurring at a lower speed than the value $\sqrt{(gf)}$ given by the first approximation.

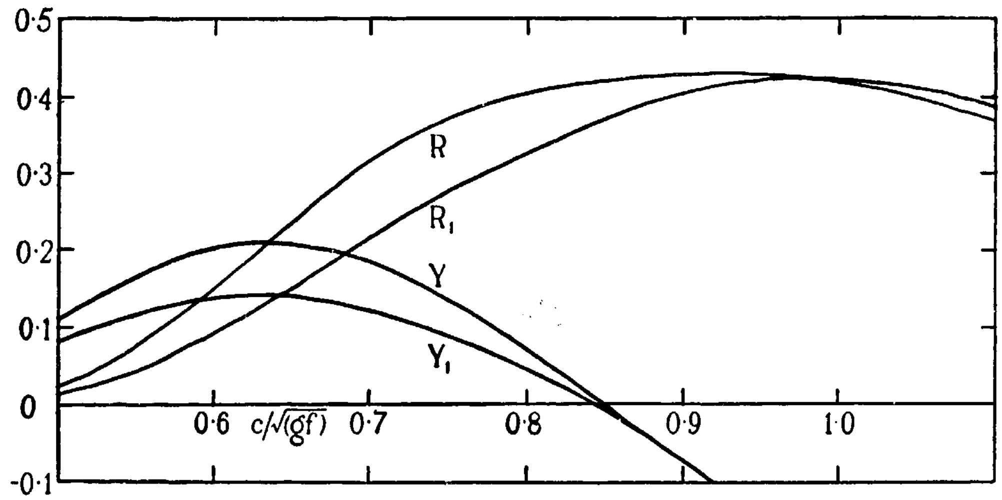

FIG. 1.

SUMMARY

A solution is given for the two-dimensional wave motion due to a circular cylinder in a uniform stream, taking fully into account the condition at the surface of the cylinder. Expressions for the horizontal and vertical forces on the cylinder are obtained in the form of infinite series in ascending powers of a certain parameter. Numerical calculations are made from these and compared with the known first approximations. The main effect of the additional terms upon the wave resistance is to increase the calculated value at low velocities and to decrease it slightly at high velocities.

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The Resistance of a Ship among Waves

By T. H. Havelock, F.R.S.

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1—The wave resistance of a ship advancing in still water may be calculated under certain assumptions, which amount to supposing the forced wave motion to be small so that squares of the fluid velocity may be neglected; moreover, the ship is supposed to advance with constant velocity in a horizontal line. It does not appear to have been noticed that we may superpose on the solution so obtained free surface waves of small amplitude, and that the addition to the resistance may be calculated, to a similar degree of approximation, as the horizontal resultant of the additional fluid pressures due to the free surface waves; this additional resistance, which may be negative, depends upon the position of the ship among the free waves. Various calculations are now made from this point of view. We consider first transverse following waves moving at the same speed as the ship, and then a ship moving in the waves left by another ship in advance moving at the same speed; finally, we examine the more general case of a ship moving through free transverse waves of any wave-length. All the cases are discussed with reference to such experimental results as are available.

2—We treat the problem first as one of steady motion with the ship at rest in a uniform stream of velocity c in the negative direction of Ox; we take the origin O in the undisturbed water surface, and Oz vertically upwards. The velocity potential is given by

$$\phi = cx + \phi_1, \tag{1}$$

where ϕ_1 represents the disturbance due to the ship. This, on the usual approximations, may be regarded as due to a source distribution over the longitudinal section of the ship; the source strength per unit area is $(c/2\pi)\,\partial y/\partial x$, with $y = f(x, z)$ as the equation of the surface of the ship, and it is to be noted that $\partial y/\partial x$ is assumed to be small.

We now take
$$\phi = cx + \phi_1 + \phi',$$
$$\phi' = hce^{\kappa_0 z}\cos(\kappa_0 x - \beta), \tag{2}$$

where $\kappa_0 = g/c^2$. The additional term represents standing surface waves of elevation $h\sin(\kappa_0 x - \beta)$. We should, of course, require further terms in order to satisfy exactly the condition at the surface of the ship; but such

terms would be of a smaller order of magnitude, and of a similar order to those which have already been neglected in obtaining an expression for ϕ_1 on the assumption that the angle between the tangent plane and the zx-plane is always small. The pressure equation is now

$$p/\rho = \text{const.} - gz - c\frac{\partial \phi_1}{\partial x} - c\frac{\partial \phi'}{\partial x}; \tag{3}$$

and the wave resistance is given by

$$R = -2\iint p\frac{\partial y}{\partial x}\,dx\,dz, \tag{4}$$

taken over the longitudinal section of the ship.

The term in ϕ_1 in (3) gives from (4) the expression for the wave resistance for the ship advancing into still water; we shall denote this by R_1. We give, for reference later, the expression for R_1 in terms of the equivalent surface distribution σ_1, namely,

$$R_1 = 16\pi\kappa_0^2\rho\int_0^{\frac{1}{2}\pi}(P_1^2+Q_1^2)\sec^3\theta\,d\theta,$$

where
$$P_1 + iQ_1 = \iint \sigma_1 e^{\kappa_0 z\sec^2\theta + i\kappa_0 x\sec\theta}\,dx\,dz. \tag{5}$$

The term in ϕ' in (3) gives from (4) the additional resistance R' due to the standing waves; we have

$$\begin{aligned} R' &= 2\rho c\iint \frac{\partial \phi'}{\partial x}\frac{\partial y}{\partial x}\,dx\,dz \\ &= -2g\rho h\iint \frac{\partial y}{\partial x}e^{\kappa_0 z}\sin(\kappa_0 x - \beta)\,dx\,dz. \end{aligned} \tag{6}$$

3—Consider a simple form of model, of uniform draft d and length $2l$, whose surface for $y > 0$ is given by

$$y = b(1 - z^2/d^2)(1 - x^2/l^2). \tag{7}$$

From (6) we obtain, after carrying out the integrations,

$$R' = \frac{8g\rho bhl}{\kappa_0^3 l^3}\left\{1 - \frac{2}{\kappa_0^2 d^2} + 2\left(\frac{1}{\kappa_0 d} + \frac{1}{\kappa_0^2 d^2}\right)e^{-\kappa_0 d}\right\}(\sin\kappa_0 l - \kappa_0 l\cos\kappa_0 l)\cos\beta. \tag{8}$$

The factor $\cos\beta$ in (8) shows how R' varies with the position of the ship among the waves; for $\beta = 0$ or $\beta = \pi$, the surface elevation is anti-symmetrical with respect to the mid-section of the ship. Further, the factor

$$(\sin\kappa_0 l - \kappa_0 l\cos\kappa_0 l)/(\kappa_0 l)^3 \tag{9}$$

gives the variation of R' with the ratio of the length of the model to the wave-length. It is obvious that the greatest positive values of R' will occur when there is a crest near the bow and a trough near the stern, and conversely for negative values of R'. The stationary values of (9) give the corresponding values of $\kappa_0 l$, or $2\pi l/\lambda$; one such value gives $\lambda/2l = 0{\cdot}55$ approximately, and for this velocity R' is negative if $\beta = 0$ and positive if $\beta = \pi$.

4—For numerical calculations we shall consider a model for which theoretical and experimental values of the wave resistance in still water are known; this is Model 1302 investigated by Wigley at the National Physical Laboratory, the results being given in these *Proceedings* (Wigley 1934). The form of the model is given by the following:

From $z = 0$ to $z = -\frac{1}{2}d$,

$$y = b\{1-(x+a)^2/l^2\}, \quad y = b, \quad y = b\{1-(x-a)^2/l^2\}$$

for x ranging from $-l-a$ to $-a$, $-a$ to a, a to $l+a$ respectively;

From $z = -\frac{1}{2}d$ to $z = -d$,

$$y = \tfrac{4}{3}b(1-z^2/d^2)\{1-(x+a)^2/l^2\}, \quad y = \tfrac{4}{3}b(1-z^2/d^2),$$

$$y = \tfrac{4}{3}b(1-z^2/d^2)\{1-(x-a)^2/l^2\}$$

for the respective ranges for x of

$$-l-a \text{ to } -a, \quad -a \text{ to } a, \quad a \text{ to } l+a. \tag{10}$$

The dimensions, all in feet, were $a = 0{\cdot}5$, $b = 0{\cdot}484$, $l = 7{\cdot}5$ and $d = 2$.

Carrying out the integrations of (6) over the longitudinal section of the model, we obtain

$$R' = \frac{8\rho bc^2h}{\kappa_0^2 l^2}\left\{1-\frac{4}{3}\left(\frac{1}{\kappa_0 d}+\frac{2}{\kappa_0^2 d^2}\right)e^{-\frac{1}{2}\kappa_0 d}+\frac{8}{3}\left(\frac{1}{\kappa_0 d}+\frac{1}{\kappa_0^2 d^2}\right)e^{-\kappa_0 d}\right\}$$

$$\times\{\sin\kappa_0(l+a)-\kappa_0 l\cos\kappa_0(l+a)-\sin\kappa_0 a\}\cos\beta. \tag{11}$$

We shall take $\beta = 0$ so as to obtain maximum effects as far as the position of the model relative to the waves is concerned. In the following table values of R'/c^2h are shown for several different velocities, R' being in lb. with c in ft./sec. and h in ft. The column R_1/c^2 gives the corresponding theoretical values for the wave resistance in still water, taken from fig. 6 of Wigley's paper.

R_1 has maxima and minima according to the interference of bow and stern waves; while R' oscillates between positive and negative values in

accordance with the factor in (11) which involves the quantity $\kappa_0 l$. One method of expressing these results is to find what height h of the imposed waves would give R' the same numerical value as R_1 at each velocity, regardless of whether R' is positive or negative. This is given in the last column of the table as h/λ, the ratio of amplitude to wave-length such that R' is numerically equal to R_1. In comparing these figures with values from observation or experiment, it should be noted that usually the height of a sea wave is measured from trough to crest, and is equal to $2h$ of these calculations. The point made now is that for quite ordinary values of the ratio of wave-height to wave-length the additional resistance, positive or negative, is of the same order as the wave resistance of the model in still water.

$\kappa_0 l$	$c/\sqrt{(gl)}$	R'/c^2h	R_1/c^2	h/λ
8	0·353	0·5	0·042	0·014
7	0·378	−0·264	0·03	0·017
6	0·408	−0·92	0·072	0·01
5	0·447	−0·78	0·1	0·013
4	0·5	0·62	0·053	0·007
3	0·577	1·05	0·12	0·007
2	0·707	1·13	0·233	0·008

5—An interesting form of the problem is the case of a model in the waves left by another model at a fixed distance in advance and moving at the same speed; it is a case for which some experimental results are available.

Instead of (2) we now have

$$\phi = cx + \phi_1 + \phi_2, \tag{12}$$

where ϕ_1 represents the disturbance due to the rear model and ϕ_2 that due to the leading model. We may replace the models by source distributions σ_1, σ_2 over their respective longitudinal sections, and the usual first approximation is taken for σ in each case, namely $\sigma = (c/2\pi)\,\partial y/\partial x$.

The resultant horizontal force on each model has been considered from this point of view in a previous paper (Havelock 1936), and a general discussion is given there in §§ 4, 5. The resistance of each model consists of various terms: the resistance of each as if isolated, mutual actions between the two models which are equal and opposite and may be assigned to local disturbances of the fluid motion, and forces due to wave interference acting on the rear model only. It is easily seen, from approximate calculations, that the mutual actions due to local effects diminish rapidly with the distance between the models, and we shall neglect these terms in what follows.

The resistances R_1, R_2 of the two models when far apart are given by (5) with P_1+iQ_1 in terms of σ_1 and P_2+iQ_2 in terms of σ_2.

In addition the rear model experiences a resistance R_{12} which, from (7) and (13) of the paper just quoted, is given by

$$R_{12} = 32\pi\kappa_0^2\rho\iint\sigma_1 dx_1 dz_1\iint\sigma_2 dx_2 dz_2$$

$$\times\int_0^{\frac{1}{2}\pi} e^{\kappa_0(z_1+z_2)\sec^2\theta}\cos\{\kappa_0(x_2-x_1)\sec\theta\}\sec^3\theta\, d\theta, \quad (13)$$

the integrations extending over the two distributions. This may be put into a form involving the same quantities P_1, Q_1, P_2, Q_2 as are required for R_1 and R_2, namely,

$$R_{12} = 32\pi\kappa_0^2\rho\int_0^{\frac{1}{2}\pi}(P_1P_2+Q_1Q_2)\sec^3\theta\, d\theta. \quad (14)$$

We now simplify the problem by supposing the two models to be similar in all respects; then if k is the distance from the bow of the leading model to the bow of the rear model, we have

$$P_2+iQ_2 = (P_1+iQ_1)\, e^{i\kappa_0 k\sec\theta}. \quad (15)$$

This gives $$R_{12} = 32\pi\kappa_0^2\rho\int_0^{\frac{1}{2}\pi}(P_1^2+Q_1^2)\cos(\kappa_0 k\sec\theta)\sec^3\theta\, d\theta. \quad (16)$$

Finally, we carry out the integrations for a model of great draft and of uniform horizontal cross-section given by

$$y = b(1-x^2/l^2). \quad (17)$$

The results may be expressed in terms of P functions used in previous investigations and defined by

$$P_{2n}(p) = (-1)^n\int_0^{\frac{1}{2}\pi}\cos^{2n}\theta\sin(p\sec\theta)\, d\theta,$$

$$P_{2n+1}(p) = (-1)^{n+1}\int_0^{\frac{1}{2}\pi}\cos^{2n+1}\theta\cos(p\sec\theta)\, d\theta. \quad (18)$$

(I am indebted to the Superintendent of The William Froude Laboratory for graphs of the first nine of this series of functions.) We obtain then for

the resistance R_2 of the leading model and for the resistance R_1+R_{12} of the rear model the expressions

$$R_1 = R_2 = \frac{32\rho b^2c^2}{\pi\kappa_0^4l^4}\{\tfrac{8}{15}+\tfrac{2}{3}\kappa_0^2l^2+\kappa_0^2l^2P_3(2\kappa_0l)-2\kappa_0lP_4(2\kappa_0l)+P_5(2\kappa_0l)\}, \quad (19)$$

$$\begin{aligned} R_{12} = \frac{64\rho b^2c^2}{\pi\kappa_0^4l^4}[&\tfrac{1}{2}\kappa_0^2l^2\{P_3(\kappa_0k_1)+2P_3(\kappa_0k)+P_3(\kappa_0k_2)\} \\ &+\kappa_0l\{P_4(\kappa_0k_1)-P_4(\kappa_0k_2)\} \\ &+\tfrac{1}{2}P_5(\kappa_0k_1)-P_5(\kappa_0k)+\tfrac{1}{2}P_5(\kappa_0k_2)], \end{aligned} \quad (20)$$

where $k_1 = k-2l$, $k_2 = k+2l$.

6—Before making numerical calculations we may refer to experiments made by Barrillon on models in tandem and other formations (Barrillon 1926). In a classical series of experiments W. Froude examined the interference between the bow and stern waves of a ship by introducing into the model varying lengths of parallel middle body between the same bow and stern. Barrillon made an interesting variation by running two models in tandem at the same speed and measuring the resistance of each model. The results were similar in character to those outlined in the previous section; for instance, the resistance of the rear model was found to be an oscillating function of its distance from the leading model, in general agreement with what would be expected from its position relative to the waves left by the leading model. We noted also that the action between the two models is made up of a mutual action and reaction due to local effects together with a wave effect upon the rear model; and the former has been neglected in the present calculations. Barrillon found, for his models at a certain speed, that the action upon the leading model was insensible if the distance apart exceeded 1 m., while the action upon the rear model was appreciable up to a distance of 14 m.; and further that, apart from its oscillations, the action upon the rear model only diminished slowly with the distance.

With a view to making corresponding calculations from (19) and (20) we notice in particular two measurements. (I am indebted to Professor Barrillon for these and other details of his investigations.) The velocity of the two models was 2 m./sec. and the length of the rear model was 2·2 m. Turning the results into the present notation, with $k = 13{\cdot}47$ m. and 16·19 m. the experimental values of the ratio R_{12}/R_1 were $-0{\cdot}224$ and $-0{\cdot}2$ respectively; these two values of k gave consecutive positions of maximum reduction of resistance of the rear model, the relative reduction being of the order of 20 %.

We now use these measurements solely in order to take a corresponding velocity and corresponding distances in the expressions (19) and (20) and so to calculate the ratio R_{12}/R_1.

We have $\kappa_0 l = gl/c^2 = 2{\cdot}7$. For the two values of k, the corresponding values of $\kappa_0 k$ are 33·07, 39·74 respectively. With these we obtain from (19) and (20) the values $-0{\cdot}24$, $-0{\cdot}3$ respectively for the ratio R_{12}/R_1, a relative reduction of resistance of between 20 and 30 %.

7—We have considered so far only wave motion which is stationary relative to the ship, and we examine now a ship advancing through free transverse waves which are moving with the velocity appropriate to their wave-length.

Suppose first that the waves are moving in the same direction as the ship. With a fixed origin O we now have, instead of (2),

$$\phi = \phi_1(x-ct, y, z) + hVe^{\kappa z}\cos\kappa(x-Vt), \tag{21}$$

where $V^2 = g/\kappa$, and the additional surface elevation due to the free waves is $h\sin\kappa(x-Vt)$.

The variable part of the pressure is $\rho\,\partial\phi/\partial t$, or

$$-\rho c\frac{\partial\phi_1}{\partial x} + g\rho h e^{\kappa z}\sin\kappa(x-Vt). \tag{22}$$

To calculate R from (4) and (22), transfer to an origin moving with the ship. Then the first term in (22) gives the same expression for R_1 as in (5), while for the additional resistance due to the second term we have

$$R' = -2g\rho h\iint\frac{\partial y}{\partial x}e^{\kappa z}\sin\{\kappa x - \kappa(V-c)\,t\}\,dx\,dz. \tag{23}$$

This is the same as in (6) for relatively stationary waves, except that κ_0, c are replaced by κ, V respectively, and that the phase β has now the varying value $\kappa(V-c)\,t$.

For transverse waves $h\sin\kappa(x+Vt)$ moving in the opposite direction (21) is replaced by

$$\phi = \phi_1(x-ct, y, z) - hVe^{\kappa z}\cos\kappa(x+Vt), \tag{24}$$

and it is easily seen that we get the same result as before with the phase β equal to $-\kappa(V+c)\,t$.

The result is that the additional resistance depends only upon the instantaneous position of the ship relative to the waves. This might have been anticipated from the various approximations which have been made. We

have assumed the free waves to be small, the corresponding quantities being of the same order as those of the forced waves due to the ship; moreover, we have neglected any direct disturbance of the free waves by the surface of the ship for the same reason as for omitting terms of a like order in obtaining expressions for the forced waves. The additional pressures are therefore simply those due to the free waves, and the additional resistance is the horizontal resultant of these pressures acting upon the ship's surface. Taking as an example the model described in (10), the resistance is given by $R_1 + R'$, where R_1 is the resistance in still water, and

$$R' = \frac{8\rho b V^2 h}{\kappa^2 l^2}\left\{1 - \frac{4}{3}\left(\frac{1}{\kappa d} + \frac{2}{\kappa^2 d^2}\right)e^{-\frac{1}{2}\kappa d} + \frac{8}{3}\left(\frac{1}{\kappa d} + \frac{1}{\kappa^2 d^2}\right)e^{-\kappa d}\right\}$$

$$\times \{\sin \kappa(l+a) - \kappa l \cos(\kappa + a) - \sin \kappa a\} \cos \beta, \tag{25}$$

where $$V^2 = g/\kappa, \quad \beta = \kappa(c \pm V)t = 2\pi t/T.$$

In this expression, T is the period of encounter of the ship with the waves.

In experiments on models in artificially produced waves, a critical condition occurs when the wave-length is about equal to the length of the model. We take therefore as a numerical example $\lambda = 2\pi/\kappa = 2(l+a)$, and $V^2 = g\lambda/2\pi = g(l+a)/\pi$.

Putting in the numerical values for this model, we get from (25)

$$R' = 89{\cdot}1 h \cos(2\pi t/T), \tag{26}$$

R' being in lb. with the amplitude h in ft.

In experiments a usual assumption is a wave-height of 6 ft. for a wave of length 400 ft.; this ratio gives for the wave-length 16 ft. the value $h = 0{\cdot}12$ ft. In that case we have

$$R' = 10{\cdot}7 \cos(2\pi t/T)\,\text{lb.} \tag{27}$$

Values of R_1 for this model are known. For instance, for model speeds of 7·08, 9·22, 11·04 ft./sec. we have $R_1 = 4{\cdot}9$, 13, 29 lb. respectively; the total resistance, wave-making and frictional, at these speeds was 14·54, 30·11, 52·15 lb. respectively. We see that, for quite a moderate ratio of wave-height to wave-length, R' represents an alternating force of relatively large amplitude. It should be noted, however, that this is for the particular case when the wave-length of the free waves is equal to the length of the model.

8—It is necessary to emphasize the basis of the present calculations. It is assumed that the model is maintained in the same relation to the un-

disturbed water surface and that it is driven forward horizontally at constant speed.

In experiments on models in waves, such as those made by Kent at The National Physical Laboratory (Kent 1922), the conditions are different, being naturally designed to reproduce to some extent conditions for ships at sea. In these experiments the model is free to pitch, and obviously an important factor is the relation of the pitching period to the period of encounter with the waves. Moreover, the model can move fore and aft within certain limits under the influence of the waves. Thus Kent makes the statement: "When the model was towed through a regular series of advancing waves, it experienced periodic fluctuations in its resistance as it met each succeeding wave. Each fluctuation in resistance was partially absorbed by the inertia of the model, but a portion of it was recorded by the resistance pen. The fluctuations were of small amplitude when the waves were of short length in comparison with the length of the model, but became much larger when the wave-length was increased." The actual results given were for a certain mean resistance over the whole experiment in each case. The precise relation between this mean resistance and the horizontal forces acting on the model at each instant would require a detailed examination of the conditions of the experiment and of the recording apparatus. However that may be, the present calculations serve to estimate some of these forces and indicate how large the fluctuating part of the resistance due to them may be under certain conditions.

A point which arises is the dependence of the amplitude of the fluctuations upon the ratio of the wave-length to the length of the model. This is given, for the model considered here, by the factor of (25) which involves κl. Taking the simpler case of that model with no parallel middle body, that is with $a = 0$, the factor concerned is

$$(\sin u - u \cos u)/u^3, \tag{28}$$

where $u = \pi L/\lambda$, with L the total length of the model, and λ the wave-length.

An interesting result is that there are certain values of the ratio λ/L for which (28) is zero; for these, the additional resultant horizontal force due to the waves is zero independently of the position of the model among the waves. For this particular model, these values are given by the roots of the equation $\tan u = u$; the corresponding values of λ/L are 0·7, 0·41, 0·29, ... Intermediate values of the ratio give maximum values for the amplitude of the fluctuations in resistance.

Summary

The wave resistance of a ship in still water can be calculated to a certain degree of approximation after making various assumptions. Similar calculations are now made for a ship among free surface waves of small height; the additional resistance, which may be negative, is considered as, to a similar degree of approximation, the horizontal resultant of the additional pressures due to the free surface waves.

The cases considered are (i) when the waves are stationary relative to the model, free transverse waves moving at the same speed, and also the case of a model on the waves left by another model in advance and moving at the same speed, (ii) a model, not free to pitch, in transverse waves moving with the speed appropriate to their wave-length.

It is shown that the additional horizontal forces may be of the same order as the wave resistance in still water even when the ratio of wave-height to wave-length has only a moderate value.

The various cases are discussed in relation to available experimental results.

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The lift and moment on a flat plate in a stream of finite width

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1. The problem of the lift on a flat plate in a stream between parallel rigid walls has been solved in an exact form, by using a suitable conformal transformation, by Tomotika (1934), who also gives an expansion for the lift in the particular case when the mid-point of the plate is midway between the walls; a similar solution for the moment on the plate does not seem to have been given. The method used in the following paper is quite different and is, perhaps, of sufficient interest to justify further examination of the problem. The flat plate is treated as the limiting case of an elliptic cylinder, and the method of solution leads directly to expansions for the lift and for the moment suitable for any position of the plate subject to the parameters being within the range necessary for convergence. Moreover, by a simple modification, expansions for lift and moment are obtained when the stream is bounded by parallel free surfaces, taking the boundary condition in an approximate form; and a further modification gives the corresponding results when one surface is rigid and the other free. A brief examination is also made of the moment for an elliptic cylinder.

General expressions

2. Consider the two-dimensional motion due to a cylinder placed in a uniform stream bounded by plane parallel walls, including circulation round the cylinder. Let C be the contour of the cross-section of the cylinder, and take the origin O so that the parallel walls are given by $y = a$, and $y = -b$, respectively. To simplify the argument, we assume that a position can be found for O such that a circle can be drawn, with centre O, entirely in the liquid and enclosing the contour C.

With w for the complex potential function, we take

$$\frac{dw}{dz} = c + \sum_0 \frac{i^{n+1} n!\, A_n}{z^{n+1}} + \frac{dw_1}{dz}. \tag{1}$$

In (1), c is the velocity of the stream, in the negative direction of Ox, the series is a suitable expansion for the singularities of the potential

function within the contour C', and the last term is to be determined so as to satisfy the boundary conditions on the walls. These conditions are

$$I\frac{dw}{dz} = 0; \quad z = x+ia,\ z = x-ib, \tag{2}$$

where I denotes the imaginary part.

To satisfy these conditions we replace the series in (1) by

$$\int_0^\infty F(\kappa)\,e^{i\kappa z}d\kappa, \quad \text{for } z = x+ia,$$

$$-\int_0^\infty F(-\kappa)\,e^{-i\kappa z}d\kappa, \quad \text{for } z = x-ib,$$

where
$$F(\kappa) = \sum_0 A_n\kappa^n. \tag{3}$$

We may build up an expression for dw_1/dz by successive images. Taking the expressions in (3), a single reflexion at a plane wall changes $F(\kappa)$ into the conjugate complex $F^*(\kappa)$; if the reflexion is at the upper wall ($y = a$) the contribution to dw_1/dz valid in the liquid is

$$\int_0^\infty F^*(\kappa)\,e^{-i\kappa z-2\kappa a}\,d\kappa, \tag{4}$$

while if the reflexion is at the lower wall ($y = -b$), the corresponding form is

$$-\int_0^\infty F^*(-\kappa)\,e^{i\kappa z-2\kappa b}d\kappa. \tag{5}$$

Taking successive reflexions at the two walls, the contributions of the infinite sets of image systems may be summed, and we obtain finally

$$\begin{aligned}\frac{dw}{dz} = c+\sum_0 \frac{i^{n+1}n!\,A_n}{z^{n+1}} &+\int_0^\infty \frac{F^*(\kappa)\,e^{-i\kappa z-2\kappa a}-F^*(-\kappa)\,e^{i\kappa z-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa \\ &-\int_0^\infty \frac{F(-\kappa)\,e^{-i\kappa z-2\kappa d}-F(\kappa)\,e^{i\kappa z-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa,\end{aligned} \tag{6}$$

where $d = a+b$, and $F(\kappa) = \Sigma A_n\kappa^n$.

It may easily be verified directly, by using (4) and (5), that (6) satisfies the boundary conditions (2).

3. We now calculate the forces on the cylinder from the expression

$$X - iY = \tfrac{1}{2}\rho i \oint \left(\frac{dw}{dz}\right)^2 dz. \tag{7}$$

We write (6) in the form

$$\frac{dw}{dz} = c + \Sigma \frac{i^{n+1} n!\, A_n}{z^{n+1}} + \Sigma B_n z^n, \tag{8}$$

where
$$B_n = \int_0^\infty \frac{(-i)^n F^*(\kappa)\, e^{-2\kappa a} - i^n F^*(-\kappa)\, e^{-2\kappa b}}{1 - e^{-2\kappa d}} \frac{\kappa^n}{n!} d\kappa$$
$$- \int_0^\infty \frac{\{(-i)^n F(-\kappa) - i^n F(\kappa)\}\, e^{-2\kappa d}}{1 - e^{-2\kappa d}} \frac{\kappa^n}{n!} d\kappa. \tag{9}$$

From (7) and (8), we obtain

$$X - iY = -2\pi\rho(icA_0 + \Sigma i^{n+1} n!\, A_n B_n). \tag{10}$$

If Γ is the circulation round the cylinder, we have $\Gamma = 2\pi A_0$; further, using (9), we easily obtain $X = 0$, and

$$Y = \rho c\Gamma + 2\pi\rho \int_0^\infty \frac{F(\kappa)\, F^*(\kappa)\, e^{-2\kappa a} - F(-\kappa)\, F^*(-\kappa)\, e^{-2\kappa b}}{1 - e^{-2\kappa d}} d\kappa. \tag{11}$$

For the moment about the origin we have

$$M = -\tfrac{1}{2}\rho R \oint z \left(\frac{dw}{dz}\right)^2 dz \tag{12}$$

$$= 2\pi\rho R i\{cA_1 + \Sigma i^n (n+1)!\, A_{n+1} B_n\}, \tag{13}$$

where R denotes the real part.

Using (9), this may be expressed in the form

$$M = 2\pi\rho R i \Bigg[cA_1 + \int_0^\infty \frac{F'(\kappa)\, F^*(\kappa)\, e^{-2\kappa a} - F'(-\kappa)\, F^*(-\kappa)\, e^{-2\kappa b}}{1 - e^{-2\kappa d}} d\kappa$$
$$- \int_0^\infty \frac{\{F'(\kappa)\, F(-\kappa) - F'(-\kappa)\, F(\kappa)\}\, e^{-2\kappa d}}{1 - e^{-2\kappa d}} d\kappa \Bigg]. \tag{14}$$

To complete the solution of the problem in any given case we have to determine the function $F(\kappa)$ so that the boundary condition of zero normal velocity is satisfied over the contour C.

ELLIPTIC CYLINDER

4. We take the contour C to be an ellipse, of semi-axes a' and b', with its major axis making an acute angle θ with the positive direction of Ox, and we take the origin at the centre of the ellipse.

In terms of a complex variable $\zeta\ (=\xi+i\eta)$, we take

$$z = pe^{i\theta}\cosh\zeta, \tag{15}$$

and the contour C is given by

$$\xi=\xi_0;\ p\cosh\xi_0 = a';\ p\sinh\xi_0 = b'. \tag{16}$$

We now write

$$\frac{dw}{d\zeta} = cpe^{i\theta}\sinh\zeta+\frac{i\Gamma}{2\pi}-\sum_1 i^n b_n e^{-n\zeta}+pe^{i\theta}\sinh\zeta\frac{dw_1}{dz}, \tag{17}$$

the second and third terms being in a suitable form in the elliptic coordinates; to obtain $F(\kappa)$ in terms of the new coefficients b_n we have to compare these two terms with the series in (6), noting that

$$dz/d\zeta = pe^{i\theta}\sinh\zeta.$$

For this purpose we put the series in (6) into the form in (3) valid for the upper surface; under the same condition it can be shown that

$$\frac{i\Gamma}{2\pi}-\Sigma i^n b_n e^{-n\zeta} = pe^{i\theta}\sinh\zeta\int_0^\infty\left\{\frac{\Gamma}{2\pi}J_0(\kappa pe^{i\theta})+i\Sigma b_n J_n(\kappa pe^{i\theta})\right\}e^{i\kappa z}\,d\kappa. \tag{18}$$

Hence, by comparison, we obtain

$$F(\kappa) = \frac{i\Gamma}{2\pi}J_0(\kappa pe^{i\theta})+i\Sigma b_n J_n(\kappa pe^{i\theta}), \tag{19}$$

and
$$\begin{aligned}\frac{dw}{d\zeta} &= cpe^{i\theta}\sinh\zeta+\frac{i\Gamma}{2\pi}-\Sigma i^n b_n e^{-n\zeta}\\ &\quad+pe^{i\theta}\sinh\zeta\int_0^\infty\frac{F(\kappa)e^{-i\kappa z-2\kappa a}-F^*(-\kappa)e^{i\kappa z-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa\\ &\quad-pe^{i\theta}\sinh\zeta\int_0^\infty\frac{\{F(-\kappa)e^{-i\kappa z}-F(\kappa)e^{i\kappa z}\}e^{-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa.\end{aligned} \tag{20}$$

We now express this in the form

$$\frac{dw}{d\zeta} = \Sigma(C_n e^{n\zeta}+D_n e^{-n\zeta}). \tag{21}$$

by substituting in (20) the expansions

$$\kappa p e^{i\theta} e^{\pm i\kappa z} \sinh \zeta = 2 \sum_{1} (\pm i)^{n-1} n J_n(\kappa p e^{i\theta}) \sinh n\zeta, \tag{22}$$

We obtain, for $n > 1$,

$$C_n = \int_0^\infty \frac{(-i)^{n-1} F^*(\kappa) e^{-2\kappa a} - i^{n-1} F^*(-\kappa) e^{-2\kappa b}}{1 - e^{-2\kappa d}} n J_n(\kappa p e^{i\theta}) \frac{d\kappa}{\kappa}$$
$$- \int_0^\infty \frac{\{(-i)^{n-1} F(-\kappa) - i^{n-1} F(\kappa)\} e^{-2\kappa d}}{1 - e^{-2\kappa d}} n J_n(\kappa p e^{i\theta}) \frac{d\kappa}{\kappa}, \tag{23}$$

$$D_n = -i^n b_n - \int_0^\infty \frac{(-i)^{n-1} F^*(\kappa) e^{-2\kappa a} - i^{n-1} F^*(-\kappa) e^{-2\kappa b}}{1 - e^{-2\kappa d}} n J_n(\kappa p e^{i\theta}) \frac{d\kappa}{\kappa}$$
$$+ \int_0^\infty \frac{\{(-i)^{n-1} F(-\kappa) - i^{n-1} F(\kappa)\} e^{-2\kappa d}}{1 - e^{-2\kappa d}} n J_n(\kappa p e^{i\theta}) \frac{d\kappa}{\kappa}, \tag{24}$$

while for $n = 1$, C_1 has the additional term $\frac{1}{2} c p e^{i\theta}$ and D_1 the additional term $-\frac{1}{2} c p e^{i\theta}$.

The boundary condition on the contour C is that the real part of $dw/d\zeta$ should be zero for $\xi = \xi_0$; this gives

$$D_n^* = -e^{2n\xi_0} C_n. \tag{25}$$

Using this in (23) and (24) we obtain an infinite set of equations for the coefficients b_n; these are, for $n > 1$,

$$i b_n = \int_0^\infty \frac{H(\kappa)}{1 - e^{-2\kappa d}} \frac{d\kappa}{\kappa},$$

$$\begin{aligned} H(\kappa) = &\{n q^n F(\kappa) J_n(\kappa p e^{-i\theta}) + (-1)^n n F^*(\kappa) J_n(\kappa p e^{i\theta})\} e^{-2\kappa a} \\ &+ \{(-1)^n n q^n F(-\kappa) J_n(\kappa p e^{-i\theta}) + n F^*(-\kappa) J_n(\kappa p e^{i\theta})\} e^{-2\kappa b} \\ &- n[\{q^n F^*(-\kappa) + (-1)^n q^n F^*(\kappa)\} J_n(\kappa p e^{-i\theta}) \\ &+ \{(-1)^n F(-\kappa) + F(\kappa)\} J_n(\kappa p e^{i\theta})] e^{-2\kappa d}, \end{aligned} \tag{26}$$

with a similar expression for ib_1 including an additional term

$$-\tfrac{1}{2} c p e^{i\theta} + \tfrac{1}{2} c p q e^{-i\theta},$$

and with $q = e^{2\xi_0}$.

By using (19), these results may be combined into an integral equation for the function $F(\kappa)$; it is

$$F(\kappa) = \frac{\Gamma}{2\pi} J_0(\kappa p e^{i\theta}) - \tfrac{1}{2} c p (e^{i\theta} - q e^{-i\theta}) J_1(\kappa p e^{i\theta})$$
$$+ \int_0^\infty \frac{\{F(v) G_1 + F^*(v) G_2\} e^{-2va} + \{F^*(-v) G_3 + F(-v) G_4\} e^{-2vb}}{1 - e^{-2vd}} \frac{dv}{v}$$
$$- \int_0^\infty \frac{\{F^*(-v) G_1 + F(-v) G_2 + F(v) G_3 + F^*(v) G_4\} e^{-2vd}}{1 - e^{-2vd}} \frac{dv}{v}, \tag{27}$$

with
$$\left.\begin{aligned} G_1 &= \Sigma n q^n J_n(vpe^{-i\theta}) J_n(\kappa p e^{i\theta}), \\ G_2 &= \Sigma(-1)^n n J_n(vpe^{i\theta}) J_n(\kappa p e^{i\theta}), \\ G_3 &= \Sigma n J_n(vpe^{i\theta}) J_n(\kappa p e^{i\theta}), \\ G_4 &= \Sigma(-1)^n n q^n J_n(vpe^{-i\theta}) J_n(\kappa p e^{i\theta}). \end{aligned}\right\} \tag{28}$$

Flat plate between parallel walls

5. We consider the limiting case obtained by making ξ_0 zero, that is, by putting $q = 1$ in the previous results. The cylinder reduces to a flat plate of width $2p$, at an angle θ to the direction of the stream, and with its mid-point at distances a, b from the upper and lower boundaries, respectively.

We write
$$\Gamma = 2\pi kcp\sin\theta; \quad F(\kappa) = cp\sin\theta f(\kappa). \tag{29}$$

The equation for $f(\kappa)$ is
$$\begin{aligned} f(\kappa) = kJ_0(\kappa p e^{i\theta}) - iJ_1(\kappa p e^{i\theta}) \\ + \int_0^\infty \frac{\{f(v)\,G_1 + f^*(v)\,G_2\}\,e^{-2va} + \{f^*(-v)\,G_3 + f(-v)\,G_4\}\,e^{-2vb}}{1-e^{-2vd}}\frac{dv}{v} \\ - \int_0^\infty \frac{\{f^*(-v)\,G_1 + f(-v)\,G_2 + f(v)\,G_3 + f^*(v)\,G_4\}\,e^{-2vd}}{1-e^{2vd}}\frac{dv}{v}, \end{aligned} \tag{30}$$
G_1, G_2, G_3, G_4 being given by (28) with $q = 1$.

We approximate to $f(\kappa)$ by successive substitution of approximations for $f(\kappa)$ in the integrals of (30), repeating the process as far as may be desired. Our object is to obtain the various quantities ultimately in power series in p/d, or alternatively in p/a or p/b, assuming these ratios to be less than unity. The expansion for $f(\kappa)$ is most readily obtained by replacing the Bessel-functions in (30) by their power series as far as necessary so as to give all terms up to a required order in the final results. We shall develop these expansions up to terms of order $(p/d)^4$; except for the length of the expressions, the expansions could readily be taken to a higher order. It is sufficient, for the present purpose, to take as the first approximation
$$f(v) = k - \tfrac{1}{2}ivpe^{i\theta} - \tfrac{1}{4}kv^2p^2e^{2i\theta} + \tfrac{1}{16}iv^3p^3e^{3i\theta} + \tfrac{1}{64}kv^4p^4e^{4i\theta}. \tag{31}$$

Further, to this order, it is sufficient to replace G_1 by
$$\begin{aligned} G_1 = \tfrac{1}{2}\kappa p e^{i\theta}(\tfrac{1}{2}vpe^{-i\theta} - \tfrac{1}{16}v^3p^3e^{-3i\theta}) \\ + \tfrac{1}{8}\kappa^2p^2e^{2i\theta}(\tfrac{1}{4}v^2p^2e^{-2i\theta}) - \tfrac{1}{16}\kappa^3p^3e^{3i\theta}(\tfrac{1}{2}vpe^{-i\theta}), \end{aligned} \tag{32}$$
and G_2, G_3, G_4 by similar expressions.

We now obtain the result of putting in the integrals of (30) a typical term $v^n p^n e^{in\theta}$ or $iv^n p^n e^{in\theta}$ instead of $f(v)$; we then apply these results to each of the terms in (31) and repeat the process until we have obtained all the terms of the required order. For the integrations with respect to v which occur in the process we use the notation

$$r_n = \int_0^\infty \frac{e^{-2va} - (-1)^n e^{-2vb}}{1 - e^{2vd}} v^n dv, \tag{33}$$

$$r_n = \int_0^\infty \frac{e^{-2vd}}{1 - e^{-2vd}} v^n dv, \tag{34}$$

n not being zero in the second case; these integrals may be evaluated in finite form.

We now give the result of this process; we obtain

$$f(\kappa) = k - \tfrac{1}{2} i\kappa p B_1 e^{i\theta} - \tfrac{1}{4}\kappa^2 p^2 B_2 e^{2i\theta} + \tfrac{1}{16} i\kappa^3 p^3 B_3 e^{3i\theta} + \tfrac{1}{64}\kappa^4 p^4 B_4 e^{4i\theta} + \ldots, \tag{35}$$

with

$$\left.\begin{aligned} B_1 &= 1 + kpr_0 \sin\theta + \tfrac{1}{2}p^2(r_1 - 2r_1' \cos 2\theta) \\ &\quad + \tfrac{1}{8}kp^3\{r_2(4\sin^3\theta - \sin\theta) + 4r_0(r_1 - 2r_1'\cos 2\theta)\sin\theta\} \\ &\quad - \tfrac{1}{8}p^4\{r_3 \cos 2\theta - 2r_3' \cos 4\theta - 2(r_1 - 2r_1' \cos 2\theta)^2\} + \ldots, \\ B_2 &= k - \tfrac{1}{4}kp^2(r_1 - 2r_1')\cos 2\theta + \tfrac{1}{8}p^3 r_2 \sin\theta + \ldots, \\ B_3 &= 1 + kpr_0 \sin\theta + \ldots, \\ B_4 &= k + \ldots. \end{aligned}\right\} \tag{36}$$

6. We consider in particular the case in which the circulation is such that the fluid velocity remains finite at the rear edge of the plate; the condition for this in an infinite stream is

$$\Gamma = 2\pi cp \sin\theta, \quad \text{or } k = 1.$$

Returning to the expression (21) for the elliptic cylinder, the condition requires that the imaginary part of $dw/d\zeta$ should be zero for $\zeta = \xi_0 + \pi i$. This gives, after putting $\xi_0 = 0$ for the flat plate,

$$\frac{i\Gamma}{2\pi} + \Sigma(-1)^n (C_n - C_n^*) = 0. \tag{37}$$

From (23) we obtain

$$\begin{aligned} \Sigma(-1)^n C_n = &-\tfrac{1}{2}cpe^{i\theta} \\ &- \Sigma\int_0^\infty \frac{i^{n-1}F^*(\kappa)e^{-2\kappa a} - (-i)^{n-1}F^*(-\kappa)e^{-2\kappa b}}{1 - e^{-2\kappa d}} nJ_n(\kappa pe^{i\theta})\frac{d\kappa}{\kappa} \\ &+ \Sigma\int_0^\infty \frac{\{i^{n-1}F(-\kappa) - (-i)^{n-1}F(\kappa)\}e^{-2\kappa d}}{1 - e^{-2\kappa d}} nJ_n(\kappa pe^{i\theta})\frac{d\kappa}{\kappa}. \end{aligned} \tag{38}$$

Hence, writing
$$\left.\begin{aligned} A_1 &= J_0(\kappa p e^{i\theta}) + iJ_1(\kappa p e^{i\theta}),\\ A_2 &= J_0(\kappa p e^{i\theta}) - iJ_1(\kappa p e^{i\theta}), \end{aligned}\right\} \tag{39}$$
this gives
$$\Sigma(-1)^n C_n = -\tfrac{1}{2}cpe^{i\theta} - \tfrac{1}{2}pe^{i\theta}\int_0^\infty \frac{A_1 F^*(\kappa)\,e^{-2\kappa a} - A_2 F^*(-\kappa)\,e^{-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa$$
$$+\tfrac{1}{2}pe^{i\theta}\int_0^\infty \frac{\{A_1 F(-\kappa) - A_2 F(\kappa)\}\,e^{-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa. \tag{40}$$

Hence the equation for k, to give the required circulation $2\pi kcp\sin\theta$, is
$$k = 1 + Ipe^{i\theta}\int_0^\infty \frac{A_1 f^*(\kappa)\,e^{-2\kappa a} - A_2 f^*(-\kappa)\,e^{-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa$$
$$-Ipe^{i\theta}\int_0^\infty \frac{\{A_1 f(-\kappa) - A_2 f(\kappa)\}\,e^{-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa, \tag{41}$$
with A_1, A_2 given by (39).

We substitute (35) in (41) and also use the power series for A_1 and A_2: carrying out the integrations and using the same notation as before, we find
$$\begin{aligned} k = {} & 1 + kpr_0\sin\theta + \tfrac{1}{2}p^2 r_1(k\cos 2\theta + B_1) - p^2 r_1'(k+B_1)\cos 2\theta \\ & -\tfrac{1}{4}p^3 r_2(k\sin 3\theta + B_1\sin\theta - B_2\sin\theta) \\ & -\tfrac{1}{16}p^4 r_3(k\cos 4\theta + 2B_1\cos 2\theta + 2B_2 + B_3\cos 2\theta) \\ & +\tfrac{1}{8}p^4 r_3'(k + 2B_1 + 2B_2 + B_3)\cos 4\theta + \dots. \end{aligned} \tag{42}$$

Finally we substitute from (36) and solve the equation for k; we obtain
$$\left.\begin{aligned} k &= 1 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4 + \dots,\\ a_1 &= r_0\sin\theta,\\ a_2 &= r_0^2\sin^2\theta + r_1\cos^2\theta - 2r_1'\cos 2\theta,\\ a_3 &= r_0^3\sin^3\theta + 2r_0 r_1\sin\theta\cos^2\theta - \tfrac{1}{4}r_2\sin 3\theta - 4r_0 r_1'\sin\theta\cos 2\theta,\\ a_4 &= r_0^4\sin^4\theta + 3r_0^2 r_1\sin^2\theta\cos^2\theta - \tfrac{1}{2}r_0 r_2\sin\theta\sin 3\theta\\ &\quad + \tfrac{1}{4}r_1^2(3 - 6\sin^2\theta + 4\sin^4\theta) - \tfrac{1}{16}r_3(2 + 3\cos 2\theta + \cos 4\theta)\\ &\quad - 6r_0^2 r_1'\sin^2\theta\cos 2\theta - 3r_1 r_1'\cos^2\theta\cos 2\theta + 3r_1'^2\cos^2 2\theta + \tfrac{3}{4}r_3'\cos 4\theta. \end{aligned}\right\} \tag{43}$$

7. We may now obtain the lift from (11) and we express it in terms of the corresponding lift in an infinite stream; that is, we write
$$Y = L;\quad Y_0 = L_0 = 2\pi\rho c^2 p\sin\theta.$$

Using (35) in (11) we obtain

$$L/L_0 = k + k^2 p r_0 \sin\theta + k p^2 r_1 B_1 \sin^2\theta - \tfrac{1}{4} p^3 r_2 (2kB_2 \cos 2\theta - B_1^2) \sin\theta$$
$$- \tfrac{1}{8} p^4 r_3 (kB_3 \sin 3\theta - 2B_1 B_2 \sin\theta) \sin\theta + \dots \quad (44)$$

Substituting from (36) and (43) and collecting the terms we obtain

$$\left.\begin{aligned}
L/L_0 &= 1 + b_1 p + b_2 p^2 + b_3 p^3 + b_4 p^4 + \dots,\\
b_1 &= 2r_0 \sin\theta,\\
b_2 &= 3r_0^2 \sin^2\theta + r_1 - 2r_1' \cos 2\theta,\\
b_3 &= 4r_0^3 \sin^3\theta + 2r_0 r_1 (2\sin\theta - \sin^3\theta) - r_2 \sin\theta \cos 2\theta - 8 r_0 r_1' \sin\theta \cos 2\theta,\\
b_4 &= 5r_0^4 \sin^4\theta + 3r_0^2 r_1 (3\sin^2\theta - 2\sin^4\theta) + \tfrac{3}{4} r_1^2 - \tfrac{3}{8} r_3 \cos 2\theta\\
&\quad - \tfrac{1}{2} r_0 r_2 (7\sin^2\theta - 12\sin^4\theta) - 18 r_0^2 r_1' \sin^2\theta \cos 2\theta - 3 r_1 r_1' \cos 2\theta\\
&\quad + 3 r_1'^2 \cos^2 2\theta + \tfrac{3}{4} r_3' \cos 4\theta.
\end{aligned}\right\} \quad (45)$$

The integrals given in (33) and (34) give for the coefficients,

$$\left.\begin{aligned}
r_0 &= \frac{\pi}{2d} \tan\alpha,\\
r_1 &= \left(\frac{\pi}{2d}\right)^2 \sec^2\alpha,\\
r_2 &= 2\left(\frac{\pi}{2d}\right)^3 \sec^2\alpha \tan\alpha,\\
r_3 &= 2\left(\frac{\pi}{2d}\right)^4 \sec^2\alpha (\sec^2\alpha + 2\tan^2\alpha),\\
r_1' &= \frac{\pi^2}{24 d^2}; \quad r_3' = \frac{\pi^4}{240 d^4},
\end{aligned}\right\} \quad (46)$$

with $\alpha = \pi(b-a)/2d$.

We may derive limiting cases from (45). If we make b and d infinite, we have a semi-infinite stream bounded by an upper plane wall; the limiting values of the coefficients are then

$$r_0 = \frac{1}{2a}; \quad r_1 = \frac{1}{4a^2}; \quad r_2 = \frac{1}{4a^3}; \quad r_3 = \frac{3}{8a^4}; \quad r_1' = r_3' = 0. \quad (47)$$

With these values we obtain

$$L/L_0 = 1+\frac{1}{2}\left(\frac{2p}{a}\right)\sin\theta+\frac{1}{16}\left(\frac{2p}{a}\right)^2(1+3\sin^2\theta)$$
$$+\frac{1}{32}\left(\frac{2p}{a}\right)^3(\sin\theta+3\sin^3\theta)-\frac{1}{512}\left(\frac{2p}{a}\right)^4(3-13\sin^2\theta-22\sin^4\theta)+\dots \tag{48}$$

This agrees with the expansion which may be found by the same method applied directly to this case. If we make a and d infinite, the stream is bounded by a lower plane wall. In this case the coefficients have the same numerical values, but r_0 and r_2 are now negative, and we see that the result is the same as (48) but with the terms in the odd powers of $2p/a$ negative.

Another special case is when the mid-point of the plate is midway between the walls, or $a = b = \frac{1}{2}d$. In this case

$$r_0 = 0;\quad r_1 = \pi^2/4d^2;\quad r_2 = 0;\quad r_3 = \pi^4/8d^4;$$
$$r_1' = \pi^2/24d^2;\quad r_3' = \pi^4/240d^4, \tag{49}$$

and we obtain

$$L/L_0 = 1+\frac{\pi^2}{24}\left(\frac{2p}{d}\right)^2(1+\sin^2\theta)-\frac{\pi^4}{7680}\left(\frac{2p}{d}\right)^4(11-53\sin^2\theta-22\sin^4\theta)+\dots \tag{50}$$

This agrees with the expansion given by Tomotika (1934) for this particular case.

In general, calculations may be made from (45) and (46), and the variation in lift examined as the plate is moved across the channel. The following values illustrate this for one particular case:

$\theta = 10°$; $2p/d = 0{\cdot}2$

a/d	0·3	0·4	0·5	0·6	0·7
L/L_0	1·071	1·037	1·017	1·002	0·989

8. We now obtain a similar expansion for the moment of the forces about the origin. If M_0 is the moment in an infinite stream,

$$M_0 = \pi\rho c^2p^2\sin\theta\cos\theta. \tag{51}$$

Using (35) in (14), we obtain, after some reduction,

$$M/M_0 = B_1+kpr_0B_1\sin\theta+2kp^2r_1B_2\sin^2\theta$$
$$-4p^2r_1'(kB_2-\tfrac{1}{2}B_1^2)\sin^2\theta-\tfrac{1}{8}p^3r_2\{3kB_3(\sin\theta-4\sin^3\theta)-2B_1B_2\sin\theta\}$$
$$-\tfrac{1}{8}p^4r_3(4kB_4\cos2\theta-2B_1B_3)\sin^2\theta$$
$$+p^4r_3'(kB_4-2B_1B_3+2B_2^2)\sin^2\theta\cos2\theta+\dots \tag{52}$$

Substituting from (36) and (43), and collecting the various terms, we obtain

$$M/M_0 = 1+c_1 p+c_2 p^2+c_3 p^3+c_4 p^4+\dots \tag{53}$$

$$\left.\begin{aligned}
c_1 &= 2r_0\sin\theta,\\
c_2 &= 3r_0^2\sin^2\theta+\tfrac{1}{2}r_1(1+4\sin^2\theta)-r_1',\\
c_3 &= 4r_0^3\sin^3\theta+r_0r_1(3\sin\theta+2\sin^3\theta)\\
&\quad -\tfrac{1}{4}r_2(\sin\theta-8\sin^3\theta)-2r_0r_1'(3\sin\theta-4\sin^3\theta),\\
c_4 &= 5r_0^4\sin^4\theta+\tfrac{15}{2}r_0^2r_1\sin^2\theta-2r_0r_2(\sin^2\theta-3\sin^4\theta)\\
&\quad +\tfrac{1}{4}r_1^2(1+14\sin^2\theta-12\sin^4\theta)-\tfrac{1}{8}r_3(1-8\sin^4\theta)\\
&\quad -3r_0^2r_1'(5\sin^2\theta-8\sin^4\theta)-r_1r_1'(1+10\sin^2\theta-20\sin^4\theta)\\
&\quad +r_1'^2(1+6\sin^2\theta-16\sin^4\theta)+\tfrac{1}{4}r_3'(1-4\sin^2\theta).
\end{aligned}\right\} \tag{54}$$

When b and d are made infinite, this reduces to the expression for a semi-infinite stream with an upper plane boundary, namely

$$M/M_0 = 1+\frac{1}{2}\left(\frac{2p}{a}\right)\sin\theta+\frac{1}{32}\left(\frac{2p}{a}\right)^2(1+10\sin^2\theta)$$
$$+\frac{5}{128}\left(\frac{2p}{a}\right)^3(\sin\theta+4\sin^3\theta)-\frac{1}{512}\left(\frac{2p}{a}\right)^4(1-14\sin^2\theta-40\sin^4\theta)+\dots \tag{55}$$

There is also a similar reduction for a lower plane boundary.

With $a = b = \frac{1}{2}d$, the mid-point of the plate being midway between the walls, we have, from (49),

$$M/M_0 = 1+\frac{\pi^2}{48}\left(\frac{2p}{a}\right)^2(1+6\sin^2\theta)$$
$$-\frac{\pi^4}{23040}\left(\frac{2p}{d}\right)^4(11-174\sin^2\theta-170\sin^4\theta)+\dots \tag{56}$$

For the general case, we have (54) with the coefficients given by (46). As a numerical example, we obtain the following values:

$\theta = 10°$; $2p/d = 0{\cdot}2$

a/d	0·3	0·4	0·5	0·6	0·7
M/M_0	1·059	1·030	1·010	0·994	0·977

Flat plate between free surfaces

9. These results may easily be modified to give approximate expressions when the stream is bounded by parallel free surfaces. At a free surface the

resultant velocity is constant; we shall take the usual approximate form of this condition which amounts to assuming the deformation of the free surface to be small and making the tangential component of the fluid velocity constant.

Thus, instead of (2), we have the boundary conditions

$$R\frac{dw}{dz} = c; \quad z = x+ia; \quad z = x-ib. \tag{57}$$

Following out the same process as in § 2, the appropriate form is now

$$\frac{dw}{dz} = c+\Sigma\frac{i^{n+1}n!A_n}{z^{n+1}}-\int_0^\infty \frac{F^*(\kappa)\,e^{-i\kappa z-2\kappa a}-F^*(-\kappa)\,e^{i\kappa z-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa$$
$$-\int_0^\infty \frac{F(-\kappa)\,e^{-i\kappa z-2\kappa d}-F(\kappa)\,e^{i\kappa z-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa, \tag{58}$$

and it may be verified directly that this form satisfies the boundary conditions (57).

It follows that the expressions for the lift and moment are now

$$Y = \rho c\Gamma-2\pi\rho\int_0^\infty \frac{F(\kappa)\,F^*(\kappa)\,e^{-2\kappa a}-F(-\kappa)\,F^*(-\kappa)\,e^{-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa, \tag{59}$$

$$M = 2\pi\rho Ri\Bigg[cA_1-\int_0^\infty \frac{F'(\kappa)\,F^*(\kappa)\,e^{-2\kappa a}-F'(-\kappa)\,F^*(-\kappa)\,e^{-2\kappa b}}{1-e^{-2\kappa d}}\,d\kappa$$
$$-\int_0^\infty \frac{\{F'(\kappa)\,F(-\kappa)-F'(-\kappa)\,F(\kappa)\}\,e^{-2\kappa d}}{1-e^{-2\kappa d}}\,d\kappa\Bigg]. \tag{60}$$

It is clear that we may write down the expansions from those in the previous sections by replacing each coefficient r_n by $-r_n$, and leaving the coefficients r'_n unaltered in sign.

Hence, instead of (45) we have,

$$L/L_0 = 1+b_1\,p+b_2\,p^2+b_3\,p^3+b_4\,p^4+\dots. \tag{61}$$

$$\left.\begin{aligned}
b_1 &= -2r_0\sin\theta,\\
b_2 &= 3r_0^2\sin^2\theta-r_1-2r'_1\cos 2\theta,\\
b_3 &= -4r_0^3\sin^3\theta+2r_0r_1(2\sin\theta-\sin^3\theta)\\
&\quad+r_2\sin\theta\cos 2\theta+8r_0r'_1\sin\theta\cos 2\theta,\\
b_4 &= 5r_0^4\sin^4\theta-3r_0^2r_1(3\sin^2\theta-2\sin^4\theta)+\tfrac{3}{4}r_1^2+\tfrac{3}{8}r_3\cos 2\theta\\
&\quad-\tfrac{1}{2}r_0r_2(7\sin^2\theta-12\sin^4\theta)-18r_0^2r'_1\sin^2\theta\cos 2\theta\\
&\quad+3r_1r'_1\cos 2\theta+3r'^2_1\cos^2 2\theta+\tfrac{3}{4}r'_3\cos 4\theta.
\end{aligned}\right\} \tag{62}$$

With b and d infinite, this gives, for a semi-infinite stream with an upper free surface,

$$L/L_0 = 1-\frac{1}{2}\left(\frac{2p}{a}\right)\sin\theta-\frac{1}{16}\left(\frac{2p}{a}\right)^2(1-3\sin^2\theta)$$
$$+\frac{1}{32}\left(\frac{2p}{a}\right)^3(3\sin\theta-5\sin^3\theta)+\frac{1}{512}\left(\frac{2p}{a}\right)^4(6-41\sin^2\theta+46\sin^4\theta)+\dots \tag{63}$$

With $a = b = \frac{1}{2}d$, we obtain

$$L/L_0 = 1-\frac{\pi^2}{24}\left(\frac{2p}{d}\right)^2(2-\sin^2\theta)$$
$$+\frac{\pi^4}{7680}\left(\frac{2p}{d}\right)^4(64-97\sin^2\theta+66\sin^4\theta)+\dots \tag{64}$$

For numerical comparison, we take the same case as before, and obtain the following:

$\theta = 10°$; $2p/d = 0{\cdot}2$

a/d	0·3	0·4	0·5	0·6	0·7
L/L_0	0·924	0·951	0·969	0·983	0·994

Similarly, for the moment, we have

$$M/M_0 = 1+c_1 p+c_2 p^2+c_3 p^3+c_4 p^4+\dots, \tag{65}$$

$$\left.\begin{aligned}
c_1 &= -2r_0\sin\theta,\\
c_2 &= 3r_0^2\sin^2\theta-\tfrac{1}{2}r_1(1+4\sin^2\theta)-r_1',\\
c_3 &= -4r_0^3\sin^3\theta+r_0r_1(3\sin\theta+2\sin^3\theta)\\
&\quad+\tfrac{1}{4}r_2(\sin\theta-8\sin^3\theta)+2r_0r_1'(3\sin\theta-4\sin^3\theta),\\
c_4 &= 5r_0^4\sin^4\theta-\tfrac{15}{2}r_0^2r_1\sin^2\theta-2r_0r_2(\sin^2\theta-3\sin^4\theta)\\
&\quad+\tfrac{1}{4}r_1^2(1+14\sin^2\theta-12\sin^4\theta)+\tfrac{1}{8}r_3(1-8\sin^4\theta)\\
&\quad-3r_0^2r_1'(5\sin^2\theta-8\sin^4\theta)+r_1r_1'(1+10\sin^2\theta-20\sin^4\theta)\\
&\quad+r_1'^2(1+6\sin^2\theta-16\sin^4\theta)+\tfrac{1}{4}r_3'(1-4\sin^2\theta).
\end{aligned}\right\} \tag{66}$$

With b and d infinite, we obtain

$$M/M_0 = 1-\frac{1}{2}\left(\frac{2p}{a}\right)\sin\theta-\frac{1}{32}\left(\frac{2p}{a}\right)^2(1-2\sin^2\theta)+\frac{1}{128}\left(\frac{2p}{a}\right)^3(7\sin\theta-12\sin^3\theta)$$
$$+\frac{1}{256}\left(\frac{2p}{a}\right)^4(1-8\sin^2\theta+8\sin^4\theta)+\dots \tag{67}$$

and with $a = b = \frac{1}{2}d$, we have

$$M/M_0 = 1-\frac{\pi^2}{24}\left(\frac{2p}{d}\right)^2(1+3\sin^2\theta) + \frac{\pi^4}{11520}\left(\frac{2p}{d}\right)^4(32+237\sin^2\theta-395\sin^4\theta)+\ldots. \qquad (68)$$

For numerical comparison with previous sections, we take the same numerical case:

$\theta = 10°$; $2p/d = 0{\cdot}2$

a/d	0·3	0·4	0·5	0·6	0·7
M/M_0	0·938	0·965	0·982	0·998	1·011

Plane boundary and free surface

10. Although the problem is not, perhaps, of practical interest, we may note that the same method can be extended to the case when one boundary, say the lower, is a rigid plane while the other, upper, boundary is a free surface; we note, again, that for a free surface the boundary condition is taken here in an approximate form.

Considering, as in § 2, the image systems formed by successive reflexions in the two surfaces, we see that these infinite series of images now consist of terms of alternate signs; summing these series we obtain

$$\frac{dw}{dz} = c+\Sigma\frac{i^{n+1}n!\,A_n}{z^{n+1}}-\int_0^\infty\frac{F^*(\kappa)\,e^{-i\kappa z-2\kappa a}+F^*(-\kappa)\,e^{i\kappa z-2\kappa b}}{1+e^{-2\kappa d}}d\kappa + \int_0^\infty\frac{\{F(-\kappa)\,e^{-i\kappa z}-F(\kappa)\,e^{i\kappa z}\}\,e^{-2\kappa d}}{1+e^{-2\kappa d}}d\kappa. \qquad (69)$$

It may be verified directly that (69) satisfies the boundary conditions

$$\left.\begin{aligned} R\frac{dw}{dz} &= c; \quad z = x+ia,\\ I\frac{dw}{dz} &= 0; \quad z = x-ib. \end{aligned}\right\} \qquad (70)$$

The expressions for the lift and the moment are

$$Y = \rho c\Gamma-2\pi\rho\int_0^\infty\frac{F(\kappa)\,F^*(\kappa)\,e^{-2\kappa a}+F(-\kappa)\,F^*(-\kappa)\,e^{-2\kappa b}}{1+e^{-2\kappa d}}d\kappa, \qquad (71)$$

$$M = 2\pi\rho Ri\left[cA_1-\int_0^\infty\frac{F'(\kappa)\,F^*(\kappa)\,e^{-2\kappa a}+F'(\kappa)\,F^*(-\kappa)\,e^{-2\kappa b}}{1+e^{-2\kappa d}}d\kappa + \int_0^\infty\frac{\{F'(\kappa)\,F(-\kappa)-F'(-\kappa)\,F(\kappa)\}\,e^{-2\kappa d}}{1+e^{-2\kappa d}}d\kappa\right]. \qquad (72)$$

We use the notation

$$\left.\begin{aligned} s_n &= \int_0^\infty \frac{e^{-2ra}+(-1)^n e^{-2rb}}{1+e^{-2rd}}\, r^n dr, \\ s_n' &= \int_0^\infty \frac{e^{-2rd}}{1+e^{-2rd}}\, r^n dr. \end{aligned}\right\} \tag{73}$$

By comparison with the expressions for the flat plate between two rigid boundaries, it is easily seen that we may write down the corresponding results by replacing the coefficients r_n by $-s_n$ and r_n' by $-s_n'$.

Thus we obtain, making these changes in (45),

$$\left.\begin{aligned} L/L_0 &= 1+b_1 p+b_2 p^2+b_3 p^3+b_4 p^4+\dots, \\ b_1 &= -2s_0\sin\theta, \\ b_2 &= 3s_0^2\sin^2\theta - s_1 + 2s_1'\cos 2\theta, \\ b_3 &= -4s_0^3\sin^3\theta + 2s_0 s_1(2\sin\theta-\sin^3\theta) + s_2\sin\theta\cos 2\theta - 8s_0 s_1'\sin\theta\cos 2\theta, \\ b_4 &= 5s_0^4\sin^4\theta - 3s_0^2 s_1(3\sin^2\theta - 2\sin^4\theta) + \tfrac{3}{4}s_1^2 + \tfrac{3}{8}s_3\cos 2\theta \\ &\quad - \tfrac{1}{2}s_0 s_2(7\sin^2\theta - 12\sin^4\theta) + 18s_0^2 s_1'\sin^2\theta\cos 2\theta - 3s_1 s_1'\cos 2\theta \\ &\quad + 3s_1'^2\cos^2 2\theta - \tfrac{3}{4}s_3'\cos 4\theta. \end{aligned}\right\} \tag{74}$$

Similarly, for the moment,

$$\left.\begin{aligned} M/M_0 &= 1+c_1 p+c_2 p^2+c_3 p^3+c_4 p^4+\dots, \\ c_1 &= -2s_0\sin\theta, \\ c_2 &= 3s_0^2\sin^2\theta - \tfrac{1}{2}s_1(1+4\sin^2\theta) + s_1', \\ c_3 &= -4s_0^3\sin^3\theta + s_0 s_1(3\sin\theta+2\sin^3\theta) \\ &\quad + \tfrac{1}{4}s_2(\sin\theta - 8\sin^3\theta) - 2s_0 s_1'(3\sin\theta - 4\sin^3\theta), \\ c_4 &= 5s_0^4\sin^4\theta - \tfrac{15}{2}s_0^2 s_1\sin^2\theta - 2s_0 s_2(\sin^2\theta - 3\sin^4\theta) \\ &\quad + \tfrac{1}{4}s_1^2(1+14\sin^2\theta - 12\sin^4\theta) + \tfrac{1}{8}s_3(1-8\sin^4\theta) \\ &\quad + 3s_0^2 s_1'(5\sin^2\theta - 8\sin^4\theta) - s_1 s_1'(1+10\sin^2\theta - 20\sin^4\theta) \\ &\quad + s_1'^2(1+6\sin^2\theta - 16\sin^4\theta) - \tfrac{1}{4}s_3'(1-4\sin^2\theta). \end{aligned}\right\} \tag{75}$$

From (73), the coefficients are given by

$$\left.\begin{aligned} s_0 &= \frac{\pi}{2d}\sec\alpha; \quad s_1 = \left(\frac{\pi}{2d}\right)^2 \sec\alpha\tan\alpha, \\ s_2 &= \left(\frac{\pi}{2d}\right)^3 \sec\alpha(\sec^2\alpha + \tan^2\alpha), \\ s_3 &= \left(\frac{\pi}{2d}\right)^4 \sec\alpha\tan\alpha(5\sec^2\alpha + \tan^2\alpha), \\ s_1' &= \pi^2/48d^2; \quad s_3' = 7\pi^4/1920d^4, \end{aligned}\right\} \tag{76}$$

with $\alpha = \pi(b-a)/2d$.

It may be verified that if we make b and d infinite, or a and d infinite, these expressions reduce to the former results for a semi-infinite stream bounded, respectively, by an upper free surface or by a lower rigid plane.

For the particular case, $a = b = \frac{1}{2}d$, we obtain

$$\begin{aligned} L/L_0 = 1 &- \frac{\pi}{2}\left(\frac{2p}{d}\right)\sin\theta + \frac{\pi^2}{96}\left(\frac{2p}{d}\right)^2(1+18\sin^2\theta) \\ &+ \frac{\pi^3}{192}\left(\frac{2p}{d}\right)^3(\sin\theta - 14\sin^3\theta) \\ &- \frac{\pi^4}{122880}\left(\frac{2p}{d}\right)^4(11 + 832\sin^2\theta - 3712\sin^4\theta) + \dots, \end{aligned} \tag{77}$$

and

$$\begin{aligned} M/M_0 = 1 &- \frac{\pi}{2}\left(\frac{2p}{d}\right)\sin\theta + \frac{\pi^2}{192}\left(\frac{2p}{d}\right)^2(1+36\sin^2\theta) \\ &- \frac{\pi^3}{256}\left(\frac{2p}{d}\right)^3(\sin\theta + 32\sin^3\theta) \\ &- \frac{\pi^4}{368640}\left(\frac{2p}{d}\right)^4(11 + 936\sin^2\theta - 12800\sin^4\theta) + \dots. \end{aligned} \tag{78}$$

The following numerical values may be compared with those in the previous sections:

$\theta = 10^\circ;\ 2p/d = 0{\cdot}2$

a/d	0·3	0·4	0·5	0·6	0·7
L/L_0	0·924	0·942	0·953	0·956	0·960
M/M_0	0·928	0·943	0·949	0·951	0·948

In this case the relative variation near the middle of the channel is much less than when the boundaries are of the same kind.

Elliptic cylinder

11. The expressions for a flat plate have been obtained as limiting cases of those for an elliptic cylinder. We shall consider now the general case when the cylinder is in a stream between plane parallel walls, and we shall examine the moment of the forces; further, in order to simplify the calculations, we assume in this case that there is no circulation.

Referring to § 4, we have to determine $F(\kappa)$ from (27) and (28) with $\Gamma = 0$.

The process of approximation is carried out as before, and we record the result up to terms necessary to give the moment to the required approximation. We obtain

$$F(\kappa) = -\tfrac{1}{2}cp(\tfrac{1}{2}\kappa p B_1 e^{i\theta} + \tfrac{1}{16}\kappa^2 p^2 B_2 e^{2i\theta} - \tfrac{1}{16}\kappa^3 p^3 B_3 e^{3i\theta} \\ - \tfrac{1}{64}\kappa^4 p^4 B_4 e^{4i\theta} + \tfrac{1}{384}\kappa^5 p^5 B_5 e^{5i\theta} + \ldots), \qquad (79)$$

$$\left.\begin{aligned} B_1 &= e^{i\theta} - qe^{-i\theta} + \tfrac{1}{4}p^2 r_1(2qe^{i\theta} - q^2 e^{-i\theta} - e^{-i\theta}) \\ &\quad - \tfrac{1}{2}p^2 r_1'(e^{3i\theta} - qe^{i\theta} - qe^{-3i\theta} + q^2 e^{-i\theta}) \\ &\quad - \tfrac{1}{32}p^4 r_3\{2q(e^{-i\theta} + e^{3i\theta}) - (1+q^2)(e^{i\theta} + e^{-3i\theta})\} \\ &\quad + \tfrac{1}{16}p^4 r_1^2\{(1+3q^2)e^{i\theta} - q(3+q^2)e^{-i\theta}\} \\ &\quad + \tfrac{1}{8}p^4 r_1 r_1'\{4qe^{3i\theta} - (1+3q^2)(e^{i\theta} + e^{-3i\theta}) + 2q(1+q^2)e^{-i\theta}\} \\ &\quad + \tfrac{1}{4}p^4 r_1'^2\{qe^{-5i\theta} - q^2 e^{-3i\theta} + q(1+q^2)e^{-i\theta} - 2q^2 e^{i\theta} + qe^{3i\theta} - e^{5i\theta}\} \\ &\quad + \tfrac{1}{8}p^4 r_3'(qe^{-5i\theta} + q^2 e^{-3i\theta} - qe^{3i\theta} + e^{5i\theta}) + \ldots, \\ B_3 &= e^{i\theta} - qe^{-i\theta} + \ldots, \\ B_5 &= e^{i\theta} - qe^{-i\theta} + \ldots. \end{aligned}\right\} \qquad (80)$$

B_2 and B_4 are of order p^3 and do not contribute to the value of the moment up to terms in p^4.

Using (79) and (80) in (14), we obtain, after some reduction,

$$M/\pi\rho c^2 p^2 \sin\theta\cos\theta = 1 + p^2\{\tfrac{1}{2}qr_1 + r_1'(q - 2\cos 2\theta)\} \\ + \tfrac{1}{16}p^4\{r_1^2(1+3q^2) + r_3(1+q^2-4\cos 2\theta) + 4r_1 r_1'(1+3q^2-8\cos 2\theta) \\ + 4r_1'^2(3+3q^2-8q\cos 2\theta+6\cos 4\theta) + 2r_3'(3+q^2-8q\cos 2\theta+6\cos 4\theta)\} + \ldots. \qquad (81)$$

In this expression, θ is the angle the major axis makes with the direction of the stream, a, b are the distances of the centre of the ellipse from the two walls, and the coefficients r are given in (46); further, if a', b' are the semi-axes of the ellipse, we have $p^2 = a'^2 - b'^2$ and $q = (a'+b')/(a'-b')$.

The moment for a flat plate in a stream between plane walls, and without circulation, has been obtained, by conformal transformation, by Tomotika (1933), who also gives an expansion for the case $a = b = \frac{1}{2}d$; it is

$$M/M_0 = 1 + \frac{\pi^2}{48}\left(\frac{2p}{d}\right)^2 (1 + 2\sin^2\theta) - \frac{\pi^4}{23040}\left(\frac{2p}{d}\right)^4 (11 - 106\sin^2\theta - 66\sin^4\theta) + \dots \quad (82)$$

If in the general result (81) we put $q = 1$ and use the values of the coefficients given in (46) and (49) we obtain again this particular result.

We shall use (81) to illustrate one point, namely the change in the moment when a flat plate is replaced by an elliptic cylinder whose major axis is of length equal to the width of the plate; thus we examine the effect of rounding the edges of the plate and giving it a finite thickness.

To simplify the calculation, we take the cylinder in the position given by $a = b = \frac{1}{2}d$. Then (81) gives

$$M/\pi\rho c^2 a'^2 \sin\theta\cos\theta = \lambda\left[1 + \frac{\pi^2\lambda}{1^2}\left(\frac{a'}{d}\right)^2 (2q - \cos 2\theta) + \frac{\pi^4\lambda^2}{23040}\left(\frac{a'}{d}\right)^4 \{99 + 168q^2 - (300 + 44q)\cos 2\theta + 33\cos 4\theta\} + \dots\right], \quad (83)$$

where $\lambda = 1 - b'^2/a'^2, \quad q = (a' + b')/(a' - b')$.

We begin with a flat plate of width $2a'$, and then keeping a' constant we increase b'; to simplify the calculations we have taken the position given by $\theta = 45^\circ$ and the following table shows the result of the calculation for various values of the ratio a'/d.

a'/d	0	0·1	0·2	0·3	0·4
b'/a'					
0	1·0	1·0165	1·0673	1·1561	1·2885
0·05	0·9977	1·0159	1·0714	1·1690	1·3149
0·09	0·9917	1·0113	1·0717	1·1734	1·3361
0·13	0·9830	1·0038	1·0679	1·1799	1·3484
0·2	0·9600	0·9829	1·0535	1·1773	1·3640
0·5	0·7500	0·7780	0·8655	1·0227	1·2664

For an infinite stream ($a'/d = 0$), this process of increasing the ratio b'/a' with a' constant gives a moment which steadily decreases to zero when $b' = a'$. An interesting point which arises from these calculations is that in a stream of finite width, with plane walls, the moment rises to a maximum

before decreasing to zero; with decreasing width of the channel, this maximum increases in amount and occurs at a higher value of the ratio b'/a'.

Summary

The paper gives a new treatment of the problem of a flat plate in a stream bounded by plane parallel walls, including circulation round the plate. The plate is considered as the limiting case of the elliptic cylinder; an integral equation is obtained, whose solution by continued approximation leads to expansions for the lift and moment on the plate. The solution is modified to give similar results when the stream is bounded by parallel free surfaces, taking the condition at a free surface in an approximate form; and a further modification gives the case when one boundary of the stream is a plane wall and the other is a free surface. The problem of the elliptic cylinder in general is also considered with reference to the moment of the forces when the stream is bounded by plane walls and when there is no circulation.

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Note on the sinkage of a ship at low speeds.

By T. H. Havelock in Newcastle-on-Tyne.

Zusammenfassung. Um einen Anhalt für die Zunahme des Tiefganges eines Schiffes bei genügend kleinen Geschwindigkeiten zu haben, ersetzt Verf. den eintauchenden Teil des Schiffes durch ein Halbellipsoid, dessen ebene Grenzfläche mit den Halbachsen a und b in der Höhe des Wasserspiegels liegt. Um diesen Körper nimmt er eine Potentialströmung an, für die die Wasseroberfläche eben bleibt. Aus dieser wird die Abnahme Q des Druckes nach oben berechnet und die Zunahme h des Einsinkens mittels der Gleichung $Q = a\,b\,\pi \cdot \varrho\, g \cdot h$ bestimmt. Die so gefundenen numerischen Resultate stimmen mit denen aus einer empirischen Formel von Horn für wirkliche Schiffskörper der Größenordnung nach gut überein. Weiter geht Verf. auf eine andere Hornsche Näherungsformel ein, die es erlaubt, aus dem Einsinken die Zunahme des Reibungswiderstandes eines Modelles, verglichen mit dem einer ebenen Platte, abzuschätzen.

1. The general problem of the position of relative equilibrium of a ship in uniform motion is a complicated one, and the following note deals only with a simplified form of the problem suitable for low speeds. It is generally assumed that at sufficiently low speeds the fluid motion approximates to the stream-line flow round the ship, neglecting the disturbance of the surface of the water; the sinkage is then due to the defect of vertical pressure caused by the fluid motion and should be proportional to the square of the speed. There do not seem to have been any calculations made to test whether these assumptions lead to results of the right order of magnitude. Such calculations might be carried out numerically for ordinary ship forms, but it is sufficient for the present purpose to take a simple form. We

assume the submerged part of the ship to be ellipsoidal. The solution of the corresponding potential problem is well-known, and an exact expression can easily be found for the total defect of vertical pressure, and hence we obtain a certain equivalent sinkage.

The problem is of some interest since Professor Horn[1]) has proposed to estimate the so-called form effect upon resistance by an approximate formula involving the sinkage at low speeds. The expressions obtained here for ellipsoidal forms are compared numerically with these results and with other experimental data.

2. A solid, whose surface is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1; \qquad a>b>c \quad \ldots \quad (1);$$

is moving through an infinite liquid with velocity U parallel to the axis Ox. The velocity potential of the fluid motion is given by

$$\varphi=\frac{a\,b\,c}{2-\alpha_0}\,U\,x\int_{\lambda}^{\infty}\frac{d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{1/2}(c^2+\lambda)^{1/2}} \quad \ldots \quad (2),$$

in which (x, y, z) are given in terms of orthogonal coordinates (λ, μ, ν) by

$$x^2=\frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)} \quad \ldots \quad (3),$$

with similar expressions for y and z.

In these coordinates the ellipsoid (1) is given by $\lambda=0$; and we have also

$$\alpha_0=a\,b\,c\int_0^{\infty}\frac{d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{1/2}(c^2+\lambda)^{1/2}}=\frac{2\,a\,b\,c}{(a^2-b^2)(a^2-c^2)^{1/2}}(F-E) \quad \ldots \quad (4),$$

F, E being elliptic integrals with parameters given by

$$\sin\alpha=\{(a^2-b^2)/(a^2-c^2)\}^{1/2};\ \sin\chi=(a^2-c^2)^{1/2}/a \quad \ldots \quad (5).$$

The fluid pressure is given by

$$p=p_0+\varrho\frac{\partial\varphi}{\partial t}-\frac{1}{2}\varrho\,q^2=p_0-\varrho\,U\frac{\partial\varphi}{\partial x}-\frac{1}{2}\varrho\,q^2 \quad \ldots \quad (6).$$

If (l, m, n) are the direction-cosines of the normal at any point of the ellipsoid, the required total defect of resolved pressure Q is given by

$$Q=\varrho\iint\left(U\frac{\partial\varphi}{\partial x}+\frac{1}{2}\,q^2\right)n\,dS \quad \ldots \quad (7),$$

the integral being taken over the half surface of the ellipsoid lying on one side of the xy-plane. Using well-known properties of the coordinates λ, μ, ν (as given, for example, in Lamb's, Hydrodynamics, p. 149), it can be shown that, on the ellipsoid $\lambda=0$, we have

$$\frac{\partial\varphi}{\partial x}=\frac{U}{2-\alpha_0}\left\{\alpha_0-\frac{2\,b^2c^2(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)\,\mu\,\nu}\right\},$$

and

$$\left.\begin{aligned} q^2=\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2 &=U^2\frac{b^2c^2(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)\,\mu\,\nu}\\ &+\left(\frac{\alpha_0}{2-\alpha_0}\right)^2U^2\left\{\frac{(b^2+\mu)(c^2+\mu)}{\mu(a^2+\mu)(\mu-\nu)}+\frac{(b^2+\nu)(c^2+\nu)}{\nu(a^2+\nu)(\nu-\mu)}\right\}\frac{a^2(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}\end{aligned}\right\} \quad \ldots \quad (8).$$

Further, we also have

$$n\,dS=\frac{1}{4}\,a\,b\left\{\frac{(\mu-\nu)(\nu-\mu)}{(c^2-a^2)(c^2-b^2)(a^2+\mu)(b^2+\mu)(a^2+\nu)(b^2+\nu)}\right\}^{1/2}d\mu\,d\nu. \quad (9)$$

[1]) Horn, Intern. Tagg. der Leiter der Schleppversuchsanstalten, 1937, S. 20.

Hence we obtain

$$Q = \frac{a\,b\,\varrho\,\overline{U}^2}{\{(a^2-c^2)(b^2-c^2)\}^{1/2}} \int\limits_{-c^2}^{-b^2} d\mu \int\limits_{-b^2}^{a^2} d\nu \left[\frac{\alpha_0}{2-\alpha_0} - \left(\frac{\alpha_0}{2-\alpha_0}+\frac{1}{2}\right)\frac{b^2 c^2 (a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)\,\mu\,\nu}\right.$$
$$\left. + \left(\frac{\alpha_0}{2-\alpha_0}\right)^2 \frac{a^2(a^2+\mu)(a^2+\nu)}{2(a^2-b^2)(a^2-c^2)} \left\{\frac{(b^2+\mu)(c^2+\mu)}{\mu(a^2+\mu)(\mu-\nu)} + \frac{(b^2+\nu)(c^2+\nu)}{\nu(a^2+\nu)(\nu-\mu)}\right\}\right]$$
$$\cdot \left[\frac{(\mu-\nu)(\nu-\mu)}{(a^2+\mu)(b^2+\mu)(a^2+\nu)(b^2+\nu)}\right]^{1/2} \quad \ldots \quad (10).$$

Carrying out the integrations, and writing

$$Q = \pi\, g\, \varrho\, a\, b\, h \quad \ldots \ldots \quad (11),$$

we obtain, for $a > b > c$,

$$\frac{g\,h}{U^2} = \frac{\alpha_0}{2-\alpha_0} + \frac{2+\alpha_0}{2(2-\alpha_0)}\frac{b\,c^2}{(a+b)(a^2-c^2)} + \left(\frac{\alpha_0}{2-\alpha_0}\right)^2 \frac{a(a^2+a\,b-c^2)}{2(a+b)(a^2-c^2)}$$
$$- \frac{2}{(2-\alpha_0)^2}\frac{a\,b\,c^2}{(a^2-c^2)^{3/2}(b^2-c^2)^{1/2}} \log \frac{b(a^2-c^2)^{1/2}+a(b^2-c^2)^{1/2}}{c(a^2-c^2)^{1/2}+c(b^2-c^2)^{1/2}} \quad \ldots \quad (12).$$

3. We require also the corresponding expression for an ellipsoid with $a > c > b$. This may be deduced directly from (12); or, alternatively, we may proceed as in the previous section but replacing $n\,dS$ in (7) by $m\,dS$, given by

$$m\,dS = \frac{1}{4}\,a\,c \left\{\frac{(\mu-\nu)(\nu-\mu)}{(b^2-c^2)(b^2-a^2)(a^2+\mu)(a^2+\nu)(c^2+\mu)(c^2+\nu)}\right\}^{1/2} d\mu\, d\nu \quad \ldots \quad (13).$$

After carrying out the integrations, we interchange b and c so that we may express the result by means of (11).

We obtain, for $a > c > b$

$$\frac{g\,h}{U^2} = \frac{\alpha_0}{2-\alpha_0} + \frac{2+\alpha_0}{2(2-\alpha_0)}\frac{b\,c^2}{(a+b)(a^2-c^2)} + \left(\frac{\alpha_0}{2-\alpha_0}\right)^2 \frac{a(a^2+a\,b-c^2)}{2(a+b)(a^2-c^2)}$$
$$- \frac{2}{(2-\alpha_0)^2}\frac{a\,b\,c^2}{(a^2-c^2)^{3/2}(c^2-b^2)^{1/2}} \arctan \frac{\{(a^2-c^2)(c^2-b^2)\}^{1/2}}{a\,b+c^2} \quad \ldots \quad (14).$$

In this case, instead of (4) and (5), we have

$$\alpha_0 = \frac{2\,a\,b\,c}{(a^2-c^2)(a^2-b^2)^{1/2}}(F-E),$$
$$\sin\alpha = \{(a^2-c^2)/(a^2-b^2)\}^{1/2}, \qquad \sin\chi = (a^2-b^2)^{1/2}/a \quad \ldots \ldots \quad (15).$$

4. The prolate spheroid may be considered separately, or may be deduced from the two previous cases. Taking limiting values, both (12) and (14) reduce to the expression for this case.

For $a > b$; $b = c$, we obtain

$$\frac{g\,h}{U^2} = \frac{\alpha_0}{2-\alpha_0} - \frac{2+\alpha_0}{2(2-\alpha_0)}\frac{b^2}{(a+b)^2} + \left(\frac{\alpha_0}{2-\alpha_0}\right)^2 \frac{a(a+2\,b)}{2(a+b)^2} \quad \ldots \quad (16)$$

and in this we have

$$\alpha_0 = \frac{2(1-e^2)}{e^3}\left(\frac{1}{2}\log\frac{1+e}{1-e} - e\right); \qquad e^2 = 1 - b^2/a^2 \quad \ldots \ldots \quad (17).$$

5. To apply these results to the problem under consideration we imagine a ship for which the immersed portion is ellipsoidal, the $x\,y$-plane being the water surface and the sides of the ship above water being vertical. Owing to the defect of buoyancy, which has been denoted by Q, the ship will sink in the water. This will, of course, alter the fluid motion; but for approximate comparison with experimental results, we define the equivalent sinkage h so that Q is equal to the weight of a volume of water of height h and of cross section equal to the section of the ship by the water surface; that is, h is defined by (11).

If the length, beam and draft of the ship are L, B, D respectively, then $L = 2a$, $B = 2b$, $D = c$; for $B >, <, = 2D$ we use the expressions (12), (14) and (16) respectively. The numerical values shown in Table I have been calculated from these formulae.

Table I. Values of $gh/\overline{U}^2$

B/D	$L/D = 10$	$L/D = 16$
1	0.0253	0.0138
2	.0453	.0231
3	.0612	.0318
4	.0735	.0397

6. The measured sinkage of ship models at low speeds has been analysed by Horn[1]), who has given an empirical formula derived as an average from available data for many different forms of model. His expression for the sinkage is, in the present notation

$$h = \frac{1}{200}(0.35 + P)\left\{\frac{1}{5}\left(11.25 - \frac{L}{B}\right)^2 + 2.5\right\}\left(1.3 - \frac{B}{10\,D}\right)\frac{\overline{U}^2}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where L, B, D are length, beam and draft respectively, and P is the prismatic coefficient of the form; the formula is valid, as an average, for suitable ranges of these parameters.

It should be noted that this formula is for actual measured sinkage, and is probably derived from velocities rather higher than those for which the preceding simple calculation is valid; moreover, the ellipsoid is not one of the ship forms included in the data. However we may use it to test the order of magnitude of the results. If we apply (18) to an ellipsoidal form with $L/B = 8$ and $B/D = 2$, we obtain $h = 0.0283\,\overline{U}^2/g$; this compares with the value $0.0231\,\overline{U}^2/g$ for this case given in Table I.

Horn[1]) has suggested using the sinkage at low speeds to estimate the increased frictional drag for a model compared with a flat plate; his formula for the percentage increase in the resistance R is

$$100\,\Delta R/R = 200\,g\,h/\overline{U}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19).$$

For the prolate spheroid with $L/B = 8$, the value of h in Table I gives, according to the formula (19), an increase of 4.6 per cent in the resistance.

Amtsberg[2]) has recently determined the resistance of a submerged prolate spheroid experimentally; he gives two values for the increase, namely 5.2 per cent and 3.7 per cent, the smaller value being obtained after applying certain corrections. Amtsberg also investigated certain other surfaces of revolution, for which the velocity potential is given by a source distribution along the axis. He gives numerical values of the ordinates of the surface and of the theoretical distribution of velocity along the contour; from these, it is possible to evaluate numerically the integral we have denoted by Q in the preceding sections. Taking, for example, the values given by Amtsberg for his model R 1257, we obtain approximately $Q = 0.0284\,\varrho\,\overline{U}^2$ (area of section). This gives an equivalent sinkage of $0.0284\,\overline{U}^2/g$ and, according to (19), an increase of resistance of about 5.7 per cent; the values deduced by Amtsberg from his experimental results are 7.3 and 4.9 per cent, the latter being the corrected value.

It is well-known that in models of this type the measured distribution of pressure over the surface only differs appreciably from the theoretical value near the rear end of the model. Hence the effect of this divergence upon the resolved vertical pressure will only be a small correction; taking, for example, model R 1257 and using Amtsberg's measured values of the pressure instead of the theoretical values, a rough approximation gives a factor of 0.0288 instead of 0.0284.

7. Summary. The sinkage of a model at sufficiently low speeds is assumed to be due to stream line fluid motion round the submerged part of the model, neglecting the disturbance of the water surface. Taking an ellipsoidal form for the submerged part, exact expressions are found for the total defect of vertical pressure and hence for a certain equivalent sinkage. The results are compared numerically with available data and are found to be of the right order of magnitude. Further, reference is made to Horn's approximate formula connecting the sinkage with the increase of resistance of the model compared with that of a flat plate.

949

[2]) Amtsberg, Jahrb. der Schiff-bautechn. Gesellsch., Bd. 38, 1937, S. 177.

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Waves produced by the Rolling of a Ship.

By T. H. HAVELOCK, F.R.S.

1. THE first part of the following paper deals with the surface waves produced by an elliptic cylinder, or a flat plate, submerged in water and performing small linear or rotational oscillations. The second part contains a short discussion of the energy dissipated in wave motion by a rolling ship, together with an estimate of the magnitude of this effect obtained from the preceding results.

Submerged Elliptic Cylinder.

2. The method adopted is to replace the oscillating body by some suitable distribution of sources and sinks or of doublets. Although the analysis could be extended to three-dimensional problems, we limit consideration at present to two-dimensional motion in a frictionless liquid. We begin with the solution for a horizontal doublet which was obtained for an oscillating circular cylinder (Havelock, 1917). Take the origin O in the free surface of deep water, Ox horizontal and Oy vertically upwards. Let there be a horizontal doublet of oscillating moment M cos σt at the point $(0, -f)$ in the liquid. The velocity potential ϕ is given by

$$\phi=\frac{\mathrm{M}x}{r_1^2}e^{i\sigma t}+\mathrm{M}e^{i\sigma t}\int_0^\infty\frac{g\kappa+\sigma^2-i\mu'\sigma}{g\kappa-\sigma^2+i\mu'\sigma}e^{-\kappa(f-y)}\sin\kappa x\,d\kappa, \quad . \quad . \quad (1)$$

with $r_1^2=x^2+(y+f)^2$. The real part of the expression is to be taken, and, further, the limiting value when the positive quantity μ' tends to zero; this latter process ensures that, at great distances from the origin, the motion will reduce to waves travelling outwards on either side. We may write (1) in the form

$$\phi=\frac{\mathrm{M}x}{r_1^2}e^{i\sigma t}+\frac{\mathrm{M}x}{r_2^2}e^{i\sigma t}+2\kappa_0\mathrm{M}e^{i\sigma t}\int_0^\infty\frac{e^{-\kappa(f-y)}\sin\kappa x}{\kappa-\kappa_0+i\mu}d\kappa, \quad . \quad . \quad . \quad (2)$$

with $r_2^2=x^2+(f-y)^2$ and $\mu=\mu'\sigma/g$, $\kappa_0=\sigma^2/g$.

The integral in (2) may be transformed by taking κ to be a complex

variable and integrating round suitable contours according as x is positive or negative. We obtain, taking the real part and making μ zero,

$$\phi = \frac{\mathrm{M}x}{r_1^2}\cos\sigma t + \frac{\mathrm{M}x}{r_2^2}\cos\sigma t \mp 2\kappa_0\pi\mathrm{M}e^{-\kappa_0(f-y)}\cos(\sigma t\mp\kappa_0 x)$$

$$\mp 2\kappa_0\mathrm{M}\cos\sigma t\int_0^\infty \frac{\kappa_0\cos\kappa(f-y)+\kappa\sin\kappa(f-y)}{\kappa^2+\kappa_0^2}e^{\mp\kappa x}d\kappa, \quad (3)$$

the upper or lower signs to be taken according as x is positive or negative.

The corresponding surface elevation η is given by

$$\eta = \mp\frac{2\pi\kappa_0\sigma\mathrm{M}}{g}e^{-\kappa_0 f}\sin(\sigma t\mp\kappa_0 x) - \frac{2\mathrm{M}\sigma}{g}\frac{x}{x^2+f^2}\sin\sigma t$$

$$\pm\frac{2\kappa_0\mathrm{M}\sigma}{g}\sin\sigma t\int_0^\infty\frac{\kappa_0\cos\kappa f+\kappa\sin\kappa f}{\kappa^2+\kappa_0^2}e^{\mp\kappa x}d\kappa. \quad (4)$$

The first term represents the regular waves, while the other two terms give a local oscillation whose magnitude diminishes with increasing distance from the centre of disturbance.

Similar expressions may be obtained for a source of oscillating magnitude or for a doublet with its axis in any direction. It may be remarked that for a doublet at a given point in the liquid, so far as the regular waves are concerned the direction of the axis affects only the phase of the waves and not their amplitude.

3. Consider the motion produced by an elliptic cylinder moving through an infinite liquid. If the motion of the cylinder is one of translation, it is well known that the fluid motion is that due to a certain distribution of doublets along the line joining the foci of the elliptic section of the cylinder; a similar proposition may also be readily proved when the motion is one of rotation.

In particular, let the cylinder be moving with velocity V parallel to the minor axis of the section; let S, S′ be the foci of the section and h the distance of a point on SS′ from the centre C. The moment per unit length of the doublet distribution is

$$a\mathrm{V}(a^2e^2-h^2)^{\frac{1}{2}}/\pi(a-b), \quad \ldots \ldots \ldots \quad (5)$$

in the usual notation, the axes of the doublets being perpendicular to SS′.

If the cylinder is rotating round C with angular velocity ω, the moment per unit length along SS′ is

$$\omega(a+b)h(a^2e^2-h^2)^{\frac{1}{2}}/2\pi(a-b), \quad \ldots \ldots \ldots \quad (6)$$

the axes being perpendicular to SS′.

Combining (5) and (6) with a suitable value of V, we may obtain the distribution when the cylinder is rotating about any point on the major axis.

4. Suppose that the cylinder is wholly immersed in liquid with the axis of the cylinder horizontal and at a depth f below the free surface of the liquid, and let the cylinder be making small rotational oscillations about its axis. Let the angle θ between the major axis of the section and the vertical be given by

$$\theta=\theta_0 \sin \sigma t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where θ_0 is small.

For a first approximation we neglect the effect of the free surface. The angular velocity of the cylinder is $\sigma\theta_0 \cos \sigma t$, and the velocity potential is that due to a certain distribution of doublets along the instantaneous position of the major axis. We shall make a further approximation for small oscillations and assume that this distribution is along the mean position of the major axis, that is, the vertical through the centre of the ellipse. Thus we consider the motion to be due to a distribution of horizontal doublets of oscillating magnitude along the line between the foci of the ellipse in its mean position. From (6), the moment per unit length at a distance h from the centre of the ellipse is

$$\frac{\sigma\theta_0(a+b)}{2\pi(a-b)}h(a^2e^2-h^2)^{\frac{1}{2}}\cos \sigma t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

the limits for h being $\pm ae$.

We replace M in (3) by this expression, write $f+h$ for f, and integrate with respect to h; we obtain then the velocity potential for the given distribution when the condition at the free surface is satisfied. Similarly, from (4) we may obtain complete expressions for the corresponding surface elevation. This consists of a local oscillation whose amplitude diminishes rapidly with distance from the cylinder, together with regular waves travelling out on either side. We shall examine here only the amplitude of these regular waves; from (4) and (8) the amplitude A of these waves, that is, the coefficient of $\sin(\sigma t-\kappa_0 x)$ for positive values of x, is given by

$$\begin{aligned} \mathrm{A} &= -\kappa_0^2\theta_0\frac{a+b}{a-b}\int_{-ae}^{ae} h(a^2e^2-h^2)^{\frac{1}{2}}e^{-\kappa_0(f+h)}dh \\ &= -\kappa_0^2\sqrt{(a^2-b^2)}(a+b)^2\theta_0 e^{-\kappa_0 f}\int_0^{\pi}\sin^2\theta\cos\theta\, e^{-\kappa_0 ae\cos\theta}d\theta. \quad . \quad . \qquad (9) \end{aligned}$$

This may be expressed in terms of the modified Bessel function $\mathrm{I}_n(x)$, and we obtain

$$\mathrm{A}=\frac{\pi\theta_0(a+b)^{\frac{3}{2}}}{(a-b)^{\frac{1}{2}}}\{\kappa_0 ae\mathrm{I}_0(\kappa_0 ae)-2\mathrm{I}_1(\kappa_0 ae)\}e^{-\kappa_0 f}. \quad . \quad . \quad . \quad (10)$$

If $\kappa_0 ae$ is small, that is, if the wave-length is large compared with the linear dimensions of the cylinder, the first term in the expansion of (10) gives, as an approximation,

$$\mathrm{A}=\tfrac{1}{8}\pi(a-b)(a+b)^3\kappa_0^3\theta_0 e^{-\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Consider a cylinder of given vertical dimension $2a$, with varying breadth $2b$. Naturally, for the circular cylinder ($b=a$) the disturbance is zero. It is of interest to note that for the approximation (11) the maximum wave amplitude occurs for $b=\frac{1}{2}a$, its value being then nearly twice the value for the limiting case of the flat plate ($b=0$).

Suppose that the cylinder has its major axis vertical, and is making linear horizontal oscillations in which the displacement is $d \sin \sigma t$. Then from (4) and (5) the amplitude of the regular waves is

$$A = \frac{2\kappa_0{}^2 a d}{a-b} \int_{-ea}^{ae} (a^2e^2-h^2)^{\frac{1}{2}} e^{-\kappa_0(f+h)} dh$$

$$= 2\pi\kappa_0 a d \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} I_1(\kappa_0 ae) e^{-\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If $\kappa_0 ae$ is small, this gives, approximately,

$$A = \pi\kappa_0^2 a(a+b) d e^{-\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Finally, combining (10) and (12) with $d=a\theta_0$, we obtain the amplitude for an elliptic cylinder with its major axis vertical in its mean position, and making small angular oscillations given by $\theta=\theta_0 \sin \sigma t$ about the upper end of its major axis; in this case we obtain

$$A = \pi\theta_0 \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} \{\kappa_0 ae(a+b) I_0(\kappa_0 ae) - 2(a+b+\kappa_0 a^2) I_1(\kappa_0 ae)\} e^{-\kappa_0 f}. \quad (14)$$

For $\kappa_0 ae$ small, the first term in the expansion is the same as (13) with $d=a\theta_0$.

In all these cases the expressions take simpler forms in the limiting case of the flat plate, for which we made b zero; but it should be noted that the ideal solution then implies infinite fluid velocity at the edges of the plate. In particular, consider a plate of height $2a$, making small oscillations about its upper edge, the centre of the plate being at a depth f. If $\kappa_0 a$ is small, the first term in the expansion of (14) gives

$$A = \pi\kappa_0{}^2 a^3 \theta_0 e^{-\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

This, naturally, is equivalent to replacing the oscillating plate by a single doublet at its centre. If, in addition, $\kappa_0 f$ is small, we may take $\pi\kappa_0{}^2 a^3 \theta_0$ as a first approximation for the amplitude of the regular waves. A similar approximation could be made for a cylinder of any cross-section, using the corresponding inertia coefficient for linear motion and the mean horizontal velocity of the cylinder.

Rolling Ship.

5. The expressions given in the previous section are approximations suitable for wholly submerged bodies; it is not permissible, in general, to apply them to the oscillations of floating bodies. The approximation

used for the doublet distribution loses accuracy with diminishing depth of submergence of the body; moreover, when the surface of the body cuts the free surface some of the expressions for the surface elevation may take infinite values. It may be noted, however, that such infinities generally occur in the local part of the disturbance, the expressions for the amplitude of the regular waves at a distance from the body remaining finite.

For the rolling ship the period is such that the wave-length of the corresponding waves is large compared with the draught of the ship. Thus if we consider the analogous problem of the oscillating plate with its upper edge in the surface, the quantity $\kappa_0 a$ of the previous section is small, in most cases about 0·1. In these circumstances, treating the motion as two-dimensional, we propose to regard the ship as a single oscillating doublet at a depth which is small compared with the wave-length; thus, from (4), we take $2\pi\kappa_0\sigma M/g$ as a first approximation for the amplitude of the waves at a distance from the ship. Further, as we cannot expect more than an estimate of the order of magnitude from this assumption, we shall regard the ship as a plank, of length L and draft D, oscillating about the water-line through an angle θ_0 on either side of the vertical; using the result given at the end of § 4 and writing T for the complete period of rolling, this gives for the height of the regular waves

$$h = \frac{4\pi^5 D^3}{g^2 T^4}\theta_0. \qquad (16)$$

It should be noted that the wave-height, as the term is commonly used, is measured from trough to crest and is twice the amplitude.

6. Before applying this result, we may review briefly calculations which have been made from a different point of view.

The part played by wave propagation in causing resistance to rolling was first recognized by W. Froude (1872) and was advocated by him in a series of papers. Froude showed that the energy propagated outwards in the wave motion corresponds to a resisting couple proportional to the angular velocity of rolling, and also that the energy actually dissipated in rolling, or a large part of it, could be accounted for by waves of extremely small height; in one case, for example, his calculation gave a height of $1\frac{1}{4}$ inches for waves 320 ft. long. The same method has been applied by other writers subsequently, and it may be worth while repeating the argument in a somewhat different form from that in which it is usually given.

Suppose the ship to be rolling about a horizontal axis through its centre of gravity, and take the equation of motion in its simplest form as

$$I\ddot{\theta} + N\dot{\theta} + Wm\theta = 0, \qquad (17)$$

where I is the moment of inertia, W the weight, m the metacentric height, and $N\dot{\theta}$ the resisting couple.

The exact solution of (17) gives damped oscillations with a damping coefficient $h = N/2I$, and the rate of dissipation of energy is $N\dot{\theta}^2$. Suppose now that the dissipation is small and assume an undamped motion $\theta = \theta_0 \sin \sigma t$, holding approximately for a sufficient time, with

$$\sigma^2 = Wm/I = 4\pi^2/T^2.$$

With this assumption, the average rate of dissipation of energy

$$\tfrac{1}{2}N\sigma^2\theta_0^2 = Wmh\theta_0^2.$$

In the usual notation for the rolling of ships, $a\theta_0$ is the decrement of rolling angle for one swing; hence $a = \frac{1}{2}hT$. Thus the average rate of dissipation of energy is $2Wma\theta_0^2/T$. Assume, with Froude, that when the ship is rolling, regular straight-crested waves are sent out on either side, the breadth of each train being approximately equal to the length L of the ship; further, let A be the amplitude of the waves, λ the wave-length, T the period, with $\lambda = gT^2/2\pi$. In each train energy is propagated outwards at half the wave velocity V, that is, at an average rate $\frac{1}{4}g\rho A^2VL$ on each side. Hence, equating the average rate of dissipation of energy to the average rate of propagation of energy outwards in the waves on both sides, we have

$$2Wma\theta_0^2/T = \tfrac{1}{2}g\rho A^2V\lambda,$$

or

$$Wma\theta_0^2 = \tfrac{1}{4}g\rho A^2\lambda L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This is, in effect, the equation given by Froude and used by later writers, the left-hand side of (18) being the loss of energy in one swing; the other side of Froude's equation was, however, twice that given in (18), owing apparently to neglect of the difference between group velocity and wave velocity. The statement given here, besides including this correction, shows the various assumptions and brings the argument into line with the usual method of approximating to the damping coefficient in isochronous damped oscillations when the damping is sufficiently small. Froude recognized that his solution was not in any sense rigorous and hoped that it would be supplemented by some direct estimate, even if with no greater exactness, of the wave-making property of a ship when rolling; it is also of interest that he proposed to attempt direct observation of the waves produced by the rolling of models of simple form. However, nothing further seems to have been done on this particular aspect of the problem since that time. Other writers have used Froude's expressions to estimate the wave height, and it appears to be accepted that wave motion accounts for a large part of the dissipation of energy in rolling, that due to fluid friction or eddy-making being relatively small apart

from exceptional resistance due to bilge keels; it has been remarked, for instance, that no reasonable values of head resistance and skin friction coefficient account for more than one-third of the actual decrement obtained by experiment, and in one case such a calculation gave only one-seventeenth of it (Baker, 1914). Nevertheless, no attempt appears to have been made to compute the wave resistance to rolling from the characteristics of the ship.

7. We shall now compare wave heights calculated from (16) with various cases to which Froude's energy method has been applied.

In the case examined in Froude's first paper already quoted, the data are T=8 sec.; $\theta_0=5\cdot65°$. The draught of the ship was not stated, but we may assume D=15 ft. With these values, (16) gives $h=1\cdot2$ inch. Froude's estimate from energy dissipation was a wave height of $1\frac{1}{4}$ inch. Other writers who have used the same formula assume that that part of the resisting couple which is proportional to the angular velocity of rolling may be attributed to the loss of energy in surface-waves. Thus Sir W. White (1895), for the rolling of H.M.S.'Revenge' without bilge keels, deduced a wave height of about $1\frac{1}{2}$ inch. In this case T=15·5 sec.; $\theta_0=13°$; D=27 ft.; and these give from (16) a wave height of just over 1 inch.

L. Spears (1898), from the rolling of U.S.S. 'Oregon,' deduced a wave height of 0·62 inch. Here T=15·2 sec.; $\theta_0=12°$; D=23 ft.; and (16) gives a wave height of 0·67 inch.

It should be remarked that in all these cases Froude's formula was used; according to the argument given in § 6 and expressed in equation (18), these estimates of wave height should be increased by a factor $\sqrt{2}$. A final example is taken from a recent paper by G. S. Baker (1939) on the rolling of ships under way. We take the data for model R 8(a), for rolling at zero speed ahead, given in Tables 1 and 3 of the paper; in the notation already used

$$W=10{,}150 \text{ ton}; \quad m=4\cdot4 \text{ ft.}; \quad T=11\cdot52 \text{ sec.};$$
$$\lambda=680 \text{ ft.}; \quad L=400 \text{ ft.}; \quad D=23\cdot2 \text{ ft.}; \quad a=0\cdot022.$$

In this case we shall use equation (18) to see what height of waves would suffice to account for the whole of the dissipation of energy, neglecting for the moment any due to friction or eddy making. With the given values we find from (18), $h=2A=2\cdot65$ inch. Again, using the values of D and T in (16), we find $h=1\cdot58$ inch.

It should be noted that (16) was derived by regarding the ship as a thin plank. The formula could be modified in an empirical manner to take into account the displaced volume and the inertia coefficient of the ship; this might be represented by multiplying (16) by a factor whose probable

value would lie between 1 and 2, but the modification is not worth while at this stage.

Both the energy method and the present calculation are no more than first approximations, and therefore we may not attach any great accuracy to the estimates by either method ; nevertheless, it is interesting that both methods give results of the same order of magnitude. On the theoretical side the problem should be treated as three-dimensional, and also the boundary conditions at the surface of the ship satisfied more closely ; in addition, the actual motion of the ship and its axis of rotation are important factors in a more detailed investigation. On the other hand, it would be desirable to have experiments on models of suitable form designed to provide better estimates of frictional and eddy-making resistance to rolling, and so to afford more reliable knowledge of the amount left to be accounted for by wave propagation.

Summary.

Expressions are obtained for the surface disturbance produced by a cylinder, of elliptic cross-section, submerged in water and making small oscillations. A simple form of these results is used as a first approximation for the height of the waves, supposed two-dimensional, sent out on either side by a rolling ship. Numerical calculations are made for cases for which a similar estimate has been made by an energy method due to W. Froude ; the results by the two methods are of the same order of magnitude.

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The pressure of water waves upon a fixed obstacle

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The diffraction of plane water waves by a stationary obstacle with vertical sides is examined, in particular the variation of amplitude along the sides and the average steady pressure due to the wave motion. Results similar to those in other diffraction problems are obtained for an infinite plane and for cylinders of circular or parabolic section, and approximations are made for sections of ship form. The examination was made in view of possible applications in the problem of a ship advancing through a train of waves, and the results are discussed in relation to the average additional resistance in such circumstances. It appears that the mean pressure obtained on diffraction theory from the second order terms can only account, in general, for a small proportion of the observed effect; the motions of the ship, and in particular its oscillations, are essential factors in the problem.

1. The problem to be considered is the resultant fluid pressure upon an obstacle held in position in a train of plane waves advancing over the surface of the water. In a previous paper (1937) I considered the additional resistance on a ship moving through waves, the work being restricted to the first order effect, a purely periodic force which may have an amplitude comparable with the resistance to the ship in still water; further, for the type of ship considered, the usual approximations were made and these included neglecting the effect of reflected or scattered waves as being of the

Vol. 175. A. (18 July 1940)

second order. One purpose of the present work is to examine that assumption; the approximate method is extended in a certain case to give the variation in amplitude of the surface oscillation along the side of the ship.

The view has been put forward recently that the mean extra resistance to a ship advancing through waves is due to the reflexion of the waves by the sides of the ship, being in fact analogous to the pressure of radiation: it has been stated, for instance, that the resultant amplitude at the bow is about one-third greater, and that at the stern one-third less, than the amplitude of the incident waves, and empirical formulae for the pressure have been constructed on that basis. The problem requires, however, a consideration of second order terms which does not appear to have been made for water waves even in simple cases. We consider total reflexion, normal or oblique, by a plane wall, and diffraction by a cylinder of circular or parabolic section, together with approximations for a section of ship form: the results are discussed in relation to the ship problem.

Diffraction of water waves

2. Consider a fixed cylindrical obstacle in the water, the sides vertical and extending down to an infinite depth; let C be the contour of any horizontal cross-section. We suppose plane waves of amplitude h to be travelling in the negative direction of Ox; the origin O is in the free surface and Oz is vertically upwards. The velocity potential of the fluid motion is of the form

$$\phi = \frac{gh}{\sigma} e^{i(\sigma t+\kappa x)+\kappa z} + e^{i\sigma t+\kappa z}\phi'(x, y). \tag{1}$$

The pressure condition at the free surface is satisfied, to the usual first order terms, by $\sigma^2 = g\kappa$. Further, we have

$$\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2}+\kappa^2\phi' = 0, \tag{2}$$

and $\partial\phi/\partial\nu = 0$ on the contour C. The potential may be expressed in terms of a source distribution over the surface of the cylinder, but that is, in general, merely a restatement of the problem. We are concerned meantime with an approximate solution when the contour C is of ship form: that is, we assume C to be a contour of small breadth compared with its length. We take Ox in the direction of the length and to be an axis of symmetry of the contour. The approximation is the same as that used in determining the waves produced by a moving ship. We take the source strength at any

point to be determined by the horizontal fluid velocity in the primary motion and by the gradient of the surface at the point. We then replace the obstacle by a plane distribution of sources over the vertical section by the zx-plane. The primary fluid motion in the present case is that of the plane waves. Thus, if $(\xi, 0, -f)$ is a point on the vertical section, and if $\partial y/\partial\xi$ is the gradient at the corresponding point on the contour, the required distribution of sources over the vertical section is of strength per unit area given by

$$-\frac{i\kappa gh}{2\pi\sigma}\frac{\partial y}{\partial\xi}e^{i(\sigma t+\kappa\xi)-\kappa f}. \tag{3}$$

Consider now a point source $m\cos\sigma t$ in the liquid at the point $(0, 0, -f)$. The velocity potential was obtained by Lamb (1922) and we use his result with a slight change of notation. The surface elevation ζ is given by $g\zeta = \partial\phi/\partial t$ with $z = 0$; we have

$$\zeta = \frac{2i\sigma m}{g}e^{i\sigma t}\left\{\frac{1}{(r^2+f^2)^{\frac{1}{2}}} - i\pi\kappa e^{-\kappa f}H_0^{(2)}(\kappa r)\right.$$
$$\left. - \frac{2\kappa}{\pi}\int_0^\infty\int_0^\infty \frac{v\sin fv+\kappa\cos fv}{\kappa^2+v^2}e^{-vr\cosh u}\,du\,dv\right\}, \tag{4}$$

where $r^2 = x^2+y^2$, $\kappa = \sigma^2/g$, $H_0^{(2)} = J_0 - iY_0$, and the real part of the expression is to be taken.

Let there be a vertical line source extending from the origin downwards, the source strength per unit length at depth f being $me^{-\kappa f}$. We substitute this value for m in (4) and integrate with respect to f from 0 to ∞. For the last term in (4) this integration gives

$$-\frac{2\kappa}{\pi}\int_0^\infty\int_0^\infty \frac{e^{-vr\cosh u}}{\kappa^2+v^2}\,du\,dv = -\frac{2\kappa}{\pi}\int_0^\infty \frac{K_0(vr)}{\kappa^2+v^2}\,dv$$
$$= -\int_0^\infty \frac{e^{-\kappa f}}{(r^2+f^2)^{\frac{1}{2}}}\,df. \tag{5}$$

Hence the terms in (4) which represent the local oscillations disappear from the integrated result for this particular vertical source distribution; and we obtain the simple result

$$\zeta = \frac{\pi\sigma m}{g}e^{i\sigma t}H_0^{(2)}(\kappa r) = \frac{2i\sigma m}{g}\int_0^\infty e^{i(\sigma t-\kappa r\cosh u)}\,du, \tag{6}$$

representing circular waves diverging from the origin. Returning to (3)

we see that the source distribution is made up of vertical line sources of this type, and we obtain for the complete surface elevation

$$\zeta = ihe^{i(\sigma t+\kappa x)} - \tfrac{1}{2}i\kappa he^{i\sigma t}\int H_0^{(2)}(\kappa r)\frac{\partial y}{\partial \xi}\,e^{i\kappa\xi}\,d\kappa. \tag{7}$$

In (7), the first term represents the incident waves; further, $r^2 = (x-\xi)^2 + y^2$, and the integration extends over the axial length of the form. It should be noted that this result is comparatively simple because we have taken the obstacle to be of infinite draft; for a ship of finite draft there would be terms representing a local surface elevation in addition to the diverging waves from each element. Further, the result is only an approximation and assumes, in fact, that the additional surface elevation is relatively small.

3. We shall apply (7) to one case only, so as to estimate the magnitude of the effect due to the scattering of waves by a narrow ship of great draft and of form similar to those for which previous calculations of wave resistance have been made.

The model is of symmetrical form with straight sides, of total length $2l$, beam $2b$, and with a parallel middle body of length $2a$; the bow and stern are equal wedges of axial length $l-a$ and of semi-angle α, where $\tan\alpha = b/(l-a)$. We take the origin at the centre of the axis, with the positive direction of Ox from stern to bow. Thus we have

$$\begin{aligned} \partial y/\partial\xi &= \alpha, && \text{for } -l<\xi<-a \\ &= 0, && \text{for } -a<\xi<a \\ &= -\alpha, && \text{for } a<\xi<l. \end{aligned} \tag{8}$$

From (7), the surface elevation at any point (x, y) is given by

$$\zeta = ihe^{i(\sigma t+\kappa x)} - \tfrac{1}{2}i\kappa h\alpha e^{i\sigma t}\int_{-l}^{-a} H_0^{(2)}(\kappa r)\,e^{i\kappa\xi}\,d\xi + \tfrac{1}{2}i\kappa h\alpha e^{i\sigma t}\int_{a}^{l} H_0^{(2)}(\kappa r)\,e^{i\kappa\xi}\,d\xi. \tag{9}$$

We shall use this only for the elevation along the axis $y = 0$, as in the corresponding calculation of wave profiles for a ship. We note that in these expressions the quantity r is essentially positive. As an example, for a point in the bow wedge, that is for $a<x<l$, we have

$$\begin{aligned} \zeta e^{-i\sigma t} = ihe^{i\kappa x} &- \tfrac{1}{2}i\kappa h\alpha\int_{-l}^{-a} H_0^{(2)}\{\kappa(x-\xi)\}\,e^{i\kappa\xi}\,d\xi \\ &+ \tfrac{1}{2}i\kappa h\alpha\int_{a}^{x} H_0^{(2)}\{\kappa(x-\xi)\}\,e^{i\kappa\xi}\,d\xi + \tfrac{1}{2}i\kappa h\alpha\int_{x}^{l} H_0^{(2)}\{\kappa(\xi-x)\}\,e^{i\kappa\xi}\,d\xi. \end{aligned} \tag{10}$$

These expressions may be evaluated in terms of two integrals which may be shown to have the following values:

$$\int_0^p H_0^{(2)}(u)\,e^{-iu}\,du = pe^{-ip}\{H_0^{(2)}(p)+iH_1^{(2)}(p)\}+\frac{2}{\pi}. \tag{11}$$

$$\int_0^p H_0^{(2)}(u)\,e^{iu}\,du = pe^{ip}\{H_0^{(2)}(p)-iH_1^{(2)}(p)\}-\frac{2}{\pi}. \tag{12}$$

We shall write (11) as $L(p)+2/\pi$, and (12) as $M(p)-2/\pi$. We also put

$$p_1 = 2\kappa l;\quad p_2 = \kappa(l+a);\quad p_3 = \kappa(l-a);\quad p_4 = 2\kappa a. \tag{13}$$

We select five points at which to make the calculations, the bow, stern, shoulders and amidships; and, in the notation indicated, we have

$$\left.\begin{aligned}
\zeta(l) &= ihe^{i(\sigma t+\kappa l)}\left[1-\tfrac{1}{2}\alpha\left\{L(p_1)-L(p_2)-L(p_3)-\frac{2}{\pi}\right\}\right],\\
\zeta(a) &= ihe^{i(\sigma t+\kappa a)}\left[1-\tfrac{1}{2}\alpha\left\{L(p_2)-L(p_4)-M(p_3)+\frac{2}{\pi}\right\}\right],\\
\zeta(0) &= ihe^{i\sigma t}[1-\tfrac{1}{2}\alpha\{L(\tfrac{1}{2}p_1)-L(\tfrac{1}{2}p_4)-M(\tfrac{1}{2}p_1)+M(\tfrac{1}{2}p_4)\}],\\
\zeta(-a) &= ihe^{i(\sigma t-\kappa a)}\left[1-\tfrac{1}{2}\alpha\left\{L(p_3)+M(p_2)-M(p_4)+\frac{2}{\pi}\right\}\right],\\
\zeta(-l) &= ihe^{i(\sigma t-\kappa l)}\left[1-\tfrac{1}{2}\alpha\left\{M(p_3)+M(p_1)-M(p_2)-\frac{2}{\pi}\right\}\right].
\end{aligned}\right\} \tag{14}$$

We apply these results to Model No. 1144 of the National Physical Laboratory. This was a model of the given form used by Wigley (1931) in comparing calculated and observed wave profiles along the sides of the model when advancing through still water. For the present purpose we suppose the model held at rest while regular plane waves of amplitude h and wave-length $2\pi/\kappa$ are moving past it. The dimensions of the model were

$$l = 8\text{ ft.};\quad a = 2\cdot 19\text{ ft.};\quad b = 0\cdot 75\text{ ft.} \tag{15}$$

We calculate only one case, namely, when the wave-length is equal to the total length of the model. Thus, in the notation of (13) we have

$$p_1 = 6\cdot 28;\quad p_2 = 4\cdot 0;\quad p_3 = 2\cdot 28;\quad p_4 = 1\cdot 72. \tag{16}$$

We have also $\alpha = 0\cdot 129$. Using tables of Bessel functions, ζ may be calculated from (14). We are not concerned with the phase of the total oscillation at each point, but only with its amplitude. We find the ratio of the amplitude

to that of the incident waves at the points $x = l, a, 0, -a, -l$ to be 1·05, 1·08, 1·09, 0·99 and 0·95 respectively.

The alteration in amplitude at bow and stern would be greater for a fuller model, and especially for a bluff-ended form. Nevertheless, these approximate calculations confirm the view that for a fine model the modification caused by the reflexion of the incident waves may be treated as a second order correction. It should also be noted that these results are for a model of infinite draft; it may be presumed that the effect would be much smaller for one whose draft is small compared with the wave-length.

4. For a vertical obstacle of infinite draft, we may readily transfer results from other diffraction problems. The effect of a cylinder of elliptic section would be of special interest, but the analytical solution does not lend itself to computation when the wave-length is of the same order as the length of the axis. It is, however, worth while examining briefly two other cases from the present point of view.

Let the cylinder be circular, its water plane section being the circle $r = a$. For plane waves of amplitude h moving in the negative direction of Ox, the complete solution is given by

$$\phi = (gh/\sigma)\, e^{i\sigma t+\kappa z}\left\{J_0(\kappa r) + 2\sum_1^\infty i^n J_n(\kappa r)\cos n\theta\right\}$$
$$-(gh/\sigma)\, e^{i\sigma t+\kappa z}\left\{b_0 H_0^{(2)}(\kappa r) + 2\sum_1^\infty i^n b_n H_n^{(2)}(\kappa r)\cos n\theta\right\}, \quad (17)$$

where $\sigma^2 = g\kappa$, and $b_n = J_n'(\kappa a)/H_n^{(2)\prime}(\kappa a)$.

Putting $r = a$ in the expression for the surface elevation, and reducing by means of relations for the Bessel functions, we obtain on the cylinder

$$\zeta = -\frac{2he^{i\sigma t}}{\pi\kappa a}\left(C_0 + 2\sum_1^\infty i^n C_n \cos n\theta\right), \quad (18)$$

where $C_n = 1/H_n^{(2)\prime}(\kappa a)$.

Computation from this expression, which involves tabulation of $J_n'^2 + Y_n'^2$, can be carried out without much difficulty except when κa is large. A detailed study might be of interest, but for the present purpose the following results suffice to show the variation of amplitude round the cylinder. The numbers in table 1 give the ratio of the amplitude at each point to the amplitude of the incident waves; $\theta = 0^\circ$ corresponds to the bow and $\theta = 180^\circ$ to the stern.

TABLE 1

$\kappa a \backslash \theta$	0°	45°	90°	135°	180°
0·5	1·44	1·28	0·97	0·91	1·00
1·0	1·71	1·62	1·16	0·68	0·82
3·0	1·92	1·75	1·35	0·82	0·62
5·0	1·96	1·86	1·36	0·64	0·48

5. For the parabolic cylinder we may use the expressions given by Lamb (1906) for the diffraction of sound waves, making the necessary modifications for water waves. In this case we take the plane waves to be moving in the positive direction of Ox; the water-plane section of the cylinder is given by

$$\kappa^2 y^2 = 4\eta_0^4 + 4\kappa\eta_0^2 x. \tag{19}$$

In the parabolic co-ordinates defined by $\kappa(x+iy) = (\xi + i\eta)^2$, the section of the cylinder is given by $\eta = \eta_0$.

The velocity potential of the motion is

$$\phi = (gh/\sigma)\, e^{i(\sigma t - \kappa x) + \kappa z}\left\{1 + C\int_{\eta}^{\infty} e^{-2it^2}\,dt\right\}, \tag{20}$$

where the constant C is given by

$$2i\eta_0\left\{1 + C\int_{\eta_0}^{\infty} e^{-2it^2}\,dt\right\} - Ce^{-2i\eta_0^2} = 0. \tag{21}$$

For the surface elevation on the cylinder, we have

$$\zeta = ihe^{i(\sigma t - \kappa x)}\left\{1 + C\int_{\eta_0}^{\infty} e^{-2it^2}\,dt\right\}. \tag{22}$$

It follows that the amplitude of the oscillation is constant round the boundary. From (21) and (22) we find that this amplitude is h/p, where p is given by

$$p^2 = 1 + 2\pi^{\frac{1}{2}}\eta_0\{(\tfrac{1}{2} - c)\sin 2\eta_0^2 - (\tfrac{1}{2} - s)\cos 2\eta_0^2\} + \pi\eta_0^2\{(\tfrac{1}{2} - c)^2 + (\tfrac{1}{2} - s)^2\}, \tag{23}$$

in which c and s denote Fresnel integrals of argument $2\eta_0\pi^{-\frac{1}{2}}$.

From (19) we have $2\eta_0^2 = \kappa a$, where a is the radius of curvature of the parabola at its vertex. Table 2 gives the ratio of the amplitude of the oscillation to that of the incident waves, calculated from (23) for certain values of κa:

TABLE 2

κa	0·05	0·1	1·0	3·0	5·0
amp./h.	1·28	1·40	1·65	1·86	1·93

These may be compared with the corresponding values at $\theta = 0$ for the circular cylinder. When κa is small, we obtain from (23) the approximate value $1 + (\pi\kappa a/2)^{\frac{1}{2}}$ for the ratio of the amplitude to that of the incident waves. We may, possibly, use this to give an upper limit for the resultant amplitude at the bow of a ship if we regard the front half of the ship as a parabola with its vertex at the bow. For instance, consider the model examined in § 3. Instead of a wedge-shaped bow, suppose it is rounded off into a parabola with its vertex at the bow and joining on to the parallel middle body at a distance of 5·81 ft. from the bow, the beam at that point being 1·5 ft. With these data, and taking the same wave-length of 16 ft., we find that $\kappa a = 0{\cdot}019$. From the approximate formula, this gives a relative amplitude at the bow of 1·17. Comparing with the previous calculations, this seems a reasonable estimate, in spite of the various assumptions; the ratio would, of course, be greater for smaller wave-lengths.

The pressure of water waves

6. For the resultant pressure upon the obstacle, the first order effect is a purely periodic force with zero mean value; this was the effect considered in the previous paper (1937) and applied to a ship among waves. To obtain a steady mean force different from zero we have to proceed to second order terms; although much work was done at one time on the pressure of vibrations, water waves do not seem to have been considered in this connexion.

We begin with plane waves, and the only general result we need is that given by Rayleigh (1915), that the usual first order expression for the velocity potential is also correct to the second order, the next term being of the third order; this was shown to be the case both for progressive waves and for stationary oscillations. There are, however, second order terms in the surface elevation.

Consider plane waves incident directly upon the plane $x = 0$ as a fixed boundary. We take

$$\phi = (2gh/\sigma)\, e^{\kappa z} \cos \kappa x \sin \sigma t, \tag{24}$$

$$\zeta = 2h \cos \kappa x \cos \sigma t + 2\kappa h^2 \cos 2\kappa x \cos^2 \sigma t, \tag{25}$$

with $\sigma^2 = g\kappa$. We have $\partial\phi/\partial x = 0$ at $x = 0$; for the pressure we have

$$p = F(t) - g\rho z + \rho \frac{\partial\phi}{\partial t} - \tfrac{1}{2}\rho\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\}. \tag{26}$$

It may be verified that, with

$$p = -2g\rho\kappa h^2 \cos 2\sigma t - g\rho z + 2g\rho h e^{\kappa z} \cos \kappa x \cos \sigma t - 2g\rho\kappa h^2 e^{2\kappa z} \sin^2 \sigma t, \tag{27}$$

the pressure conditions at $z = \zeta$, given by (25), are satisfied to the second order, namely $p = 0$ and

$$\frac{\partial p}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial \phi}{\partial z}\frac{\partial p}{\partial z} = 0.$$

To the first order, (24) and (25) represent plane waves of amplitude h reflected at the plane $x = 0$. We may now evaluate the additional pressure upon this plane per unit width. We put $x = 0$ in (27) and integrate with respect to z from $-\infty$ to ζ. The first order term is the periodic force $(2g\rho h/\kappa)\cos\sigma t$; for the additional quadratic terms we obtain

$$-\tfrac{1}{2}g\rho\zeta^2 + 2g\rho h\zeta\cos\sigma t - g\rho h^2\sin^2\sigma t, \tag{28}$$

the second term in (28) coming from the expansion of $e^{\kappa\zeta}$. We put in the value of ζ from (25), noting that we only need this to the first order; and we obtain for the additional steady force P per unit width of the plane, taking mean values,

$$P = \tfrac{1}{2}g\rho h^2, \tag{29}$$

where h is the amplitude of the incident waves.

It may be remarked that instead of using the fact that the second order term in the expansion of ϕ is zero, it would have sufficed for the present purpose to assume ϕ to be purely periodic, an assumption made by Larmor (1920) in the corresponding calculation for sound waves. It is well known that waves of finite amplitude possess linear momentum in the direction of propagation; the average amount, to second order terms, is $\pi\rho h^2 V$ per wave length, V being the wave velocity. On the other hand, if we calculate the rate of transfer of linear momentum across a vertical plane, we obtain an average rate of $\tfrac{1}{4}g\rho h^2$; this gives in one period one-half the average momentum in one wave-length. The average pressure P given by (29) may be regarded as due to the reversal of this flow of momentum. We notice also that P is equal to one-half the average density of energy in the standing oscillations, and this may again be connected with the fact that the group velocity for water waves is one-half the wave velocity. For plane waves of amplitude h incident upon the plane $x = 0$ at an angle α to the plane, we may take

$$\left.\begin{aligned} \phi &= (2gh/\sigma)\,e^{\kappa z}\cos(\kappa x\sin\alpha)\sin(\sigma t - \kappa y\cos\alpha), \\ \zeta &= 2h\cos(\kappa x\sin\alpha)\cos(\sigma t - \kappa y\cos\alpha). \end{aligned}\right\} \tag{30}$$

We obtain now, instead of (28), the quadratic terms for the additional pressure as

$$\begin{aligned} &-\tfrac{1}{2}g\rho\zeta^2 + 2g\rho h^2\cos(\sigma t - \kappa y\cos\alpha) \\ &\qquad - g\rho h^2\{\cos^2\alpha\cos^2(\sigma t - \kappa y\cos\alpha) + \sin^2(\sigma t - \kappa y\cos\alpha)\}. \end{aligned} \tag{31}$$

Taking mean values, this gives

$$P = \tfrac{1}{2}g\rho h^2 \sin^2 \alpha. \tag{32}$$

7. We proceed similarly for any fixed cylinder of infinite draft with vertical sides; it is not necessary to examine the second order terms for the surface elevation, and we assume that the velocity potential is correct up to that order, or at least that any second order term is purely periodic.

Consider the solution for the circular cylinder which was given in § 4. We write it as

$$\left.\begin{aligned}\phi &= (gh/\sigma)\, e^{\kappa z}(L \cos \sigma t - M \sin \sigma t),\\ \zeta &= -h(L \sin \sigma t + M \cos \sigma t),\end{aligned}\right\} \tag{33}$$

where L, M are functions of r, θ which may be obtained from (17).

At any point on the cylinder we have

$$\begin{aligned}p = F(t) - g\rho z - g\rho h e^{\kappa z}(L \sin \sigma t + M \cos \sigma t)\\ -(\rho/2a^2)\, e^{2\kappa z}\{\kappa^2 a^2 (L \cos \sigma t - M \sin \sigma t)^2 + (L' \cos \sigma t - M' \sin \sigma t)^2\},\end{aligned} \tag{34}$$

with $r = a$, and the accent denoting $\partial/\partial\theta$.

We integrate with respect to z from $-\infty$ to ζ and expand to second order terms; then for the resultant force we multiply by $a \cos\theta d\theta$ and integrate round the circle.

It is readily seen that the first order term in the additional force is a periodic effect of amount

$$\frac{4g\rho h}{\kappa^2} \frac{J_1'(\kappa a) \sin \sigma t + Y_1' \cos \sigma t}{J_1'^2(\kappa a) + Y_1'^2(\kappa a)}. \tag{35}$$

From the quadratic terms we get, after taking mean values, the steady additional force

$$R = \tfrac{1}{4} g\rho h^2 a \int_0^\pi \left\{ L^2 + M^2 - \frac{1}{\kappa^2 a^2}(L'^2 + M'^2)\right\} \cos\theta d\theta, \tag{36}$$

where

$$L + iM = \frac{2i}{\pi\kappa a}\left(b_0 + 2\sum_1^\infty b_n \cos n\theta\right), \tag{37}$$

and $b_n = i^n / H_n^{(2)\prime}(\kappa a)$.

We have

$$\left.\begin{aligned}\int_0^\pi (L^2 + M^2)\cos\theta d\theta &= \frac{4}{\pi\kappa^2 a^2}\sum_0^\infty (b_n b_{n+1}^* + b_n^* b_{n+1}),\\ \int_0^\pi (L'^2 + M'^2)\cos\theta d\theta &= \frac{4}{\pi\kappa^2 a^2}\sum_1^\infty n(n+1)(b_n b_{n+1}^* + b_n^* b_{n+1}),\end{aligned}\right\} \tag{38}$$

where the asterisk denotes the conjugate complex. Putting in the value of b_n and using properties of the Bessel functions, these expressions may be reduced to a simple form; we obtain finally

$$R = \frac{4g\rho h^2 a}{\pi^2\kappa^3 a^3}\sum_0^\infty \left\{1 - \frac{n(n+1)}{\kappa^2 a^2}\right\}^2 \frac{1}{(J_n'^2 + Y_n'^2)(J_{n+1}'^2 + Y_{n+1}'^2)}, \tag{39}$$

the argument of the Bessel functions being κa.

The series in (39) occurs also in an expression given by Nicholson (1912) in a similar problem for electromagnetic waves. Some values had been calculated before this reference was known, and with Nicholson's values for the series we have the results in table 3.

TABLE 3

κa	0·5	1·0	2·0	3·0	5·0
$R/\frac{2}{3}g\rho h^2 a$	0·429	0·998	0·940	0·950	0·965

It was shown by Nicholson that when κa is large, the series approximates to the value $\frac{1}{6}\pi^2\kappa^3 a^3$; hence when the wave-length is small compared with the diameter of the cylinder, we have approximately

$$P = \tfrac{2}{3}g\rho h^2 a. \tag{40}$$

This agrees with the limiting value if we assume total reflexion over the front half of the cylinder and a complete shadow over the rear half, and apply to each element the expression (32) for total oblique reflexion from a plane; for we then have

$$P = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \tfrac{1}{2}g\rho h^2 a \cos^3\theta\, d\theta = \tfrac{2}{3}g\rho h^2 a. \tag{41}$$

Although this limiting value is obtained theoretically as an extreme case for short waves, it is interesting to note from the preceding table that it is practically attained for comparatively long waves of wave-length even larger than the diameter. This consideration suggests using the method to give an upper limit for cylinders whose section is more like that of a ship.

8. Consider a cylinder with vertical sides, the horizontal section being of ship form and symmetrical about Ox. We assume total reflexion by the sides of the ship from the bow back to where the sides become parallel to Ox, and we assume a complete shadow aft of that point.

For the model of § 3, in which the bow is a wedge of semi-angle α, we obtain for the total resistance

$$R = \tfrac{1}{2}g\rho h^2 B \sin^2\alpha, \tag{42}$$

where $B = 2b$ = beam.

In general, for any form of the front portion of the model, we have

$$R = \tfrac{1}{2}g\rho h^2\int_{-b}^{b}\sin^2\alpha\,dy = \tfrac{1}{2}g\rho h^2 B\,\overline{\sin^2\alpha}. \tag{43}$$

In (43), α is the angle the tangent to the form makes with Ox, and the bar denotes the mean value of $\sin^2\alpha$ with respect to the beam of the ship.

Suppose, for instance, that the section of the model is the ellipse $x^2/a^2+y^2/b^2 = 1$. It is easily shown that in this case

$$\overline{\sin^2\alpha} = \frac{b^2}{a^2-b^2}\left\{\frac{a^2}{b\sqrt{(a^2-b^2)}}\tan^{-1}\frac{\sqrt{(a^2-b^2)}}{b}-1\right\}. \tag{44}$$

This would be a full form of model. If we take $a = 8b$, as an average ratio of length to beam, we find from (44) that the mean value of $\sin^2\alpha$ is 0·17. The mean value is less for models with moderate bow angle; probably an average value would be about 0·1, with still smaller values for models with fine lines.

In a recent paper Kreitner (1939) has put forward the proposition that the extra resistance to a ship among waves is nothing else than the radiation pressure of the ocean waves. The semi-empirical formula given by Kreitner for this force upon a ship at rest in a train of waves is

$$R = g\rho h^2 B\,\overline{\sin\alpha}, \tag{45}$$

in the present notation, in which h is the amplitude of the incident waves; the last factor is a mean value for the angle of entrance not clearly defined. The derivation of this formula is not clear, but it appears to be based upon an estimate of the difference of resultant amplitude at bow and stern, and upon taking the mean value of the hydrostatic pressure due to the surface elevation. This latter assumption is incorrect; and further, we found in (43), that the last factor should be the mean value of $\sin^2\alpha$ taken across the beam. Numerically, for usual ship forms, these differences result in (43) giving about one-fifth of the value from (45).

For a certain model, a ship with full lines, the relevant data are $B = 69{\cdot}2$ ft., $L = 530$ ft., $h = 2\tfrac{1}{4}$ ft. If we assumed the fore half of the ship to be an ellipse and used (44), we should have 0·175 as the mean value of $\sin^2\alpha$; but this is certainly too large and we take a smaller value, say 0·12. With these values, (43) gives a force of 0·6 ton. This is, moreover, an upper limit and also assumes the ship to have vertical sides and to be of great draft. The recorded extra resistance for this ship is given as about 2·8 tons; but this was for a model advancing through the waves.

The steady pressures we have been considering will certainly be increased if the ship is itself in steady motion through the waves, but the problem

then becomes complicated and, in practice, many other factors must be taken into account. The wave resistance of the ship, as calculated for uniform motion through still water, is probably altered; moreover, the motion of the ship, and in particular its pitching and other oscillations, must have an important influence. It may well be that interactions between first order effects which in themselves are purely periodic may, through phase differences, give rise to steady additional resistances.

The calculations which have been made here refer to a model held at rest in a train of waves. The only reference to experiments of this nature appears to be in a paper by Kent and Cutland (1935). The model was No. 1255 of the National Physical Laboratory, and the dimensions were: length = 16 ft., beam = 1·92 ft., draft = 0·52 ft. For this model the mean value of $\sin^2 \alpha$ was probably not more than 0·1. If we suppose the wave amplitude, that is half the wave height, to be 2 in. for waves, say, 5 ft. in length, then (43) gives as an upper limit a force of 0·17 lb. The experimental results were not published, no doubt because this particular experiment was only incidental to the main investigation; but it may be taken that the calculated value obtained here is of the order of one-half the measured value for waves of the given height and length. Here, again, although the model is said to be at rest, it has necessarily a certain small amount of freedom for oscillatory motion. While such motion might be expected to diminish the magnitude of the pressures we have been considering, it may also bring other effects into operation. Further experiments of this nature, with more detailed measurements, would be of great interest.

The immediate object of the present work was to examine, in cases amenable to calculation, the magnitude of the mean force obtainable on the analogy of radiation pressure. The general conclusion is that while such a force exists as a contributory cause, it is insufficient to account for the extra resistance observed in a ship advancing through waves; in those circumstances the total effect is probably the result of several factors of approximately equal importance.

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The Drifting Force on a Ship among Waves.

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1. WHEN a ship is advancing through a train of waves it experiences an average steady resistance greater than that at the same speed in smooth water. There are no doubt several factors operative in producing this result; one, for instance, may be described as wave pressure due to the reflexion or scattering of the ocean waves by the surface of the ship (Kreitner, 1939). This must certainly be taken into account in a complete theory, but investigation of it involves second-order terms in the hydrodynamical equations and a satisfactory solution of the problem would be difficult. Certain calculations which I have given recently (1940) seem to show that this cause is not likely to account for more than a small fraction of the observed results. Experiment shows that the effect is most prominent when the period of encounter of the ship with the waves is near the natural periods of the ship's oscillations; whether directly or indirectly, the phenomenon is clearly associated with the heaving and pitching motions of the ship. In the paper already quoted (1940) it was suggested that it may well be that interactions between first-order effects which in themselves are purely periodic may, through phase differences, give rise to steady additional resistances. The object of the present note is to give some tentative calculations amplifying and illustrating this suggestion. For this purpose we fall back on the approximate theory which neglects the disturbing effect of the ship's surface upon the wave motion. In suitable cases we may perhaps regard the necessary additions for the reflected waves to be small corrections, as, for instance, for a long narrow ship (1937). This assumption was the basis of the theory developed by W. Froude in his work on the rolling of a ship among waves, in which case the wave-length is assumed large compared with the beam of the ship. It was also used explicitly by Kriloff in his well-known analysis of the heaving and pitching of a ship among waves. This latter work dealt only with the oscillations of the ship, and not with the extra resistance to motion which is now under consideration. It is true that the problem involves, in some form at least, second-order terms, and any partial separate examination of such terms is unsatisfactory; the following calculations

are therefore subject to correction by a more complete analysis, but they may serve to bring out a new point of view.

2. It is interesting to recall the development of the similar problem in rolling. Some years ago Suyehiro (1924), experimenting with a small model, announced the discovery of a drifting force sideways upon a ship when rolling in waves. This interesting result does not seem to have been studied by other workers, and either confirmed or disproved. The effect is small and probably needs suitable conditions of forced rolling in resonance with the natural period of roll. Suyehiro himself ascribed the force to reflexion of the waves by the side of the ship and supported this view by observation of the motion of the fluid particles near the ship. No calculation was made of the reflexion or scattering of the waves by the ship, and this is a problem which still awaits solution. Here, again, no doubt this form of wave pressure contributes to the result, but there is no reason to suppose it adequate in itself; moreover, the experiments showed a close association of the drifting force with the rolling of the ship. Recently an alternative theory has been put forward by Watanabé (1938). Starting from the Kriloff equations, Watanabé deduced an expression for the drifting force involving the angle of roll and the phase lag between the roll and the actuating moment; applied to Suyehiro's model, this expression gave a force of rather more than half the observed value.

In the following sections we derive similar expressions for the drifting force due to heaving and pitching when the ship is head-on to the waves; we assume throughout the usual theory of irrotational waves of small height.

3. Take the origin O in the undisturbed surface of the water, Ox horizontal and perpendicular to the wave crests and in the direction of the ship from stern to bow, Oy horizontal and Oz vertically upwards. We shall suppose first that the ship has no forward motion or, more precisely, we may suppose it constrained so that it is free to make small vertical oscillations and free also to make small rotational oscillations about a horizontal axis parallel to Oy through some point G. We consider plane waves of small amplitude h and of wave-length $2\pi/\kappa$ moving in the negative direction of Ox. To the first order the velocity potential is given by

$$\phi=(gh/\sigma)e^{\kappa z}\sin(\sigma t+\kappa x), \qquad (1)$$

with $\sigma^2=g\kappa$, and the pressure by

$$p=p_0-g\rho z+\rho\frac{\partial\phi}{\partial t} \qquad (2)$$

$$=p_0-g\rho z+g\rho h e^{\kappa z}\cos(\sigma t+\kappa x), \qquad (3)$$

p_0 being the pressure at the free surface.

A complete solution would include an addition to (1) necessary to satisfy the boundary condition at the surface of the ship in its actual motion and also the condition of constant pressure over the free surface of the water. We are, meantime, neglecting this additional term, and assuming the conditions such that for a first approximation we may calculate the resultant forces from the pressure given by (3). The resultant horizontal force backwards is given by

$$F=\iint pl\,dS, \qquad \qquad (4)$$

taken over the immersed surface of the ship in any position, (l, m, n) being the direction-cosines of the outward drawn normal at any point. This may be transformed into a volume integral taken throughout the immersed volume V of the ship, and using (3) we have

$$\begin{aligned} F&=\iiint \frac{\partial p}{\partial x}dV \\ &=-g\rho\kappa h\iiint e^{\kappa z}\sin(\sigma t+\kappa x)\,dV \qquad \qquad (5)\end{aligned}$$

Let S_0, V_0 be the immersed surface and volume, respectively, when the ship is in its equilibrium position in smooth water. If the ship is held in this position in the waves, the corresponding force F_0 calculated from (5) is a purely periodic force with mean value zero (1937). Suppose now the ship to be in a slightly displaced position S due to heaving and pitching. The additional horizontal force is given by (5) integrated throughout the volume between S_0 and S. If $\delta\nu$ is the distance from any point of S_0 normally outwards to S, we have $dV=\delta\nu\,dS_0$. Let the pitch be measured by a small angle θ of rotation round an axis through a point G on Oz at a height c above O, taking θ to be positive with the bow up; and let the heave be given by a small vertical displacement ζ upwards. Then, to the first order in ζ and θ, we have

$$\delta\nu=n\zeta+\{nx-l(z-c)\}\theta. \qquad \qquad (6)$$

Hence the horizontal force backwards in the new position is given by

$$\begin{aligned} F=F_0&-g\rho\kappa h\zeta\iint e^{\kappa z}\sin(\sigma t+\kappa x)n\,dS_0 \\ &-g\rho\kappa h\theta\iint e^{\kappa z}\sin(\sigma t+\kappa x)\{nx-l(z-c)\}\,dS_0, \qquad (7)\end{aligned}$$

where the integrals are taken over the equilibrium position of the ship's surface.

Calculations may be made directly from this expression, but we put it into another form to show that it leads to an average steady force backwards.

Let B be the extra buoyancy for the ship in its equilibrium position due to the wave motion, that is, the additional force upwards arising from

the term $\rho\partial\phi/\partial t$ in (2). Similarly, let P be the additional moment of this pressure about the axis through G in the direction of θ increasing. Then we have

$$B=-g\rho h\iint e^{\kappa z}\cos(\sigma t+\kappa x)n\,dS_0, \quad (8)$$

$$P=g\rho h\iint e^{\kappa z}\cos(\sigma t+\kappa x)\{l(z-c)-nx\}dS_0. \quad (9)$$

Hence we may write (7) as

$$F=F_0-\frac{\kappa}{\sigma}\zeta\frac{\partial B}{\partial t}-\frac{\kappa}{\sigma}\theta\frac{\partial P}{\partial t}. \quad (10)$$

The usual approximate equations for the motion of the ship are obtained by taking into account also the hydrostatic buoyancy and moment arising from the term $g\rho z$ in (3). With M, I as effective mass and moment of inertia, respectively, and assuming a simple law of damping in each case, the equations are

$$M\ddot{\zeta}+N\dot{\zeta}+g\rho A\zeta=B, \quad (11)$$

$$I\ddot{\theta}+N'\dot{\theta}+g\rho mV_0\theta=P, \quad (12)$$

A being the area of the water plane section and m the metacentric height for pitching.

When calculated from (8) and (9), B and P are of the form

$$B=B_0\sin(\sigma t+\alpha)\,;\quad P=P_0\sin(\sigma t+\alpha'), \quad (13)$$

B_0, P_0, α, α' depending upon the wave-length and the form of the ship.

To obtain from (10) quadratic terms giving a mean value different from zero, we need consider only the forced oscillations in ζ and θ. These are given by

$$\left.\begin{aligned}\zeta&=kB_0\sin(\sigma t+\alpha-\beta)\\ \theta&=k'P_0\sin(\sigma t+\alpha'-\beta')\end{aligned}\right\}, \quad (14)$$

k, k' being the usual magnification factors, and β, β' the corresponding phase lags, obtained by solving (11) and (12) for the forced oscillations. Using (13) and (14) in (10) and taking mean values of the quadratic terms, we obtain for the mean backward force

$$R=\tfrac{1}{2}\kappa kB_0^2\sin\beta+\tfrac{1}{2}\kappa k'P_0^2\sin\beta', \quad (15)$$

an expression which is essentially positive.

With ζ_0 and θ_0 the amplitude of the forced heaving and pitching, respectively, and B_0, P_0 the amplitudes of the buoyancy and pitching moment as in (13), we have from (15)

$$R=\tfrac{1}{2}\kappa B_0\zeta_0\sin\beta+\tfrac{1}{2}\kappa P_0\theta_0\sin\beta'. \quad (16)$$

4. We have only used equations (11) and (12) to show that the average steady force is a resistance. In attempting comparison with experimental results one cannot rely upon calculations from these equations,

except for general descriptive purposes. Among other reasons, there is a lack of precise information about the damping of natural heaving and pitching. A common statement is that the damping in both cases is large, the natural oscillations in uniform waves diminishing rapidly and the motion reducing to the forced oscillations. On the other hand, this is difficult to reconcile with certain experimental results when the period of encounter with the waves is near a natural period; in such cases the amplitude of the resultant oscillation has a slow periodic variation from a minimum to a maximum in a manner suggesting the superposition of natural and forced oscillations of nearly equal period. The only published estimate from experimental results appears to be that given by Horn (1936), who states that the damping of heaving and pitching is of the same order of magnitude; his estimate gives a logarithmic decrement of about 1·4, an extremely large value compared, for instance, with the damping of rolling.

We have assumed the ship to have no forward motion, but, so far as the present approximation goes, we may suppose it moving with uniform speed; the only difference is that the quantity σ in (13) is such that $2\pi/\sigma$ is the period of encounter of the ship with the waves.

We may make a rough estimate of the order of magnitude of the extra resistance given by (16). For a cargo boat of 400 ft. in waves of 500 ft. in wave-length and of height 6 ft., the amplitude P_0 of the pitching moment may be about 80,000 ft.-tons while the amplitude B_0 of the buoyancy might be, say, 300 tons. Hence from (16) we should have

$$R=1{\cdot}9\zeta_0 \sin\beta+502\,\theta_0 \sin\beta'. \qquad (17)$$

If the period of encounter is not near a natural period we might assume a total heave of 4 ft. and a pitch of 3°; whence

$$R=3{\cdot}8 \sin\beta+13 \sin\beta'. \qquad (18)$$

A value of 15° for the phase lags β, β' would not be unreasonable. This would give $R=4{\cdot}4$ tons, of which three-quarters would be associated with pitching and one-quarter with heaving.

5. For a more detailed analysis, we consider a simple form of wall-sided model of uniform draft d, with a rectangular middle body of length $2a$ and beam $2b$, and with entrance and run each of length l and of parabolic form. Thus with O at the centre of the parallel middle body, the equation of the contour from $x=a$ to $x=l+a$ is

$$y=b\{1-(x-a)^2/l^2\} \qquad (19)$$

at all depths, from $z=-d$ to $z=0$; and there is a similar equation for the run at the stern from $x=-(l+a)$ to $x=-a$.

The model is symmetrical fore and aft, and for simplicity we assume G, the position of the axis of rotation, to be at O; thus we take $c=0$ in (9). The integrations in (8) and (9) over the sides and keel of this model are readily carried out, and we obtain eventually

$$B=\frac{8g\rho bh}{\kappa p^2}e^{-q}\{\sin(p+p_1)-p\cos(p+p_1)-\sin p_1\}\cos\sigma t. \quad . \quad . \quad (20)$$

$$P=\frac{8g\rho bh}{\kappa^2p^2}[\{(p^2+p_1-3)\sin(p+p_1)-(p_1+3p)\cos(p+p_1)$$
$$+3\sin p_1-p_1\cos p_1\}e^{-q}+(1-e^{-q}-qe^{-q})\{\sin(p+p_1)$$
$$-p\cos(p+p_1)-\sin p_1\}]\sin\sigma t, \quad \cdot \quad (21)$$

where $p=\kappa l$, $p_1=\kappa a$, $q=\kappa d$.

In (21) under usual conditions, the second part is small compared with the part which is factored by e^{-q}; if this latter factor be also neglected, the expression is simply the conventional pitching moment obtained from the hydrostatic pressure due to the wave elevation integrated over the water-plane section of the ship. This form of model is not quite suitable for the approximations on which the calculations have been made, but there are no experimental results available for simple symmetrical models of small beam.

From experiments, carried out at the National Physical Laboratory, on a model of a single-screw cargo ship, Kent and Cutland (1941) have obtained some very interesting results. We give the relevant data for the ship: length 400 ft., beam 55 ft., draught 24 ft., and displacement 11,332 tons. The natural pitching and heaving periods are given as 6·2 and 7·42 sec. respectively. Measurements were made of pitch and heave, of resistance and of other quantities, under various conditions in waves of 175 ft., 350 ft., and 490 ft. In the shortest waves it is probable that a considerable part of the increased resistance arises from the reflexion of the waves by the ship. Applying an expression which I gave recently (1940) for this resistance, Kent and Cutland show that it accounts for rather more than half the measured resistance for a ship moored in the waves. That expression gave a limiting value for a ship of great draught held at rest, assuming total reflexion by the front portion of the ship. It seems reasonable, therefore, to suppose that the force arising from reflexion would be much smaller for a ship of usual form floating on the water, and especially so for the large wave-lengths. Moreover, in the short waves the pitch and heave are slight and are irregular in period: in the medium waves the periods are approximately equal to the period of encounter between ship and wave, but the amplitude changes from a minimum to a maximum in regular cycles: in the long waves the pitch and heave are approximately uniform. In the present

calculations the amplitudes in (16) are those of the forced oscillations of heave and pitch ; hence we attempt only a comparison with the results of Kent and Cutland for the 490 ft. wave-length, when presumably the motions are purely forced vibrations. Another reason for limiting comparison to waves of length greater than the ship is that the expressions for the buoyancy and pitching moment are probably better approximations than for shorter waves. Without attempting any close approximation to the form of the ship, we shall simply use the expressions (20) and (21) with

$$a=140 \text{ ft. };\quad l=60 \text{ ft. };\quad d=20 \text{ ft. };$$

$$b=27{\cdot}5 \text{ ft. };\quad h=2{\cdot}5 \text{ ft. };\quad \lambda=2\pi/\kappa=490 \text{ ft.}, \qquad (22)$$

these dimensions giving a ship of about the same displacement, and waves of 490 ft. in length and 5 ft. in height.

With these values (20) and (21) give

$$\mathrm{B}=345 \cos \sigma t\,;\quad \mathrm{P}=-83880 \sin \sigma t, \qquad (23)$$

in tons, and ft.-tons respectively.

The numerical factors in (23) are the values of B_0 and P_0 in (16). The amplitudes ζ_0 and θ_0 we shall take from the observed results, assuming that these refer to forced vibrations in the period of encounter. The remaining factors are the phase differences, and these are more uncertain. It may be noted that the effect which is under discussion arises from the damping and the phase lag produced thereby ; if there is no phase lag there is no force. On the simple theory expressed in equations (11) and (12) the phase factor, sin β or sin β', has its maximum value of unity at resonance and diminishes on either side of this period, the diminution being more rapid the less the damping. The importance of the phase of the ship's motion in relation to that of the waves from a practical point of view is well known, but there are not many precise measurements suitable for the present purpose. The problem was attacked in an early paper by Kent (1922), and the recent paper by Kent and Cutland (1941) gives further detailed observations. They give a diagram showing positions of wave crest and trough along the ship at maximum pitch with the bow down, and from this one should be able to deduce the value of β' for use in (16). However, it must be remembered that the model was not a simple symmetrical form, with the axis of pitch at the centre of the water-plane section ; in fact, the position of this axis probably varied during the motion. It is also clear that the motion is not adequately covered by the theory of equations (11) and (12). For the authors state : " In general, as the ship's pitching period was not isochronous owing to the changing resistance to pitch, successive pitches

showed a periodic movement of the wave crest position, backwards and forwards along the hull." The diagram given in the paper shows the mean positions. From this diagram, it seems that we may assume there was no measureable phase lag for the ship moored, with zero speed, in 490 ft. waves. Hence, from (16), the corresponding mean force is negligible, and this agrees with the observations. For the same wave-length when the ship had a speed of 8 knots, a rough estimate of the phase lag from the diagram is about 12·5°, and we take that value for β' in (16). As the free periods of heave and pitch are nearly equal, and the damping probably of the same order, we take the same value for β in (16). From the measurements given in the paper for 5 ft. waves, we have $\zeta_0=2\cdot1$ ft., $\theta_0=1\cdot6°$. With these values in (16) we get the extra resistance for the ship in tons; expressing the result for the model, 16 ft. long, we obtain from (16) a mean resistance of 0·63 lb. The measured value was 0·37 lb.

It is not worth while pursuing these tentative calculations further at present, but at least it seems that one can obtain results of the right order of magnitude; in fact, the calculated results are generally too high, especially at the peak values under resonance conditions, but that might have been anticipated. On the theoretical side, the various limitations and assumptions have already been sufficiently indicated in the course of the work. On the experimental side, there is a lack of suitable data obtained under conditions sufficiently approximating to the simplifications which have to be made before any calculations are possible.

6. In the present work, with the ship head-on to the waves, heaving and pitching have been considered together; for if the argument is valid for one kind of displacement it should also apply to the other. Moreover, the natural periods of heave and pitch are usually nearly equal and so resonance effects for the forced vibrations overlap. Reference has been made to Watanabé's work on the drifting force when rolling, that is, when the ship is broadside-on to the waves; in that work, as in Suyehiro's, the effect of heaving was entirely neglected. The expression (16) given here for the drifting force in heaving and pitching may also be used for heaving and rolling when the waves are broadside on, with the quantities in (16) having the appropriate values for those conditions. However, the natural periods of heave and roll usually differ considerably, and therefore the resonance effects are separated. The data for Suyehiro's model are not sufficient for these calculations to be made, otherwise one might compare the drifting forces due to heaving and rolling; it would be of interest if experiments could be devised to test whether these separate effects are observable, and to have experiments made under conditions suitable for comparison with theory.

Summary.

The problem considered is the drifting force on a ship when head-on to a regular train of waves. A satisfactory theory would have to include, among other factors, the effect of reflexion of the waves by the surface of the ship; in the present note this is neglected in order to make tentative calculations from another point of view which associates the effect directly with the oscillations of the ship. A steady average drifting force is obtained depending upon the phase differences between the heaving and pitching motions and the periodic forces and couples due to the wave motion. An examination is made of experimental results and, although available data are not suitable for detailed comparison, it appears that the calculations give drifting forces of the right order of magnitude.

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The Damping of the Heaving and Pitching Motion of a Ship.

By T. H. HAVELOCK, F.R.S.

1. IN a recent paper (1940) I discussed the damping of the rolling of a ship in still water due to the radiation of energy in the wave motion set up by the rolling. The following note is a similar examination of heaving and pitching oscillations ; an attempt is made to estimate the dissipation of energy in wave motion and comparison is made with such experimental results as are available.

The problem may be stated first in relation to heaving motion. Consider a body of mass M floating freely in water, and suppose it is acted on by a periodic force E cos pt and is making small vertical oscillations ; let ζ be the vertical displacement upwards from the equilibrium position. The equation of motion, for a frictionless liquid, is

$$M\ddot{\zeta}=X-Mg+E\cos pt, \qquad (1)$$

where X is the vertical resultant of the fluid pressures on the immersed surface. For an exact solution we should have to determine the velocity potential of the fluid motion so as to satisfy the boundary condition at the moving surface of the solid and the condition of constant pressure over the free surface of the liquid. Failing such a solution we proceed

by approximations. One part of the force X-Mg is the additional hydrostatic buoyancy, $g\rho S\zeta$ upwards, assuming the solid to have vertical sides near the water-line and S to be the area of the water plane section. Suppose now that the motion of the body is a forced vibration of frequency p and that the energy radiated is relatively small; then, as in similar problems, for instance the scattering of sound waves by a movable obstacle, it is assumed that the rest of the resultant fluid pressure may be expressed as the sum of two terms, one proportional to $\ddot{\zeta}$ and the other to $\dot{\zeta}$. The factor of the first term represents the so-called added mass, while the second corresponds to the loss of energy by propagation of waves outwards. In these circumstances, equation (1) is reduced to the form

$$M'\ddot{\zeta}+N\dot{\zeta}+g\rho S\zeta=E\cos pt, \qquad (2)$$

where M′ is the total effective mass.

2. Various empirical formulæ have been devised for the effective mass of a ship for heaving motion, and for flexural vibrations. Reference may be made in particular to Lewis (1929) for ship forms, and to Browne, Moullin and Perkins (1930) for the added mass of prisms floating in water. The basic assumption in these studies is to neglect the wave disturbance and to suppose the fluid motion to be that due to a certain solid moving in an infinite liquid, the solid being made up of the immersed part of the floating body and its reflexion in the free surface of the water. The experiments of Browne, Moullin and Perkins showed that this leads to a reasonable value of the added mass, the calculated values being rather higher than those deduced from the experiments.

It is the second term of equation (2), namely $N\dot{\zeta}$, with which the present paper is specially concerned. Instead of calculating the fluid pressures, an alternative method is to work out the mean rate of propagation of energy outwards in the wave motion, and equating this to the mean value of $N\dot{\zeta}^2$ we obtain a value for N for the given frequency. This procedure is permissible under the assumed conditions under which the motion is a forced simple harmonic vibration and the radiated energy is small. To obtain the corresponding logarithmic decrement for the damped natural vibrations, these may be taken as approximately of period $2\pi/\sigma$, with

$$\sigma^2 M'=g\rho S. \qquad (3)$$

Then the logarithmic decrement is given by $\pi N/\sigma M'$, with N having its value for the frequency σ. There is very little work, theoretical or experimental, to which reference can be made. Browne, Moullin and Perkins (1930) measured the damping for prisms vibrating in air and when immersed in water; they conclude "The damping added by the water is negligible compared with the damping due to the supports, a result which might not have been expected." But in those experiments the prisms were not floating freely and the frequency was of the order of 13 per second; it can readily be estimated that the energy in the wave

motion would then be very small. However, the experiments show that damping by fluid friction and eddies was also negligible. Reference may be made specially to work by Schuler (1936) with a vibrating prism of rectangular section, in which direct measurement was made of the amplitude of the waves. The logarithmic decrement was also measured, and it was concluded from the dimensional form of the results that the damping was due to wave motion, viscous and other damping being negligible in comparison. Schuler gives no theoretical calculation of the damping, and unfortunately the data necessary for making a comparison with theory are not recorded, such as the effective mass and restoring force and the free or forced nature of the vibrations.

Coming to the ship problem, as far as published work is concerned there is practically no accurate information about the damping of natural heaving. It is usually stated to be very large, any natural vibrations dying out very quickly. The only numerical estimate appears to be that given by Horn (1936) and said to be an average result derived from a large number of models; his estimate gives a logarithmic decrement for natural heaving of about 1·45. This is very large, and would mean that the amplitude is reduced by about one-half in each swing. It is also stated that the decrement for natural pitching oscillations is of the same order of magnitude.

3. We now examine the waves produced by an oscillating body, and we adopt the method of replacing it by some suitable distribution of alternating sources.

We consider first two-dimensional fluid motion, and we take the origin O in the free surface, Ox horizontal, and Oz vertically upwards. If there is a source of strength $m\cos pt$ per unit length at a depth f, that is at the point $(0, -f)$, the velocity potential is given by

$$\phi=-me^{ipt}\log\frac{r_1}{r_2}+2me^{ipt}\int_0^\infty\frac{e^{-\kappa(f-z)}\cos\kappa x}{\kappa-\kappa_0+i\mu}d\kappa, \quad . \quad . \quad . \quad (4)$$

where $p^2=g\kappa_0$, $r_1^2=x^2+(z+f)^2$, $r_2^2=x^2+(z-f)^2$; and we take the limiting value of the real part of the expression as μ is made zero.

This leads to a surface elevation given by

$$\zeta=\frac{2\pi mp}{g}e^{-\kappa_0 f}\cos(pt\mp\kappa_0 x)-\frac{2mp}{g}\sin pt\int_0^\infty\frac{\kappa\cos\kappa f-\kappa_0\sin\kappa f}{\kappa^2+\kappa_0^2}e^{\mp\kappa x}dk, \quad (5)$$

where the upper or lower signs are taken according as x is positive or negative.

The first term in (5) gives the regular waves propagated outwards on either side; if A is the amplitude of these waves and E the mean rate of propagation of energy outwards per unit length, we have, taking account of both sides of the origin,

$$\mathrm{E}=g\rho p\mathrm{A}^2/2\kappa_0=2\pi^2m^2\rho pe^{-2\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By summation, or integration, we can obtain the corresponding expression for any given distribution of periodic sources in the liquid.

4. Consider a long prism, of rectangular cross-section and of breadth $2b$, immersed in water to a depth f and made to perform small vertical oscillations $a \sin pt$. For an approximate solution we suppose the motion to be two-dimensional and to be that due to a uniform distribution of sources, of density $(pa/2\pi) \cos pt$, over the immersed base of the prism at its mean depth f. The regular waves on the side $x>0$ are given, from (5), by

$$\zeta=(p^2a/g)e^{-\kappa_0 f}\int_{-b}^{b} \cos \{pt-\kappa_0(x-h)\}\,dh$$

$$=2ae^{-\kappa_0 f} \sin (\kappa_0 b) \cos (pt-\kappa_0 x). \qquad (7)$$

Hence, for the mean rate of radiation of energy per unit length of the prism, we have

$$\mathrm{E}=(2g\rho pa^2/\kappa_0)e^{-2\kappa_0 f} \sin^2 (\kappa_0 b). \qquad (8)$$

If the wave-length $2\pi/\kappa_0$ is large compared with the breadth of the prism, we have the simpler forms

$$\zeta=2\kappa_0 abe^{-\kappa_0 f}. \qquad (9)$$

$$\mathrm{E}=2g\rho p\kappa_0 a^2b^2e^{-2\kappa_0 f}. \qquad (10)$$

In the experiments by Schuler (1936) a rectangular prism was used and the amplitude of the waves and other quantities measured directly. Schuler obtained the expression (7) by an indirect energy method suggested by Prandtl, and it was contrasted with the source theory of the effect ; however, we have seen that it follows from assuming a uniform distribution of sources over the base of the prism. The interesting point is that the experimental results agree reasonably well with the expression (7) for periods such that the wave-length $2\pi/\kappa_0$ is greater than the breadth $2b$ of the prism.

5. We now apply these results to the heaving of a ship in still water. We may, as in similar cases, treat the motion as two-dimensional in the first instance, an approximation which may be supported by the experiments of Browne, Moullin and Perkins. These authors also give an approximate formula for the added mass of a ship of normal form in vertical heaving motion ; this is given as $0{\cdot}95\rho b^2l$, where ρ is the density of water, $2b$ the maximum beam and l the total length of the ship.

We take an example from recent work by Kent and Cutland (1941), carried out on models at the National Physical Laboratory. The data for the corresponding ship are : length 400 ft., beam 55 ft., draught 24 ft., displacement 11,332 tons, natural heaving period 7·42 sec. From the formula just given the added mass comes out as 8200 tons ; thus the total effective mass M′ in equation (2) is about 20,000 tons. It is of interest to check this result in a different way. If $2\pi/\sigma$ is the period when damping is neglected, we have the relation given in (3) The change in period due to damping is relatively small, so we may use the recorded period ; further, estimating the water plane area S as 17,600 sq. ft., we obtain from (3) a value for M′ of about 20,000 tons.

Suppose now that the heaving motion is given by $\zeta = a \sin \sigma t$. The wave-length for a period of 7·42 sec. is about five times the beam of the ship. It is thus permissible to take a simple distribution, namely a uniform line source, of strength $m \cos \sigma t$ per unit length, extending over the length L of the ship at some suitable mean depth f. We take the value of m to correspond to the rate of alteration in displaced volume of the ship, which is $\mathrm{S}\dot{\zeta}$ or $\mathrm{S}a\sigma \cos \sigma t$. Hence we take

$$m = \mathrm{S}\sigma a / 2\pi \mathrm{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We put this value of m in (6) and, using

$$\tfrac{1}{2}\mathrm{N}\sigma^2 a^2 = \mathrm{EL}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

we obtain

$$\mathrm{N} = (\rho\sigma \mathrm{S}^2/\mathrm{L}) e^{-2\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For the corresponding logarithmic decrement we have

$$\delta = \frac{\pi \mathrm{N}}{\sigma \mathrm{M}'} = \frac{\pi \rho \mathrm{S}^2}{\mathrm{M}'\mathrm{L}} e^{-2\sigma^2 f/g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Putting in the values already given for this case and taking $f = 20$ ft. as a mean depth, (14) gives the value $\delta = 1{\cdot}4$. The agreement with Horn's estimate is, of course, merely a coincidence. Clearly, this large value of δ goes beyond the assumption on which N has been calculated, namely that the damping is small enough to allow approximately simple harmonic waves to be established. Nevertheless the calculation is sufficient to show that wave motion is quite adequate to account for the large damping which has been observed in practice.

6. There does not seem to have been any experimental work on cases of three-dimensional fluid motion. We shall examine the corresponding theory, as it will allow of more detailed calculation for other source distributions and also of application to pitching. Consider a point source of alternating strength $m \cos pt$ at a depth f below the surface, that is, with the source at the point $(0, 0, -f)$. In this case the surface elevation is given by

$$\zeta = \frac{2ipm}{g} e^{ipt} \left\{ \frac{1}{(r^2+f^2)^{\frac{1}{2}}} - \frac{2\kappa_0}{\pi} \int_0^\infty \int_0^\infty \frac{\kappa \sin \kappa f - \kappa_0 \cos \kappa f}{\kappa^2 + \kappa_0^2} e^{-\kappa r \cosh u} \, du \, d\kappa \right.$$
$$\left. - i\pi\kappa_0 e^{-\kappa_0 f} \mathrm{H}_0^{(2)}(\kappa_0 r) \right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $r^2 = x^2 + y^2$, $p^2 = g\kappa_0$, $\mathrm{H}_0^{(2)} = \mathrm{J}_0 - i\mathrm{Y}_0$, and the real part of the expression is to be taken. There is a corresponding expression for the velocity potential. The first two terms in (15) represent local standing oscillations of the surface, and the third term the symmetrical circular waves propagated outwards. For the present purpose we only require the wave motion at a great distance, and the first term in the asymptotic expansion of the Bessel functions is sufficient; hence, for this part of the velocity potential and surface elevation, we obtain

$$\phi \sim 2\pi\kappa_0 m \left(\frac{2}{\pi\kappa_0 r} \right)^{\frac{1}{2}} e^{-\kappa_0 f + \kappa_0 z} \sin\left(pt + \frac{\pi}{4} - \kappa_0 r \right) \quad . \quad . \quad . \quad (16)$$

$$\zeta \sim \frac{2\pi\kappa_0 p m}{g}\left(\frac{2}{\pi\kappa_0 r}\right)^{\frac{1}{2}} e^{-\kappa_0 f} \cos\left(pt+\frac{\pi}{4}-\kappa_0 r\right). \quad . \quad . \quad . \quad (17)$$

The rate of transmission of energy outwards is obtained from the rate of work of the fluid pressure over the surface of a vertical cylinder of radius r, that is, from

$$-\int_{-\infty}^{0} \rho \frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial r} 2\pi r\, dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Using (16), we obtain for E, the mean rate of transmission of energy outwards,

$$E = 4\pi^2\rho\kappa_0 p m^2 e^{-2\kappa_0 f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This result may be generalized to cover any given distribution over a surface S in the liquid. Let $m \cos pt$ be the source strength per unit area at a point (x', y', z') on this surface; we have to substitute for r in (15) the quantity

$$(r^2-2rx' \cos\theta-2ry' \sin\theta+x'^2+y'^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad (20)$$

and then integrate over the distribution.

It is readily seen that we only need the approximation for r large, namely

$$\phi \sim \left(\frac{8\pi\kappa_0}{r}\right)^{\frac{1}{2}} e^{\kappa_0 z}\left\{P \sin\left(pt+\frac{\pi}{4}-\kappa_0 r\right)+Q \cos\left(pt+\frac{\pi}{4}-\kappa_0 r\right)\right\}, \quad (21)$$

where $$P+iQ=\iint m(x', y', z')e^{\kappa_0 z'+i\kappa_0(x'\cos\theta+y'\sin\theta)}dS. \quad . \quad . \quad . \quad (22)$$

From these expressions, we obtain for the mean rate of outflow of energy

$$E=2\pi\rho\kappa_0 p\int_0^{2\pi}(P^2+Q^2)\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

7. Consider a circular cylinder, of radius b, immersed with its axis vertical to a depth f and making forced vertical oscillations given by $a \sin pt$. As in the two-dimensional problem, if $2\pi g/p^2$ is larger than the diameter of the cylinder, we may assume the wave motion to be due to an alternating source $m \cos pt$ at a depth f, with $4\pi m=\pi b^2 pa$. Hence, from (19), we have

$$E=\frac{\pi^2\rho}{4g} b^4 a^2 p^5 e^{-2p^2 f/g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Assuming that this may be used to evaluate N for the natural damped vibrations when the cylinder is floating freely, we obtain

$$N=\frac{\pi^2\rho}{2g} b^4\sigma^3 e^{-2\sigma^2 f/g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

From the usual hydrostatic theory, $\sigma^2=g/f$. Without attempting to evaluate the effective mass in this case, we write $M'=(1+\mu)M=(1+\mu)\pi\rho b^2 f$. Hence, with these values, the logarithmic decrement is

$$\delta=\frac{\pi N}{\sigma M'}=\frac{\pi^2 b^2 e^{-2}}{(1+\mu)f^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For instance, with $f=4b$ and neglecting μ we should have $\delta=0{\cdot}04$.

8. We may now attempt to apply these results to the pitching motion of a ship. For a long narrow ship the appropriate source distribution could be taken over the longitudinal vertical section of the ship as in the theory of wave resistance for such forms. On the other hand, the keel of the ship may play a large part in wave production in pitching. As a suitable example for calculation, we choose a rectangular form with vertical sides, of length $2l$ and beam $2b$, and floating immersed to a depth f; such a form will clearly exaggerate the wave-making effects of bow and stern. We suppose the form to have an angular pitching oscillation given by $\theta=\theta_0 \sin pt$. We neglect the effect of the vertical ends and consider only the flat base. With the present procedure, we take the source strength at each point such that $4\pi m \cos pt$ is equal to the normal velocity, that is, equal to $x'\dot{\theta}$ in the notation of (22). Hence, from (22), we have in this case

$$\mathrm{P}+i\mathrm{Q}=\frac{p\theta_0}{4\pi}\int_{-l}^{l} dx' \int_{-b}^{b} x' e^{-\kappa_0 f + i\kappa_0(x'\cos\theta + y'\sin\theta)}\, dy' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$=\frac{ip\theta_0 \sin(\kappa_0 b \sin\theta)}{\pi\kappa_0^3 \sin\theta \cos^2\theta}\{\sin(\kappa_0 l\cos\theta)-\kappa_0 l\cos\theta\cos(\kappa_0 l\cos\theta)\}e^{-\kappa_0 f}. \quad . \quad . \quad . \quad . \quad (28)$$

From (23) this gives

$$\mathrm{E}=\frac{8\rho p^3\theta_0^2}{\pi\kappa_0^5}e^{-2\kappa_0 f}\int_0^{\frac{\pi}{2}}\frac{\sin^2(\kappa_0 b\sin\theta)}{\sin^2\theta\cos^4\theta}\{\sin(\kappa_0 l\cos\theta)-\kappa_0 l\cos\theta\cos(\kappa_0 l\cos\theta)\}^2 d\theta. \quad . \quad . \quad . \quad . \quad (29)$$

For the pitching of a ship, as for heaving, $\kappa_0 b$ is a moderately small quantity; (29) then reduces to a simpler form, which might have been derived directly by assuming a line distribution of sources and sinks. We have then

$$\mathrm{E}=\frac{8\rho p^3 b^2\theta_0^2}{\pi\kappa_0^3}e^{-2\kappa_0 f}\int_0^{\frac{\pi}{2}}\{\sin(\kappa_0 l\cos\theta)-\kappa_0 l\cos\theta\cos(\kappa_0 l\cos\theta)\}^2\frac{d\theta}{\cos^4\theta}. \quad . \quad . \quad . \quad . \quad (30)$$

For pitching oscillations of a ship, the usual equation for natural pitching in still water is

$$\mathrm{I}\ddot{\theta}+\mathrm{N}\dot{\theta}+g\rho\mathrm{V}m\theta=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where I is the total effective moment of inertia of the ship, V the displaced volume, and m the longitudinal metacentric height. As before, we estimate N by equating the mean value of $\mathrm{N}\dot{\theta}^2$ to the value of E given by (30), with p equal to the natural frequency σ. There do not seem to be any direct determinations or calculations for the added moment of inertia. We shall therefore derive the effective value of I from the relation

$$\sigma^2\mathrm{I}=g\rho\mathrm{V}m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

with $2\pi/\sigma$ the natural period of pitching when damping is neglected. With this relation, the logarithmic decrement is given by

$$\delta = \frac{\pi N}{\sigma I} = \frac{\pi\sigma N}{g\rho V m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Collecting these results, and expressing the integrand in (30) in terms of Bessel functions, we find

$$\delta = \frac{8\pi\kappa_0 b^2 l^3}{mV} e^{-2\kappa_0 f} \int_0^{\frac{\pi}{2}} J^2_{3/2}(\kappa_0 l \cos\theta) \frac{d\theta}{\cos\theta}. \quad . \quad . \quad . \quad . \quad (34)$$

To obtain a numerical result we take a case from an early paper by Kent (1922). The relevant data for the ship are: length 400 ft., beam 52 ft., draught 22 ft., displacement 10,000 tons, longitudinal metacentric height 458 ft., natural pitching period 6·11 sec.

In (34) we take the distribution of the same length as the ship, that is $l=200$ ft.; we assume a mean beam of 40 ft., and we take $f=20$ ft. These values suffice for a rough approximation. The integral in (34) was computed from tables of Bessel functions; and we obtain finally the result $\delta=1{\cdot}6$. The same general remarks apply to the limitations of this calculation as in the case of heaving motion; however, it is interesting that the decrement δ comes out at about the same value in the two cases.

9. *Summary.*—Using expressions for the wave motion due to alternating sources in a liquid, application is made to the heaving and pitching motions of a ship, and, in particular, to estimating the damping from the rate of propagation of energy outwards in the wave motion. This method of approximation assumes the damping to be small, and the results obtained are too large for much importance to be attributed to the actual numerical values. Nevertheless, it may be concluded that the wave motion gives rise to large damping for both heaving and pitching, and that the decrements are probably comparable with those obtained experimentally.

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THE APPROXIMATE CALCULATION OF WAVE RESISTANCE AT HIGH SPEED

By T. H. HAVELOCK, F.R.S.

SYNOPSIS.—*The main purpose of the paper is to explore the possibility of applying the present theory of wave resistance to models whose lines are not given by mathematical equations. A brief survey of the wave theory is given and this leads to a sub-division of the ship and the corresponding source distribution ; the determination of the latter is based on sectional areas and local prismatic coefficients. For low speeds a large number of divisions is necessary for reasonable approximation and the calculations become too laborious, but results have been obtained for speeds higher than a Froude number* $\sqrt{(v/gL)}$ *of about* 0·4. *These approximations are applied first to experimental models with mathematical lines, and the results compared with those calculated from the usual integrals and with the measured resistances. Finally the method is applied to two models with non-mathematical lines, the necessary data being obtained from the plans and the results compared with measured resistances.*

Introduction

1. IN recent years the comparison of calculated and measured wave resistance has been the subject of much research and considerable success has been achieved ; but the work has necessarily been limited to relatively simple forms of model whose lines can be expressed by mathematical equations. The chief desideratum at the present stage would seem to be an extension of this comparison to a wider range of types and to more normal forms of model ; this would, no doubt, disclose deficiencies in the present theory of wave resistance but would provide a basis for further development and improvement. These considerations suggest an examination of the application of the present theory to models with non-mathematical lines, with a view to seeing whether the difficulties of the calculations can be avoided by approximations giving reasonable accuracy and consistency, even if only over some limited range of speed. The present paper is the record of an attempt to make such calculations ; whether the particular method prove useful or not, it is hoped that the general statement will stimulate interest in the problem and lead to further investigation, both experimental and mathematical.

From one point of view the problem is quite simple. If we assume the well-known integral expressions for wave resistance (4, 6), the matter is one of approximate integration over the ship's surface. The main difficulty arises from the double computation ; intermediate integrals have to be evaluated not only for a sufficient number of stations on the ship but also for a sufficient number of values of a parameter so that the final resistance integral may be computed. The labour involved has prevented any direct calculation on these lines. It is proposed here to examine the problem differently by returning to first principles of the theory of wave resistance, beginning with the simplest possible expressions and trying to find how far it is necessary to go before we get results of sufficient accuracy.

2. We consider a ship moving steadily through the water, and we neglect meantime any effects due to fluid friction. The motion of the water must satisfy the laws of fluid dynamics, together with the necessary conditions at the surface of the ship and at the free surface of the water. Although the problem can be stated thus precisely, and formulated in mathematical terms, it has not been possible to obtain an exact solution for even the simplest form of floating body ; we have therefore to approximate to a solution by successive

steps. The first step is to neglect the wave motion and consider the fluid motion produced by the ship assuming the water surface to remain plane ; the next step is to obtain the wave disturbance produced by this fluid motion while ignoring the presence of the ship. A third step would then be to evaluate the influence of the ship on the waves so calculated, and so on by successive steps. Meantime the theory has not in fact proceeded further than the first two steps.

Equivalent Source Distribution

3. The first step in the process may be expressed in another form. Consider a double ship formed of the immersed volume of the ship and its inverted reflection in the water plane, and suppose this complete solid entirely immersed in water and moving forward with uniform velocity v. Over the fore part of the ship the water is moving forwards and outwards, and over the after part it flows in to follow the motion of the ship. This fluid motion can be represented completely by a definite continuous distribution of sources and sinks over the surface of the ship at each instant and moving with the ship. Let σ be the source strength per unit area at any point of the ship's surface, σ being positive over the fore part and negative over the after part for a normal form. (The notation used in this paper is that if m is the strength of a point source, $4\pi m$ is the volume of liquid flowing out in unit time). It is clear that, since the total volume of water is unaltered, the integrated value of σ over the whole surface is zero, or the sum of the positive sources is equal to the sum of the negative sources. On the other hand, if x is the distance of any point from some transverse reference plane, say the mid-ship section, the integrated value of σx taken over the whole surface is a definite amount and is the moment M of the distribution. A simple expression for M can be derived from general principles without knowing the actual distribution. It can be shown that

$$M = (1 + k)\, Vv/4\pi \qquad (1)$$

In this expression V is the volume of the body, and k is the inertia coefficient for longitudinal motion : that is, $g\rho kV$ is the added mass due to the motion of the water.

If v_n is the component of the velocity v normally outwards at any point of the ship's surface, it is convenient to write the corresponding source distribution in the form $\sigma = (1 + k^1)v_n/4\pi$. In general, k^1 varies from point to point, but for an ellipsoid it is constant and equal to k. The added mass for longitudinal motion is not very important in ship problems and there are few estimates of its value. It is of interest to note that W. Froude investigated this effect in his well-known experiments on H.M.S. *Greyhound*. He made two sets of experiments, one with retarded motion and the other with acceleration ; the former gave a coefficient of about 20% and the latter of about 7%, and on experimental grounds Froude attached more value to the larger estimate. Whatever may be the interpretation of experimental results, we are concerned here with the theoretical coefficient for non-viscous fluid motion ; and there is reason to regard the lower value as more appropriate for normal ship forms. Although this correction should be noted for future examination, we may meantime regard it as relatively small, at least for the so-called narrow models to which the wave theory has so far been limited. The usual approximation amounts, in fact, to neglecting the inertia coefficient k for longitudinal motion ; and, in what follows, we take the source strength per unit area to be given by $\sigma = v_n/4\pi$. We can easily verify the total moment M of the distribution in this case. Imagine a horizontal cylinder of small cross section cutting the midship section in an area dS, and cutting out an area dS_1 at a point P_1 on the fore part of the ship's surface and an area dS_2 at a point P_2 on the after part. Then we have

$$\begin{aligned} \sigma_1\, dS_1 &= v_n\, dS_1/4\pi = v dS/4\pi \\ \sigma_2\, dS_2 &= -\, v dS/4\pi \end{aligned} \qquad (2)$$

Hence

$$M = \int v P_1 P_2 dS/4\pi = vV/4\pi \qquad (3)$$

the integral being taken over the midship section, and V being the immersed volume.

To sum up, with this approximation, the source distribution on the ship's surface is specified as follows : the total source strength on any portion of the surface is given by $v/4\pi$ times the area of the projection of that portion on to the midship section, with an obvious rule for determining the sign of the projected area.

Formulæ for Wave Resistance

4. We give now expressions which will be used for the calculation of wave resistance ; for general formulæ for any distribution of sources reference may be made to *Roy. Soc. Proc.* A. 138, p. 339 (1932). We take the origin O on the centre line of the form and in the water plane, Ox in the direction of motion, Oz vertically downwards, and Oy horizontally at right angles to the other two axes. We shall be concerned here with sources only in the longitudinal zx-plane. If we have any distribution of which a typical source is of strength m_r at the point (x_r, o, z_r), the corresponding wave resistance is given by

$$R = 16\pi k^2\rho\int_o^\infty (I^2 + J^2) \cosh^2 u \, du \quad . \quad . \quad . \quad (4)$$

where

$$I = \Sigma m_r e^{-kz_r \cosh^2 u} \sin (kx_r \cosh u)$$
$$J = \Sigma m_r e^{-kz_r \cosh^2 u} \cos (kx_r \cosh u) \quad . \quad . \quad . \quad (5)$$

where $k = g/v^2$ and the summation extends over the given system of sources.

If we make the assumptions for a narrow ship, outlined in the previous sections from a somewhat different point of view, it can easily be verified that these expressions lead to the usual integrals for the wave resistance. We have the same form for R with

$$I = \frac{c}{2\pi}\iint (\partial y/\partial x) \sin (kx \cosh u) e^{-kz \cosh^2 u} \, dx dz$$
$$J = \frac{c}{2\pi}\iint (\partial y/\partial x) \cos (kx \cosh u) e^{-kz \cosh^2 u} \, dx dz \quad . \quad . \quad . \quad (6)$$

The integrals are taken over the longitudinal section of the ship, and $(\partial y/\partial x)$ is taken from the equation of the surface of the ship.

First Approximations

5. After this preliminary survey we proceed to the immediate problem, namely dividing the ship into a finite number of sub-divisions. Although of no practical value, we begin with the most extreme simplification to illustrate the point of view of the present study. We have seen that the total moment of the source distribution is $Vv/4\pi$ where V is the immersed volume and v the speed. We now suppose this moment to be concentrated at a point as a source and sink doublet with its axis in the direction of motion. The longitudinal location of this doublet is immaterial so far as the resistance formula is concerned and we may suppose it to be in the midship section. For its depth we use here, and throughout the work, the principle that for a first approximation we replace any system of sources by a source of the total strength placed at the centroid of the system. Since the source strength on any element of the ship's surface is proportional to the projection of that element on the midship section, it follows at once that the depth of the centroid of the distribution is the depth h of the centroid of the midship section. Thus the first approximation is a horizontal doublet of moment $Vv/4\pi$ at a depth h. Putting these values into the expression for the wave resistance of a doublet, which may be deduced from (4), (5), we obtain

$$R = (g\rho/\pi) k^3 V^2\int_o^\infty e^{-2kh \cosh^2 u} \cosh^5 u \, du \quad . \quad . \quad . \quad (7)$$

This integral can be expressed in terms of Bessel functions, and its value obtained readily from tables of these functions.

It is clear that this extreme simplification can only be an ideal limit for very high speeds, and it is no use comparing it with experimental results. It is, however, of interest as the limit towards which the usual complete theoretical expressions should tend. Consider, for instance, the simplest type of experimental model with parabolic lines, the surface being specified by

$$y = b\,(1 - z^2/d^2)\,(1 - x^2/l^2) \qquad (8)$$

Calculations, meantime unpublished, from the complete integrals (4) and (6) have been made recently for very high speeds by Mr. W. C. S. Wigley, who has placed his results at my disposal. Taking the Froude number $f = v/\sqrt{(gL)}$, the highest value for which calculations were made was $f = 1{\cdot}77$. With length $= L = 2l = 16$ ft., beam $= 2b = 1{\cdot}5$ ft., draught $= d = 1$ ft., at this value of f the complete formula gives a wave resistance of 31·8 lb. Calculating from (7) with $V = 10\frac{2}{3}$ cub. ft., $h = \frac{3}{8}$ ft., we obtain a resistance of about 40 lb. The comparison is not so far out as might have been anticipated, and to that extent it may be taken as confirming the argument by which the simple formula was obtained.

6. The next simplest dissection of the ship is to divide it into two by the midship section. We consider the fore and aft parts separately, replacing each part by a single source at the centroid of the distribution in each case. For the positive sources on the fore part of the ship we have seen that if M is the area of the midship section the total source strength is $Mv/4\pi$. From the argument in the previous sections it is readily seen that the moment of the distribution with reference to the midship section is $V_1 v/4\pi$, where V_1 is the volume of the fore part; hence the centroid is at a distance V_1/M, or $p_1 l_1$, ahead of the midship section, where l_1 is the length of the fore part and p_1 its prismatic coefficient. Similarly the controid of the negative sources on the after part is at a distance $p_2 l_2$ astern of the midship section, where l_2 is the length of the after part and p_2 its prismatic coefficient. Thus we have a pair of sources, positive and negative, each of numerical strength $Mv/4\pi$, at the depth h of the centroid of the midship section, and at a distance pL apart, where L is the length of the ship and p its prismatic coefficient. Applying the formulæ (4), (5) to this combination, we obtain for the wave resistance

$$R = (4g\rho/\pi)\,kM^2\int_0^\infty e^{-2kh\cosh^2 u}\sin^2(\tfrac{1}{2}kpL\cosh u)\cosh^3 u\,du \qquad (9)$$

This is an interesting expression from a theoretical point of view, as it brings in factors which are admittedly of the first importance in wave resistance: the area of the midship section and the depth of its centroid, or roughly the depth of the centre of buoyancy of the ship, the length of the form and its prismatic coefficient. But it will clearly exaggerate, in general, the interference between bow and stern systems; and it is too simplified for practical purposes, except possibly for special types of model over a limited range of speed.

General Sub-division of the Ship

7. The total moment of the ship is distributed in a continuous source distribution over the surface of the ship: distributed in length, in depth, and in beam. The last of these is neglected in the usual theory and we leave it on one side meantime, noting the possibility of including it in further developments. Of the other two, the distribution in length is specially important. We now divide the ship by taking transverse sections at any required number of stations; for simplicity at first we consider complete sections, leaving subdivision in depth till later. Let S_1, S_2 be the areas of any two transverse sections, say in the fore part of the ship with $S_2 > S_1$. The total source strength on the ship's surface between these stations is

$$(S_2 - S_1)\,v/4\pi \qquad (10)$$

The ship being symmetrical with respect to the vertical longitudinal section, the centroid of the distribution lies on this median plane. Its depth is the

depth of the centroid of the area between the corresponding traces of the sections on the body plan of the ship. The longitudinal location of the centroid may be specified by a kind of local prismatic coefficient for the increase in volume in relation to the increase in cross-sectional area. It is readily seen, from the argument in the previous sections, that if $\bar{x}$ is the distance of the centroid ahead of the station S_2, x_{12} the distance between the stations and V_{12} the volume between them, we have

$$\bar{x} = (V_{12} - S_1 x_{12})/(S_2 - S_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The same construction holds if we take horizontal sections in addition, and subdivide in depth as well as in length. We replace each subdivision so formed by a single source at a certain point ; the strength of the source and its location are easily derived from the usual data for the ship, for example the curves of sectional areas and volumes, the body plan and principal dimensions. We may exhibit this information in the form of a diagram representing the longitudinal section of the ship divided into compartments ; in each compartment is placed a number for the strength of the source at a given point in that compartment. The diagram gives quantitative information about the wave-making quality of the ship, and may be useful even if we do not carry out the subsequent calculation for the wave resistance. It may be noted that we have tacitly assumed a normal form of ship, with the sources all positive on the fore part and all negative on the after part. For a bulbous bow, for instance, we should have a superposed source and sink combination which could be calculated by the same procedure. Of course, if we pursue this process far enough to arrive at very small subdivisions, we are back at the original problem of approximate evaluation of the complete theoretical integrals ; in particular, the precise location of the elementary source within its compartment would lose significance. It remains to be seen whether, with the particular method described above, a relatively small number of subdivisions will give any accuracy in calculation. It is obvious in advance that high speeds will give conditions most suitable for comparison ; roughly speaking, the deciding factor is the relation between the distance between stations and the predominant wave length, and as we come down to lower speeds it will be necessary to increase the number of stations.

Comparison with Experimental Models

8. Before applying the method to models with non-mathematical lines, we test it by comparison with experimental models of simple form. We take first the parabolic form, the equation of whose surface has been given in (8). Extensive calculations have been made for this form from the usual complete integrals and tables of the various integrals have been given by Wigley in a recent paper.* We shall take, at first, complete transverse sections at $x = 0, \pm \frac{1}{2}l, \pm \frac{3}{4}l$. The sections are all similar and their centroids, and therefore those of their differences, are all at the same depth $\frac{3}{8}d$. The sectional area is given by $S = M(1 - x^2/l^2)$. Using the formulæ (10) and (11), we obtain sources of strengths, omitting the common factor $vM/4\pi$,

$$\tfrac{1}{4},\ \tfrac{5}{16},\ \tfrac{7}{16}$$

$$\text{at } x = \tfrac{1}{3}l,\ \tfrac{19}{30}l,\ \tfrac{37}{42}l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

respectively. The model being symmetrical fore and aft, and neglecting viscosity, there are corresponding negative sources at similar negative values of x. Referring to (4) and (5), the cosine terms cancel out, and we are left with

$$R = (g\rho/\pi)kM^2 \int_0^\infty I^2 \cosh^2 u \, du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$I = 2e^{-\frac{3}{8}kd\cosh^2 u}\{ \cdot 25 \sin(\cdot 333\, kl \cosh u)$$

$$+ \cdot 3125 \sin(\cdot 633\, kl \cosh u) + \cdot 4375 \sin(\cdot 881\, kl \cosh u)\} \quad . \quad . \quad (14)$$

*"Calculated and Measured Wave Resistance of a Series of Forms defined Algebraically, the Prismatic Coefficient and Angle of Entrance being Varied independently," by W. C. S. Wigley, M.A. *I.N.A.* Vol. 84, p. 52, 1942.

Calculations have been made for the standard model with length $= 16$ ft., beam, $= 1{\cdot}5$ ft., draught $= 1$ ft. We note that $k = g/v^2 = 1/f^2L$, where f is Froude's number. The integral was evaluated by direct quadrature, and no attempt was made to attain any high degree of accuracy in the numerical values as the work is regarded mainly as an exploration of possibilities ; if necessary, more systematic methods of computation could be devised, but meantime it is hoped there are no errors serious enough to invalidate the general deductions.

For a given value of f, the sines in (14) were calculated for values of u increasing by 0·2, or in some cases by 0·1 ; it was not found necessary to go beyond $u = 4$, because of the decrease in the exponential factor. The integrand in (13) was then calculated for these values and graphed as a function of u, so that additional values could be inserted where needed ; finally the value of the integral was obtained by the usual rules for the area under the graph. In Table 1, the wave resistance in lb. calculated in this way from (13) and (14) is denoted by R_a ; the corresponding values R_c have been obtained from the tables given by Wigley, using the complete theoretical integrals and omitting any correction for viscosity.

TABLE 1

f	·265	·303	·341	·404	·522	·607	·884
R_a	1·93	3·48	2·44	8·0	20·7	23·7	27·1
R_c	1·08	3·25	2·43	7·9	20·4	23·6	26·1

The agreement in the range ·341 to ·607 is surprisingly good ; the differences, it should be stated, are well within the limits of possible error in the present numerical computations. At lower speeds it was expected that the subdivision would be too coarse-grained, and the approximation gives unreliable results due to accidental coincidences between the various sine terms. One way of expressing it is that replacing the model by a small number of finite sources introduces interference effects between these sources taken in pairs and these become important at the lower speeds ; whereas in the actual model with its continuous lines these are smoothed out. To obtain the same result by calculation we should have to increase the subdivision in length. Suppose we take, in addition, a horizontal section at half-draught, then, considering any transverse section, 11/16 of the area is above and 5/6 is below this level ; further the centroid of the upper portion is at a depth $21d/88$, and that of the lower portion is at a depth $27d/40$ below the water plane. Hence all that is necessary, for this model, is to replace the exponential factor in (14) by

$$\tfrac{11}{16}e^{-\frac{21}{88}kd\cosh^2 u} + \tfrac{5}{16}e^{-\frac{27}{40}kd\cosh^2 u}$$

Various calculations have been made in this way, and also with different transverse sections. In general it may be said that numerical values are increased by greater subdivision in depth and diminished by additional transverse sections ; increasing both enables one to increase the range of speed for which effective agreement can be obtained.

9. We have now to examine the extent to which the approximation reflects changes in form and whether it is sufficiently sensitive in that respect. In the paper already quoted, Wigley compares calculated and experimental values for a set of models defined by two parameters, the general equation of the forms being

$$y = b\,(1 - z^2/d^2)\,(1 - x^2/l^2)\,(1 + a_2x^2/l^2 + a_4x^4/l^4) \quad . \quad . \quad . \quad . \quad (15)$$

It is a simple matter to obtain general formulæ for the sectional areas and their differences for any scheme of subdivision, and for the positions of the respective centroids in accordance with (10) and (11), and they need not be reproduced here. We shall take three particular cases, for two of which experimental results are also given in the paper quoted.

Model 1,970 *B*. This is specified by

$$a_2 = \cdot 4375 \; ; \; a_4 = - \cdot 4375$$
$$2l = 1{\cdot}6 \text{ ft.} \; ; \; 2b = 1{\cdot}5 \text{ ft.} \; ; \; d = 1 \text{ ft.}$$

For comparison with the previous case we take the same subdivision : no horizontal section, and transverse sections at $x = o, \pm \frac{1}{3}l, \pm \frac{2}{3}l$. The approximate source distribution could be shown on a diagram of the longitudinal section of the model ; it is given here in Table 2, with the divisions not drawn to scale.

TABLE 2

− ·485	− ·327	− ·188	·188	·327	·485

These numbers, when multiplied by $vM/4\pi$, give the source strengths. The depth of the sources is $\frac{2}{3}d$, while the longitudinal distances from the midship section are found to be $x/l = \pm \cdot 349, \pm \cdot 638, \pm \cdot 879$.

Comparison may be made with the distribution for the previous model with $a_2 = a_4 = o$. Since we have not taken any horizontal section, the differences correspond to those on the curves of sectional area—or, rather, to the differences in the gradients of those curves. Using these coefficients for the sine series we calculate R from (13). Using the same notation as in Table 1, the comparison between calculated values from the complete integrals and from the approximation is shown in Table 3.

TABLE 3

f	·303	·341	·404	·522	·607	·884
R_a	5·82	4·23	8·67	21·74	24·9	28·4
R_c	5·34	4·31	8·83	21·7	24·6	27·9

Model 1970 *C*. In this case

$$a_2 = \cdot 8125 \; ; \; a_4 = - 1{\cdot}3125$$
$$2l = 16 \text{ ft.} \; ; \; 2b = 1{\cdot}5 \text{ ft.} \; ; \; d = 1 \text{ ft.}$$

With the same subdivision, we find the distribution shown in Table 4.

TABLE 4

− ·456	− ·385	− ·159	·159	·385	·456

For the horizontal location of the sources, we obtain.

$$x/l = \pm \cdot 377, \pm \cdot 639, \pm \cdot 870.$$

For comparison with experimental results, the calculated values have been expressed in terms of $© _r = R \{ 25{\cdot}41/\delta^{\frac{1}{3}} v \}^2$, where δ is the displacement and v the velocity. The results are shown in Fig. 1, in which the two curves have been reproduced from Wigley's paper. One curve is for the residuary resistance, obtained in the usual way by deducting from the total measured resistance the part due to skin friction calculated from Froude's coefficients ; the other curve is for the wave resistance calculated from the theoretical integrals, without viscosity correction. The results obtained by the present approximation are shown by a cross for each of the speeds for which calculations were made.

Model 2038 *C*. This model is specified by

$$a_2 = -0{\cdot}5 \; ; \; a_4 = 0 \; ;$$
$$2l = 16 \text{ ft} \; ; \; 2b = 1{\cdot}75 \text{ ft.} \; ; \; d = 0{\cdot}5 \text{ ft.}$$

In addition to the variations in the parameters, we have larger beam and only half the draught. Taking the same sections we obtain the scheme in Table 5.

The sources are at depth $\frac{3}{8}d$, and the longitudinal positions are $x/l = \pm \cdot 327$, $\pm \cdot 627$, $\pm \cdot 869$. Making calculations with this plan it was found that the values at the lower speeds tended to be too large. This is probably due to the large source strength in the middle compartments compared with the previous cases, and possibly to the shallower draught. It was decided to take additional transverse sections so as to divide each middle compartment into two of equal strength ; this can be calculated from the general formulæ (10) and (11). Thus the scheme finally adopted is $- \cdot 314$, $- \cdot 342$, $- \cdot 172$, $- \cdot 172$, $\cdot 172$, $\cdot 172$, $\cdot 342$, $\cdot 314$, with the longitudinal positions given by $x/l = \pm \cdot 227$, $\pm \cdot 418$, $\pm \cdot 627$, $\pm \cdot 869$. The depth is the same as before. The consequence is that we have now four sine terms to evaluate. The results are shown in Fig. 1, the curves being reproduced from Wigley's paper and the values from the present approximation denoted by crosses.

TABLE 5

− ·314	− ·342	− ·344	·344	·342	·314

TABLE 6

10 9 8 5 2 1 0

− ·362	− ·136	− ·110	·183	·163	·262
− ·247		− ·145	·126	·266	

The agreement shown in Fig. 1 between the two sets of calculated values is reasonably good. The four cases which have been examined, taken together, give some idea of the scope of the approximation and of the measure in which it responds to changes in the form of the model. It is not the present purpose to compare calculated results with experimental, but the latter have been included in Fig. 1 for the last two cases. It should be noted that viscosity effects have been neglected, but these are comparatively small at the speeds under consideration ; moreover, the residuary resistance has not been corrected by any allowance for form effect upon the frictional resistances, or similar refinements. It is generally considered that the main difference between calculated and experimental values of wave resistance at these speeds is due to sinkage and trim of the model. From the point of view of the present work, this would be reflected mainly in an increase in the effective area of the midship section ; and it can be seen from the formulæ that the values are very sensitive to changes in this factor.

Models with Non-mathematical Lines

10. We proceed now to apply the method to models with lines not given by mathematical equations, for which the wave resistance has not hitherto been calculated. For obvious reasons, in view of the range of speeds under consideration, it is not possible to deal with recent models. Data have, however, been obtained for two models : these include complete plans and dimensions, together with the record of actual measurements of resistance. (I am indebted to Mr. J. L. Kent, Superintendent of the William Froude Laboratory, for permission to use this material, and to Mr. W. C. S. Wigley for much valuable help.)

Model A. The body plan and other data for this model are shown in Fig. 2.

It is obvious that the problem is more complicated than for the simple model, symmetrical fore and aft and with similar transverse sections throughout. After some preliminary calculations it was decided to take the following subdivision : one horizontal section throughout at half draught ; in the upper half, transverse sections at stations 1, 2, 5, 8 and 9 ; and in the lower half, transverse sections at stations 2, 5, and 8. The various sectional areas and the depths of their centroids, and the corresponding volumes were obtained from the plans. From the sectional areas in the upper and lower halves, we obtain the source diagram shown in Table 6.

The midship section area is 0·4306 sq. ft., so the source strengths are the numbers in Table 4 multiplied by $\cdot 4306v/4\pi$, with v in ft./sec. For the depths of the sources, those in the upper row range between ·094 ft. and 1·43 ft., while in the lower row they range from ·3 ft. to ·32 ft. Instead of using all these depths, giving separate exponential factors for the terms in the formulæ, we shall use a mean depth for each row. It is obvious from the construction that the mean depth in each case is the depth of the centroid of the corresponding half of the midship section ; these depths are ·107 ft. and ·302 ft. respectively. For the horizontal positions of the sources we carry out the calculations required by (11) ; with x measured forward from the midship section, we obtain, with x in ft.,

x (upper) : $-6{\cdot}218,\ -4{\cdot}869,\ -2{\cdot}806,\ 2{\cdot}782,\ 4{\cdot}804,\ 6{\cdot}157$
x (lower) : $-5{\cdot}16,\ -2{\cdot}635,\ 2{\cdot}778,\ 5{\cdot}586.$

Since the model is not symmetrical fore and aft, we have to consider both sine and cosine series in (5). The expressions for I and J can now be written down ; each of them contains ten terms, but we simplify them further for approximate computation. We group the terms in pairs for corresponding compartments fore and aft of the midship section. For instance, in the upper row we have the pairs.

$$\cdot 183 \sin(2{\cdot}782\, q) + \cdot 110 \sin(2{\cdot}806\, q) \text{ in } I,$$

and

$$\cdot 183 \cos(2{\cdot}782\, q) - \cdot 110 \cos(2{\cdot}806\, q) \text{ in } J,$$

where we have written q for $(g/v^2)\cosh u$. We replace these by $\cdot 293 \sin(2{\cdot}794\, q)$ and $\cdot 073 \cos(2{\cdot}794\, q)$ respectively, the difference so made being unimportant. Making a similar change for all the pairs of terms, we find that the cosine terms are small compared with the sine terms ; further, their contributions to the resistance integral are proportional to their squares, and we propose to neglect the cosine terms. It has, however, been verified by approximate calculation at one or two speeds that the cosine terms would not add more than about one per cent. to the resistance. Finally, we are left with

$$\begin{aligned} I = {} & e^{-\cdot 107p}\{\cdot 293 \sin(2{\cdot}794\, q) + \cdot 299 \sin(4{\cdot}83\, q) \\ & \quad + \cdot 624 \sin(6{\cdot}187\, q)\} \\ & + e^{-\cdot 302p}\{\cdot 271 \sin(2{\cdot}706\, q) + \cdot 513 \sin(5{\cdot}373\, q)\} \end{aligned} \quad . \quad . \quad . \quad (16)$$

where $p = (g/v^2)\cosh^2 u$, $q = (g/v^2)\cosh u$.

With (16) and (4), the wave resistance has been calculated for six speeds ranging from $f = \cdot 352$ to $f = \cdot 749$. The results are shown in the dotted curve of Fig. 4 as values of R/b^2v^2, where $2b =$ beam. The experimental curve has been obtained in the usual way, the residuary resistance being the actual measured resistance less the skin friction calculated from the wetted surface at rest and the appropriate Froude coefficient. The difference between experimental and calculated values is much the same as for the previous cases. The falling off in calculated value at very high speeds is rather more than usual ; this may be due in part to the approximation, but most of it could be accounted for by the effect of sinkage and trim.

Model B. The body plan and other data are shown in Fig. 3. This model has the same displacement, length and beam as Model A, but has greater draught.

With the same sections as before, the corresponding source distribution is shown in Table 7.

TABLE 7

	10–9	9–8	8–5	5–2	2–1	1–0
	·326	·203	·075	·239	·177	·189
	·177		·219	·181	·214	

The midship section are is 0·4671 sq. ft., and the depths of the centroids of the upper and lower portions are 0·113 ft. and 0·334 ft. respectively. For the horizontal distances, in ft. we obtain

x (upper) : $-6{\cdot}11,\ -4{\cdot}89,\ -3{\cdot}09,\ 2{\cdot}71,\ 4{\cdot}763,\ 6{\cdot}11$

x (lower) : $-4{\cdot}6,\ -3{\cdot}53,\ 2{\cdot}42,\ 5{\cdot}355$

Comparing with the scheme for Model A, we see that there is a greater degree of dissymmetry between the positive and negative distributions ; this makes the calculations more troublesome, as we cannot neglect the cosine terms altogether. Grouping the terms in pairs as before, we neglect the cosine terms for the lower row of sources as unimportant, and we obtain

$$I = e^{-\cdot 113p}\{\cdot 314\sin(2{\cdot}9\,q) + \cdot 38\sin(4{\cdot}826\,q) + \cdot 515\sin(6{\cdot}11\,q)\}$$
$$+\,e^{-\cdot 334p}\{\cdot 4\sin(2{\cdot}975\,q) + \cdot 391\sin(4{\cdot}978\,q)\},$$
$$J = e^{-\cdot 113p}\{\cdot 164\cos(2{\cdot}9\,q) - \cdot 027\cos(4{\cdot}826\,q) - \cdot 137\cos(6{\cdot}11\,q)\} \quad . \quad . \quad . \quad (17)$$

The resistances have been calculated from (17) and (4) ; it was found that in this case the cosine terms add about six per cent. to the final values. The calculated and experimental curves are shown in Fig. 4 ; the calculated values are in general rather higher than might have been anticipated. For both these models, the calculated values at the lower speeds could probably be improved by a more suitable subdivision and more detailed computation.

General Remarks

11. A few notes may be added on matters left over for further investigation.

Beam. In addition to subdivision in length and depth, we might also take longitudinal sections ; for instance, suppose we take a section through the median vertical plane. Then instead of a distribution of sources in one plane, we have a space distribution which could be specified and located by the same methods ; and expressions for the wave resistance could be obtained from the general formulæ. The effect might be examined theoretically in some simple case ; but it is only likely to be of importance at low speeds where several other factors also affect the results.

Viscous Effects. One effect of viscosity is that the frictional belt round the ship makes the run and stern less effective in wave-making. This can be represented, somewhat empirically, by a reduction factor for the after part of the ship. This reduction factor, if obtained from comparison between calculated and measured resistances, will include other effects of viscosity than that just mentioned ; in fact, it will also probably include in some cases effects for non-viscous flow which have been left out of account meantime for instance, what might be called a screening effect of the bow for models with broad beam. However that may be, any empirical factor could be used in the present scheme by making the necessary reduction in the numerical magnitudes of the negative sources for the after part ; this would mean including the cosine series in the formulæ ; otherwise the calculations would be the same. At sufficiently low speeds, if we assume that—for one reason or another the stern contributes little to the wave-making, then the same number of sections as were necessary for the whole length of the ship might, if concentrated over the effective length of the bow, give a sufficiently fine subdivision for approximate calculation.

Location of Sections. Probably the best method of locating the transverse sections would be one which was to some extent related to the type of model ; there are some indications to that effect in the present work. For convenience in a first survey the sections have been taken at fixed stations, both the strengths of the sources and their positions varying from model to model. Another

plan would be to take sections giving equal differences of sectional area, and this would lighten the numerical work to some extent. On the other hand, it would be possible to locate the sections so that the sources were for the most part in fixed positions relative to the length of the model, and such a scheme would have the great advantage of allowing of tabulation of the sine and cosine terms in advance. Obviously any scheme which permits tabulation and systematic procedure in the computation would not only give greater accuracy in the calculations but would make it possible to extend their range of application.

As a general conclusion from the present work it may be said that, although the method needs further testing and systematizing, it indicates a possibility of calculating wave resistance from the plans of the model, at least for high speeds ; and that the results so obtained would agree fairly well with those that could be calculated from the usual integrals if the lines of the model were given by mathematical equations. If this should prove to be the case, it would be possible to have a greater variety of form in experimental models, so providing more material for comparison between theory and experiment and giving ultimately a better basis for application of the calculations in practice.

Fig. 1—*see next page.*

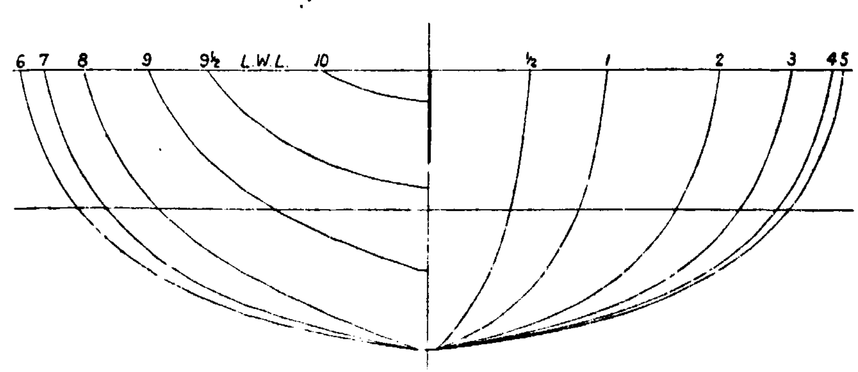

Fig. 2—*Model A.* 13·57′ × 1·28′ × 0·434′.
Displt. 259·4 lb. M.S. coefft. 0·775. Prism. coefft. 0·711.

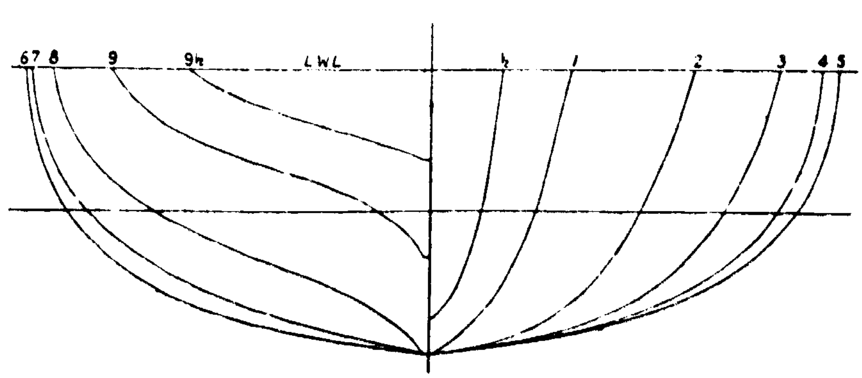

Fig. 3—*Model B.* 13·57′ × 1·28′ × 0·455′.
Displt. 259·4 lb. M.S. coefft. 0·802. Prism. coefft. 0·656.

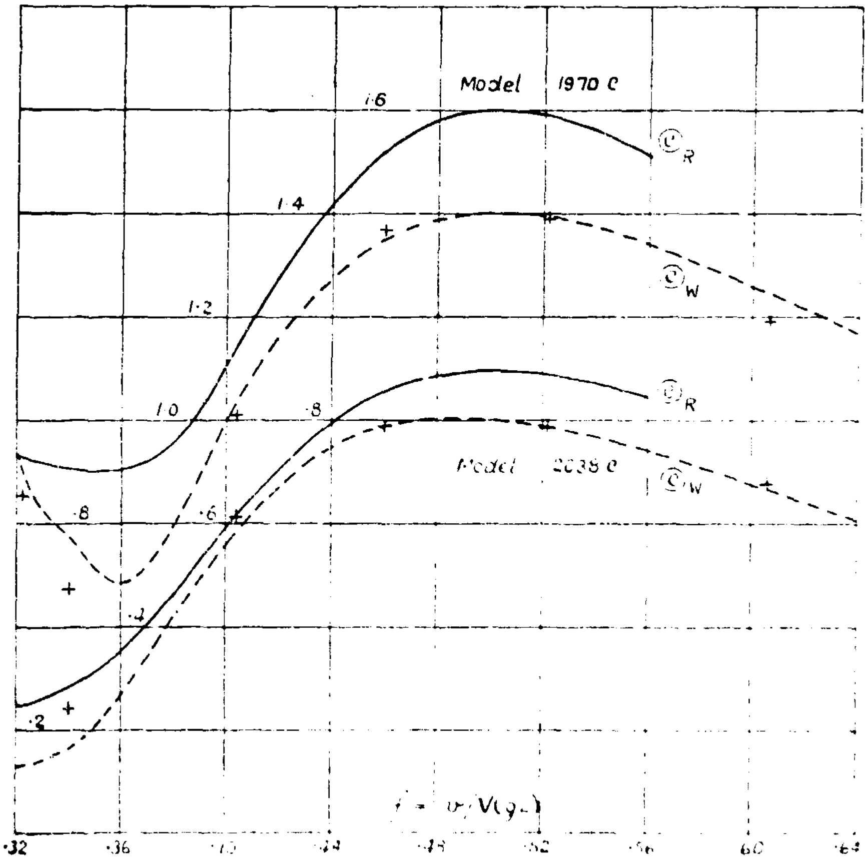

Fig. 1.—Values from Approximation compared with Curves of Measured and Calculated Values from Wigley (1942).

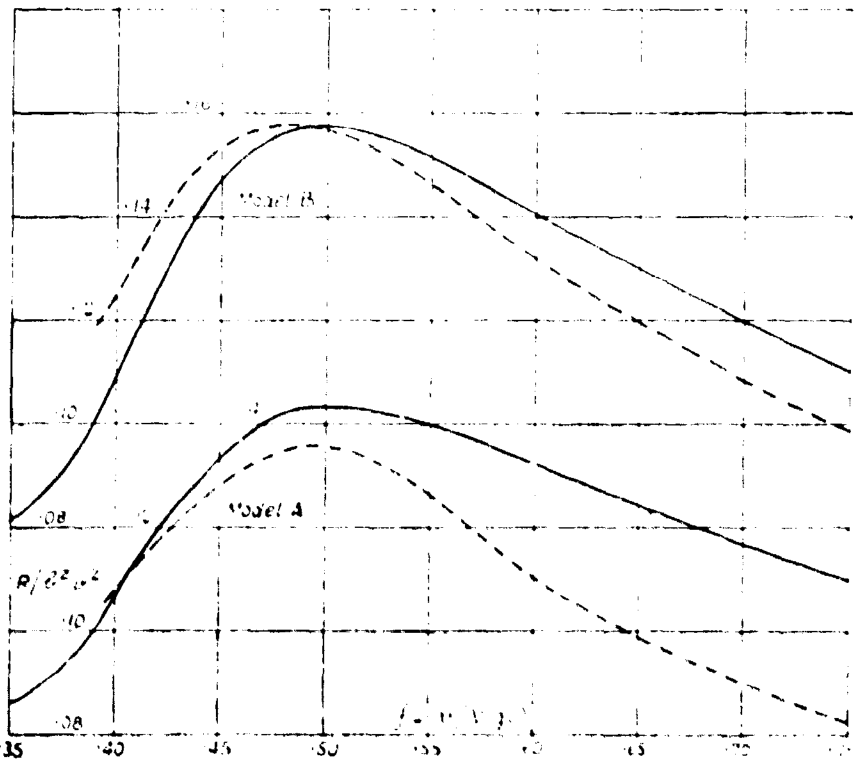

Fig. 4 ——— *from experiment.* - - - - *from approximation*

NOTES ON THE THEORY OF HEAVING AND PITCHING

By Professor T. H. HAVELOCK, M.A., D.Sc., F.R.S., Honorary Member.*

Summary

The main points in the paper are (i) a calculation of the damping of heaving and pitching due to the waves produced by the motion of the ship, (ii) an examination of the extra resistance caused by the reflection of a regular train of waves by the ship's surface, (iii) a suggested theory which gives an extra resistance more closely associated with the heaving and pitching motions.

No attempt is made to formulate a complete theory; the work is based, in the main, on the usual approximate first-order equations of motion and the hydrodynamical theory is that of potential fluid motion under gravity and neglecting viscosity. Details of mathematical analysis are given in an appendix, and the paper gives an account of the work together with numerical calculations and comparison with experimental data.

Oscillations in Smooth Water

The usual approximate equations for heaving and pitching in smooth water are

$$M\ddot{\zeta} + N_1\dot{\zeta} + g\rho S\zeta = 0 \quad . \quad . \quad . \quad (1)$$

$$I\ddot{\theta} + N_2\dot{\theta} + Wm\theta = 0 \quad . \quad . \quad . \quad (2)$$

In these equations ζ = upward displacement of the centre of gravity G, θ = angle of pitch about the transverse axis through G measured positive with bows up, S = area of water plane section, $W = g\rho V$ = displacement in the equilibrium position, m = longitudinal metacentric height. Further, it is assumed that the ship has a simple symmetrical form so that there is no coupling between heaving and pitching so far as first-order equations are concerned. N_1 and N_2 are coefficients which are considered later.

Effective Mass and Moment of Inertia.—It has been observed that the periods of heave and pitch in still water are approximately equal, and it is easily seen how this arises. Suppose at first that we neglect the damping terms in (1) and (2), and also ignore the effect of the inertia of the surrounding water. Then (1) gives for the period of heaving $2\pi\sqrt{(d/g)}$, where $d = V/s$ = mean uniform draught. Turning to equation (2), the longitudinal metacentric height is of the order of the length of the ship and a usual first approximation is to take

$$m = GM = BM = Sk^2/V = k^2/d$$

where k is the radius of gyration of the water plane section about the transverse axis.

If K is the radius of gyration of the ship about the transverse axis for pitching it can be seen that, at least for uniform loading, K^2 differs from k^2 by a quantity of the order of the square of the ratio of draught to length; thus, except for special types of form or mass distribution, we may take K^2 as approximately equal to k^2. Hence in (2), we have $I = Wk^2/g$ and $m = k^2/d$, and the result is the same approximate period $2\pi\sqrt{(d/g)}$ for pitching as for heaving.

For mean uniform draught ranging from 20 ft. to 30 ft., this means a period of from 5 sec. to 6 sec. The natural periods for usual types of cargo ship generally range from 6 sec. to 7 sec. The difference arises from two causes, damping and the inertia of the water. Even with large damping the effect on the period is comparatively small, and practically all the difference is due to the inertia of the surrounding water.

The calculation of added mass for heaving usually proceeds on the assumption that we may replace the immersed volume of the ship by a double ship wholly immersed in an infinite liquid; this underlies the work of F. M. Lewis (R.1†) and of Browne, Moullin and Perkins (R.2). There do not seem to be any similar calculations for rotation, or any with direct application to ship forms. One remark may be made about such calculations for a floating body. A complete solution, satisfying the condition of constant pressure at the free surface of the water, would include wave motion of the water. Neglecting gravity there are two alternative assumptions for the surface condition, that it is either a rigid plane surface or an open surface of constant pressure. We might take the condition at the free surface to be zero normal velocity or zero tangential velocity. The calculations on added mass have taken the latter condition. It is of interest to note that the former condition, of a rigid plane boundary, has been used by Brard (R.3) in work on the corresponding inertia effects in the rolling of a ship. In my view, the choice of appropriate boundary condition depends not only on the mode of motion of the ship, but also upon whether its oscillations are of short period or of long period. However that may be, the inertia coefficients in the present problems are generally estimated by indirect methods, or in effect by comparing observed periods with those calculated without allowing for the inertia of the water. The only difficulty that arises is that often the stated periods have not been directly observed, but have themselves been deduced indirectly. There is, however, general agreement that a normal value for the added mass for heaving would be from 80 to 100 per cent of the displacement, with even more for broad, shallow forms; while for pitching the added moment of inertia might be normally 40 to 50 per cent of the moment of inertia of the ship—reference may

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† References at end of paper.

be made, for instance, to G. S. Baker (R.4). We may examine this in a few cases from the point of view of the same approximate basic period $2\pi\sqrt{(d/g)}$ for both heaving and pitching.

With data from Kent and Cutland (R.5) for a cargo ship of 400 ft. × 55 ft. × 24 ft., we take the mean uniform draught as 21·5 ft. This gives a basic period of 5·13 sec. The natural resisted periods of pitch and heave are given as 6·20 and 7·42 sec. respectively; taking the ratio of each of these to the basic period and squaring, we get the corresponding added moment of inertia and added mass, namely about 46 per cent and 100 per cent respectively.

Similarly, from the details given for the motor ship *San Francisco* (R.6) with a mean draught of 22 ft. the basic period is 5·19 sec. The observed periods of pitch and heave are given as 6·51 and 7·34 sec.; and we deduce corresponding inertia increments of 57 and 100 per cent.

For a different type, a fast ship 400 ft. × 48 ft. × 13 ft., we have data taken from Kent and Cutland (R.7). The mean uniform draught of 10·5 ft. gives a basic period of 3·59 sec. From resonance effects in rough water the natural resisted periods of pitching and heaving were assumed to be approximately 5·4 and 5·8 sec. Accepting these values, we get an increase of moment of inertia of about 125 per cent, and of mass of about 160 per cent. These values seem too high, though increased values would naturally be expected from the greater ratio of beam to draught.

Leaving aside the approximation in using the same basic period for both pitching and heaving, the total effective mass and moment of inertia can, of course, be calculated if we know the requisite data and the observed periods; for from (1) and (2) we have $M = g\rho S T_h^2/4\pi^2$ and $I = m W T_p^2/4\pi^2$.

Damping.—We consider now the second term in equations (1) and (2), representing the damping of the natural oscillations. This arises partly from frictional effects and partly from energy lost in the wave motion produced by the oscillation. In order to evaluate the latter contribution we ignore for the present all effects due to viscosity. In the problem of rolling the association of damping with wave motion has been familiar since the time of W. Froude. Some recent calculations (R.8) have shown that it is certainly capable of accounting for a large proportion of the observed damping for a ship with zero speed of advance. The rolling problem is simpler than that of heaving and pitching in that the damping is small; on the other hand, it is more difficult to calculate the wave motion directly in terms of the form of the ship.

For damping due to heaving, reference may be made to some small-scale experimental studies. In particular, Schuler (R.9) examined the waves produced by a prism making vertical oscillations, and, among other results, deduced that the damping was due to wave motion, viscous and other damping being negligible in comparison. In the application to ship motion, Kreitner (R.10) has emphasized the importance of this kind of damping in heaving and pitching.

Calculations of the magnitude of this effect have been given in a recent paper (R.11), and also in the Appendix to the present notes.

Suppose the ship is acted on by a periodic force, say $H_0 \cos pt$, so that it is making forced heaving of period $2\pi/p$. We could write the equation of motion in the form

$$M\ddot{\zeta} + g\rho S\zeta = X + H_0 \cos pt \quad . \quad . \quad (3)$$

where we consider X as the vertical resultant of the additional fluid pressures due to the wave motion. The assumption is that if X could be calculated it would be a resistance proportional to the velocity $\dot{\zeta}$ and could be transferred to the other side of the equation and be the term $N_1\dot{\zeta}$ as in equation (1). Meantime we can only evaluate N_1 by indirect methods. The impressed force $H_0 \cos pt$ does work at a rate just sufficient to maintain the forced oscillations; if the latter are of amplitude ζ_0, this mean rate of work is $\frac{1}{2}p^2 N_1 \zeta_0^2$. This is equated to the mean rate at which energy is propagated outwards in the wave motion, and so we obtain an expression for N_1. To determine the wave motion we replace the ship by a suitable distribution of alternating sources over its surface and hence deduce an expression for the mean rate of outflow of energy (A.1 and 2).* The same argument applies to the pitching motion with reference to the forced oscillations, and we derive an expression for the corresponding factor N_2. Calculations have been made for a simplified form of ship; wall-sided, of length L, beam B, of constant draught d, the horizontal sections being the same and elliptical in shape. The expressions for N_1 and N_2 are given in A. 5, 6, 10 and 11. For numerical values we take L = 400 ft., B = 55 ft., d = 20 ft.; these dimensions giving a rough correspondence with a cargo ship of about 10,000 tons displacement. Calculations for N_1 from A. 5 and 6 have been made for six different values of the period $T = 2\pi/p$ and the results are shown in Table I in lb.-ft.-sec. units, the lb. being the unit of force.

TABLE I

T	$N_1 \times 10^{-4}$	N_1/M
5	0·27	0·20
6	0·43	0·32
7	0·54	0·40
8	0·60	0·45
9	0·67	0·50
10	0·70	0·53

For the values of N_1/M in the third column, we have assumed an added mass of 90 per cent and have taken the effective mass M to be 19,000 tons. If the ship were heaving in a natural damped motion of period T, the logarithmic decrement of the motion would be given by $N_1 T/2M$. If, for instance, the natural period is 7 sec., then taking the corresponding value from Table I, we should get a logarithmic decre-

* A. refers to the appendix, and R. to the list of references.

ment of 1·41. This is a very high degree of damping compared, for instance, with rolling. It seems probable that any numerical estimates have been deduced from resonance curves under forced heaving. The only published estimate appears to be that given by Horn (R.6). It is stated that the result of observations on various models gave an average value of 0·45 for the quantity $N T/2\pi M$, in the present notation, or a logarithmic decrement of 1·41; it is also stated that the corresponding damping coefficient for pitching was of the same order.

For pitching, calculations for the same model from A. 10 and 11 are shown in Table II.

TABLE II

T	$N_2 \times 10^{-9}$	N_2/I
5	2·46	0·21
6	3·87	0·33
7	4·61	0·39
8	6·07	0·51
9	6·06	0·51
10	4·97	0·42

For an appropriate value of I we use data from a model of Kent and Cutland (R.5), to which reference has already been made; this was a cargo ship 400 ft. × 55 ft. × 24 ft. of 11,332 tons, with a longitudinal G M = 467 ft. and a natural resisted pitching period of 6·2 sec. Using $I = T_p^2 W m/4\pi^2$, we get an effective moment of inertia $I = 11{\cdot}85 \times 10^9$. It may be noted that this gives an effective radius of gyration of 0·31 L, which seems about the right value. With this value of I, the third column in Table II gives the values of N_2/I. We notice the striking similarity in the values of N_1/M and N_2/I in Tables I and II, with some interesting differences in detail. This agrees with the statement that the damping coefficients for heaving and pitching are of the same order. For a period of between 6 and 7 sec. Table II gives a logarithmic decrement of about 1·16.

The form of model used for these calculations was chosen for simplicity to give the order of magnitude of the effect. The work could be carried out in detail for any form given by mathematical equations, with the corresponding source distribution over the surface; but such calculations are hardly worth while meantime, or at least not without corresponding experimental work on simplified forms specially arranged to test and develop the theory.

Oscillations among Waves

If, instead of being in smooth water, the ship is subject to the action of a regular train of waves, there are many new factors which should be taken into account: for instance, the disturbance of the wave train by reflection from the ship, and the wave system produced by the forward motion of the ship. The hydrodynamic forces acting on the ship will no doubt affect the amount of damping and may alter the effective periods of pitch and heave. A first approximation involves neglecting these complications and evaluating the forces on the ship from the pressures in the undisturbed train of waves. This was the simplification adopted by W. Froude in his theory of rolling, and it was also the basis of the well-known work of Kriloff on pitching and heaving. The conventional method is to suppose the ship held in its equilibrium position and to calculate the excess or defect of buoyancy and its moment from hydrostatic pressures due to the instantaneous position of the wave profile relative to the ship. We confine the discussion in this section to the first approximation, but we calculate the forces and couples directly from the pressure system in a regular train of simple harmonic waves. Reference may be made to A.§2, where results are obtained for the particular model we are using, a wall-sided ship with elliptical horizontal section. For this model, equations (1) and (2) for smooth water are replaced by

$$M\ddot{\zeta} + N_1\dot{\zeta} + g\rho S\zeta = H_0 \cos pt \quad . \quad . \quad (4)$$

$$I\ddot{\theta} + N_2\dot{\theta} + W m\theta = -P_0 \sin pt \quad . \quad . \quad (5)$$

with H_0, P_0 given by A.16 and 18.

The forced oscillations are then

$$\zeta = \zeta_0 \cos(pt - \beta_1); \; \theta = -\theta_0 \sin(pt - \beta_2) \quad (6)$$

with ζ_0, θ_0, β_1, β_2 given by A.21.

In attempting any comparison with observed results, it must be remembered that the expressions have been obtained from a very simple form of model. In general, model results are for forms not readily adapted to mathematical calculation, and moreover there are other factors arising from lack of symmetry fore and aft; in particular, if the centre of buoyancy is fore or aft of the centre of flotation there is coupling between heaving and pitching.

We make numerical calculations for the dimensions used in the previous section: L = 400 ft., B = 55 ft., d = 20 ft., and we take the wave height $2r$ = 5 ft. There are various possible methods of exhibiting the results. We choose that of graphing the total pitch $2\theta_0$ on the period of encounter as a base, in each case for a given speed of the ship. Thus for a given period of encounter at a given speed we find the corresponding wave-length λ, and then the value of P_0 from A.18. For the effective moment of inertia I we take the value used in the previous section and also the same natural period 6·2 sec. Further we obtain the corresponding value of $N_2 p/I$ from the data given in Table II.

In Fig. 1 the two curves show the graphs for the ship at rest and for a speed of 8 knots. The double humps on these curves are of interest; they arise because not only does the magnification factor have a maximum at resonance but the pitching moment P_0 has maxima depending upon the wave-length. This effect can be clearly seen in some curves of results from models; in particular, reference may be made to Kent (R.12), Fig. 3, and (R.13), Fig. 3. Of course in actual model results there would be no definite zeros of the pitching; the curves would be smoothed out by viscous and other

effects. On Fig. 1 are also shown values extracted from model results given by Kent and Cutland (R.5); these results were for wave-lengths of 175, 350 and 490 ft., in waves of 5 ft. in height. It should be noted that no attempt has been made to fit this model beyond taking the main dimensions and displacement about the same. The points marked by a cross are for zero speed of advance, and they fit fairly well into the calculated curve. Points marked by a circle are for a speed of 8 knots. In the calculated curve for 8 knots we have used the same natural pitching period as for zero speed. There seems to be some evidence that the effective natural period increases with the speed. The large divergence

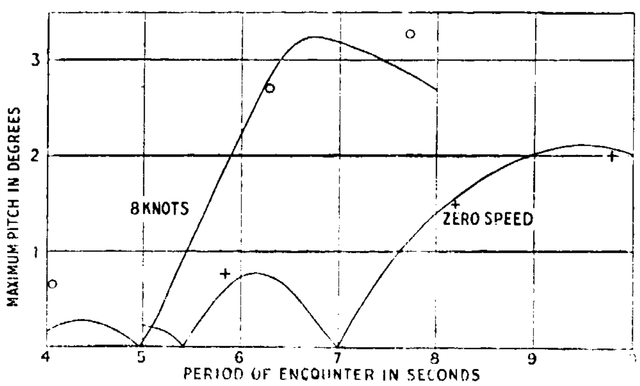

FIG. 1

at the lowest period of encounter at 8 knots may be due to various causes; for one thing the calculated pitching moment P_0 is more subject to error at the smaller wave-lengths, and for another it is probable that the pitching in the smaller wave-lengths is not the simple forced pitching to which the calculations refer.

Similar graphs could be made for heaving, but it should be remarked that observed maxima in long waves are generally greater than those given by calculation. This has been noted previously in regard to model results; it may be that the calculated buoyancy is more susceptible to change in wave-length or possibly that in long waves the damping is less—it might, for instance, be a better approximation in such cases to calculate the damping from the motion of the ship relative to the fluid motion in the wave train. Finally, in this brief review we may consider the phase lags for heave and pitch denoted by the angles β_1 and β_2 in (6). It is a simple matter, so far as the approximate equations are concerned, to determine the position of the ship in relation to the wave profile at maximum pitch or heave. This is, of course, a very important point. It has been examined by J. L. Kent in various papers; and, in particular, in Kent and Cutland (R.5) a diagram is given showing the wave crest and trough positions along the ship at maximum pitch. It is difficult to derive from this diagram results suitable for the present calculations. The model was not designed for the purpose; moreover, it is stated that successive pitches showed a periodic movement of the wave crest position backwards and forwards along the hull, the diagram giving mean positions at the instant of lowest pitch. Referring to (6), there is maximum pitch with bows down when $pt - \beta_2 = \pi/2$. From A.13, it follows that the wave profile relative to the ship at that instant is given by $\zeta = -r \sin(kx + \beta_2)$. Hence there is a trough at a distance $\frac{1}{4}\lambda - \beta_2 \lambda/2\pi$ ahead of amidships.

If T_e is the period of encounter and T_p the natural period of pitch, and if damping were entirely neglected, we should have $\beta_2 = 0$ for $T_e > T_p$, and $\beta_2 = \pi$ for $T_e < T_p$. In the former case there is a trough $\frac{1}{4}\lambda$ ahead of amidships and in the latter it is $\frac{1}{4}\lambda$ astern of amidships. The damping smooths off this sudden change of phase; but whatever the damping we should have $\beta_2 = \pi/2$ for $T_e = T_p$. Hence there should be a trough at amidships, for a simple symmetrical model, at lowest pitch at the resonance period of encounter. In the diagram referred to above, there is a trough amidships for zero speed of advance at a wave-length of about 230 ft.; this corresponds to a period of encounter of 6·77 sec., the natural period for the model being 6·2 sec. But, for various reasons, it is not possible to push the comparison so far as to determine the phase lags. The possible magnitude of surging effects, for instance, needs examination; and in various respects the theory is only a first approximation and requires amplification in conjunction with suitable experimental data.

Resistance of a Ship among Waves

A ship when moving through a regular train of waves is subject to an average steady resistance greater than that experienced at the same speed in smooth water. There are various obvious factors which may be supposed to contribute to this result: for instance, the disturbance of the wave motion by the surface of the ship, the alteration in the wave resistance due to interference with the wave train or due to altering attitude of the ship, or a more direct effect of the surging, heaving and pitching motions.

If we consider only the first order approximate equations used in the previous sections, the regular wave train supplies an alternating addition to the resistance, such as that given in A.17; a more detailed examination of this periodic force may be found in R.14.

In order to obtain an increased average resistance we have to take into account second order terms. When we are dealing with first order effects it is, generally, legitimate to consider factors separately and obtain the combined result by simple superposition; but this is not the case when we have to include second-order terms. Noting, however, that a complete theory including all second-order effects might well modify partial results, we shall examine two possible factors which lead to increased average resistance.

Wave Reflection.—The first possibility is the effect of reflection, or scattering, of the regular wave train by the surface of the ship, and this is undoubtedly a true contributing cause. It has been put forward recently as the sole basis of the extra resistance in a very interesting paper by Kreitner (R.10). The underlying hydro-

dynamical theory has been examined in a recent paper (R.15) to which reference may be made for details of the analysis. The ship problem is the reflection of the wave train by the ship, which is itself free to move and does take part to some extent in the motion of the surrounding water; it is in fact the dynamical problem of the motion of the complete system of ship and water. Leaving this on one side we consider the forces on a fixed obstacle in waves. The fundamental case is that of a regular train of waves incident normally upon a fixed vertical plane, which we may take of infinite draught. There is perfect reflection of the waves; if r is the amplitude (half wave height) of the incident train, there is an oscillation of amplitude $2r$ at the plane. The usual first order theory for waves of small height gives a periodic force of $(g \rho r \lambda/\pi) \cos pt$ for the additional force per unit width of the plane, λ being the wave-length and $2\pi/p$ the corresponding period. Carrying the theory to second-order terms, the result of the analysis is to give an additional average steady force on the plane amounting to $\frac{1}{2} g \rho r^2$ per unit width. If the waves are incident at an angle α to the plane, the corresponding average force is $\frac{1}{2} g \rho r^2 \sin^2 \alpha$ per unit width. An interesting problem would be the reflection of waves by a vertical cylinder of elliptical cross section, like the model used in the previous calculations of this paper; it is possible to obtain an analytical solution, but the functions involved have not been tabulated sufficiently to allow of numerical results. The corresponding work has been carried out for a vertical cylinder of circular cross-section, giving the variation of amplitude round the cylinder and the resultant steady force and the dependence of both these quantities on the wave-length. When the wave-length is small compared with the diameter of the cylinder, the resultant steady force approximates to the value $\frac{2}{3} g \rho r^2 a$, where a is the radius. An interesting result shown by these calculations is that this limiting value is practically attained so long as the wave-length is not greater than the diameter. We may obtain this limiting value by making an extreme assumption. Imagine the waves to be completely reflected by the front half of the cylinder, leaving smooth undisturbed water round the rear half Then treat each element of the front half as if it were part of an infinite plane upon which the waves are incident at an angle α. On this assumption we should have for the resultant force

$$R = \frac{1}{2} g \rho r^2 \int_{-a}^{a} \sin^2 \alpha \, dy \quad . \quad . \quad . \quad . \quad (7)$$

taken over the transverse diameter of the cylinder; and this gives the result $\frac{2}{3} g \rho r^2 a$.

This suggests a similar expression for a vertical cylinder of any horizontal cross-section. With the extreme assumption of reflection round the front half and smooth water round the rest, it appears that the steady average force due to wave reflection should not exceed the amount

$$R = \frac{1}{2} g \rho r^2 B \overline{\sin^2 \alpha} \quad . \quad . \quad . \quad . \quad (8)$$

where B is the maximum beam, and the last factor is the mean value of $\sin^2 \alpha$ with respect to the beam, α being the angle which the tangent at any point makes with the fore- and aft- central axis.

Kreitner (R.10) gives an expression which, in the present notation, is

$$R = g \rho r^2 B \overline{\sin \alpha} \quad . \quad . \quad . \quad . \quad (9)$$

In deriving this, it is apparently assumed that the average pressure on a plane can be calculated from the instantaneous value of the hydrostatic pressure due to the elevation of the water surface. When numerical values are obtained for ship forms, the general result is that the expression (8) gives about one-quarter or one-fifth of the value given by (9).

If we take the elliptical model used in the previous sections, an expression for the mean value of $\sin^2 \alpha$ can be readily obtained; with L = 400 ft., B = 55 ft., the value of this factor is 0·183. In waves of 5 ft. in height, (8) then gives an extra resistance of about 0·9 ton. With a normal ship form with moderate bow angle, the mean value of $\sin^2 \alpha$ would be about 0·1, reducing the extra resistance by this calculation to about ½ ton. The observed extra resistance for a ship of that type would be, on the average, about 2½ tons.

It should be noted again that the expression (8) is put forward only as an outside limit for a fixed obstacle of great draught. In the actual problem the ship is free to respond to the wave motion; further, unless the wave-length is very much less than the length of the ship, the finite draught of the ship seems likely to reduce the amount of the reflection effect. The general conclusion, so far as the present calculations go, is that, while wave reflection is a true contributory cause and must be included in a complete theory, it is only capable of accounting for a fraction of the observed extra resistance; we must, however, add the reservation that forward motion of the ship through the waves might modify that conclusion.

A possible application of the formula (8) would be to determine the mean pull on the mooring rope of a ship subjected to waves which are short in comparison with the length of the ship. This has been investigated by Kent and Cutland (R.5) and details of the comparison with model results will be found in that paper. The experimental conditions most nearly approximating to the theoretical assumptions were for a 16-ft. model moored in waves of 7 ft. in length; the height of the waves was given values ranging from 0·12 ft. to 0·32 ft. It was found that, on the average, the value calculated from (8) was about 56 per cent of the observed mean pull.

Resistance associated with Heaving and Pitching.— Another possibility is suggested by the consideration that first-order effects which in themselves are purely periodic may, through phase differences, give rise to a steady additional resistance. Such a theory would associate the resistance directly with the oscillations of surging heaving and pitching—though it is probable that the first of these plays only a minor part. There are different views of the extent to which the resistance depends upon the heaving and pitching motions; but the effect is

5

certainly most prominent when the period of encounter is near one of the natural periods and, directly or indirectly, the phenomena are closely associated. The problem involves to some extent second-order terms and the analysis is therefore subject to correction by a more complete theory; but meantime we ignore the disturbance of the wave train by reflection and use the approximate equations for heaving and pitching as in the previous sections. The analysis is given in detail elsewhere (R.16) and a short account in the Appendix to the present paper.

We calculate the force on the ship from the pressure in the undisturbed wave train; but, instead of taking the equilibrium position of the ship, we make the calculation for a displaced position, with a vertical displacement ζ due to heave and a rotation θ due to pitch. To the first order in ζ and θ, the resultant force backwards is found to be (A.25)

$$F = F_0 - \frac{2\pi}{p\lambda}\zeta\frac{\partial H}{\partial t} - \frac{2\pi}{p\lambda}\theta\frac{\partial P}{\partial t} \quad . \quad . \quad (10)$$

In this, $2\pi/p$ is the period of encounter with the waves and also the period of the forced oscillations; H and P are the buoyancy and pitching moment and are also of period $2\pi/p$. The first term F_0 is the purely periodic horizontal force to which reference has been made earlier. Taking average values of the quadratic terms in the rest of (10) we obtain for the average steady resistance

$$R = (\pi/\lambda)\, H_0\, \zeta_0 \sin\beta_1 + (\pi/\lambda)\, P_0\, \theta_0 \sin\beta_2 \quad . \ (11)$$

with H_0 and P_0 the amplitudes of the buoyancy and pitching moment, ζ_0 and θ_0 the amplitudes of the forced heaving and pitching, and β_1 and β_2 the phase lags of the oscillations.

It is of interest to recall the history of the similar problem in rolling. In 1924 Suyehiro (R.17), experimenting with a small model, measured a drifting force sideways on a ship when rolling in waves. The effect is small and probably is only appreciable in suitable conditions of forced rolling in resonance with the natural period of roll. Suyehiro himself ascribed the force to reflection of the waves by the side of the ship: however no calculations have been made of the magnitude of such an effect. In 1938 an alternative theory was put forward by Watanabe (R.18). Starting from the Kriloff equations, Watanabé deduced an expression for the drifting force involving the angle of roll and the phase lag between the roll and the actuating moment; applied to Suyehiro's model, this expression gave a force of rather more than half the observed value.

Returning to (11), consider the various factors when making numerical comparison with observed results. The values of H_0 and P_0 have to be taken from such calculations of buoyancy and pitching moment as can be made for any given form. The amplitudes ζ_0 and θ_0 we shall take from observed results, assuming, as is necessary, that these are for forced oscillations. The most uncertain factors are the phase lags. It will be noticed that these are important in that on the present view the extra resistance arises from the damping and the phase lags produced thereby; if there is no phase lag, there is no resultant steady force. Reference has been made to the diagram given by Kent and Cutland (R.5) from which the phase lag for pitching might be deduced. It is not suitable for the present purpose, however the attempt may be made so as to obtain some idea of the magnitude of the resistance given by (11).

If we take the results in waves of 490 ft. in length, the diagram shows that for zero speed of the model there was no appreciable phase lag. Hence, according to (11) there should be no resistance; and, in fact, the measured resistance under those conditions was very small. Incidentally the observed results also confirm the view that resistance due to wave reflection must be very small when the wave-length becomes greater than the length of the ship.

If we take next the same wave-length with a speed of 8 knots for the ship, a rough estimate from the position of the wave trough gives a phase lag for pitching of about 12·5 deg. We shall assume the same value for heaving, and we take $\beta_1 = \beta_2 = 12{\cdot}5$ deg. For H_0 and P_0 we take the wall-sided ship with elliptical horizontal section which has been used in the earlier calculations.

With $L = 400$ ft., $B = 55$ ft., $d = 20$ ft., $\lambda = 490$ ft., and in waves of height $2r = 5$ ft., we obtain from A. 16 and 18

$$H_0 = 358 \text{ tons}; \ P_0 = 67{,}633 \text{ ft.-tons}$$

The observed measurements in 5-ft. waves give $\zeta_0 = 2{\cdot}1$ ft. and $\theta_0 = 1{\cdot}6$ deg., approximately. With these values we get from (11) a resistance $R = 3{\cdot}66$ tons, of which about 1 ton comes from the term in the heaving motion. From the given results in the same paper, the measured resistance for the 16-ft. model was 0·37 lb. or a resistance of 2·58 tons for the full-sized ship. The measure of agreement is perhaps as much as could be expected considering the uncertainty of the data and also that no special attempt has been made to calculate values for the particular model used in the experiments. It is not worth while adding further similar calculations at the present stage: but it may be said that the suggested theory is capable of giving results of the right order of magnitude.

On the theoretical side, it is hoped that the various limitations and assumptions have been sufficiently indicated. On the experimental side, there is a lack of suitable data obtained under conditions approximating to the simplifications which have to be made before any calculations are possible: such experimental results would be a valuable and, indeed, essential aid in developing and modifying any tentative theory of such a complex problem.

Appendix

(1) *Damping in Smooth Water.*—If a ship is making forced oscillations of heaving or pitching, we may calculate the wave motion by supposing each element of the ship's surface to be the seat of an alternating source, say of strength $m \cos pt$ per unit area. Knowing the velocity potential of the distribu-

6

tion of sources, it is possible to calculate the average rate at which energy is being propagated outwards in the wave motion. It has been shown (R.11) that this mean rate of outflow of energy is given by

$$E = 2\pi\rho\,(p^3/g)\int_0^{2\pi}(P^2 + Q^2)\,d\theta \quad . \quad . \quad . \quad (1)$$

where

$$P + iQ = \iint m\,(x, y, z)\,e^{p^2/g\,.\,(z + ix\cos\theta + iy\sin\theta)}\,dS \quad (2)$$

the integral in (2) being taken over the immersed surface of the ship in its equilibrium position. The axis O x is taken along the longitudinal axis of the water plane section with the positive direction from stern to bow, O y transversely at the midship section, and O z vertically upwards. We shall assume the source strength at each point to be such that $4\pi m$ is the amplitude of the normal component of velocity of the ship's surface at each point, this being a reasonable approximation in view of usual ship dimensions. A further simplification may be made by neglecting the distribution in the transverse axis O y, since the length $2\pi g/p^2$ is usually several times as large as the beam of the ship.

Suppose the ship to be wall-sided, of uniform draught d, and with the horizontal sections ellipses of axes L and B.

Let the ship be making forced heaving oscillations of amplitude ζ_0 and period $2\pi/p$. The source strength over the flat bottom is of amplitude $p\,\zeta_0/4\pi$ per unit area; and we treat it as a line distribution at constant depth d along the central line, with strength proportional to the beam at each point. Hence from (2) we have

$$P + iQ = \frac{p\,\zeta_0}{4\pi}\,B\,e^{-p^2d/g}\int_{-\frac{1}{2}L}^{\frac{1}{2}L}\left(1 - \frac{4x^2}{L^2}\right)^{\frac{1}{2}} e^{ip^2x/g\,.\cos\theta}\,dx$$

$$= \tfrac{1}{8}\,p\,\zeta_0\,B\,L\,e^{-p^2d/g}\,J_1\,(q\cos\theta)/(q\cos\theta) \quad . \quad (3)$$

where $q = p^2\,L/2\,g$.

Hence from (1) the mean rate of propagation of energy outwards is given by

$$E = \frac{\pi\rho}{8g}\,B^2\,L^2\,\zeta_0^2\,p^5\,e^{-2p^2d/g}\int_0^{\pi/2}\left\{\frac{J_1(q\cos\theta)}{q\cos\theta}\right\}^2 d\theta \quad . \quad (4)$$

We now equate this to the mean value of $N_1\,\dot{\zeta}^2$, namely $\frac{1}{2}\,p^2\,N_1\,\zeta_0^2$, and we obtain

$$N_1 = (\pi\rho/4\,g)\,B^2\,L^2\,p^3\,e^{-2p^2d/g}\,F_1 \quad . \quad . \quad (5)$$

where F_1 has been written for the integral in (4).

This integral may be evaluated by quadrature using tables of Bessel functions. It was, however, found more satisfactory to calculate it from an equivalent series. It can be shown that

$$F_1 = \frac{\pi}{8}\sum_{m=0}^{\infty}\frac{(-1)^m\,(2m)!\,(2m+2)!}{(m!)^3\left\{(m+1)!\right\}^2\,(m+2)!}\left(\tfrac{1}{4}q\right)^{2m} \quad . \quad (6)$$

Similarly, if the ship is making forced pitching oscillations of angular amplitude θ_0 and period $2\pi/p$ we have

$$P + iQ = \frac{p\,\theta_0}{4\pi}\,B\,e^{-p^2d/g}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} x\left(1 - \frac{4x^2}{L^2}\right)^{\frac{1}{2}} e^{ip^2x/g\,.\cos\theta}\,dx \quad (7)$$

in which we have neglected the contribution of the vertical sides of the ship compared with the effect of the flat bottom.

The integral in (7) may be expressed in terms of Bessel functions, and we find

$$P + iQ = \frac{1}{16}\,p\,\theta_0\,B\,L^2\,e^{-2p^2d/g}\,J_2\,(q\cos\theta)/(q\cos\theta) \quad (8)$$

Thus for pitching motion we have

$$E = \frac{\pi\rho}{32\,g}\,B^2\,L^4\,\theta_0^2\,p^5\,e^{-2p^2d/g}\int_0^{\pi/2}\left\{\frac{J_2\,(q\cos\theta)}{q\cos\theta}\right\}^2 d\theta \quad . \quad (9)$$

Equating this to $\frac{1}{2}\,p^2\,N_2\,\theta_0^2$, we obtain

$$N_2 = (\pi\rho/16\,g)\,B^2\,L^4\,p^3\,e^{-2p^2d/g}\,F_2 \quad . \quad . \quad (10)$$

where F_2 is the integral in (9). This may be evaluated from the series

$$F_2 = \frac{\pi}{8}\sum_{m=0}^{\infty}\frac{(-1)^m\,(2m+2)!\,(2m+4)!}{m!\left\{(m+1)!\,(m+2)!\right\}^2\,(m+4)!}\left(\tfrac{1}{4}q\right)^{2m+2} \quad (11)$$

(2) *Buoyancy and Pitching Moment in Waves.*—Suppose at first that the ship has zero speed of advance, and that the waves are moving directly towards it. The velocity potential of the fluid motion is

$$\phi = (g\,r/p)\,e^{kz}\sin(p\,t + k\,x) \quad . \quad . \quad . \quad (12)$$

with $p^2 = g\,k$; this corresponds to waves of elevation given by

$$\zeta = r\cos(p\,t + k\,x) \quad . \quad . \quad . \quad . \quad (13)$$

the amplitude r being one-half the height measured from trough to crest, and the wave-length λ being $2\pi/k$. The pressure p at any point is

$$p = p_0 - g\,\rho\,z - \rho\,\frac{\partial\phi}{\partial t} \quad . \quad . \quad . \quad . \quad (14)$$

The second term is the hydrostatic pressure whose effect is included in the equations of motion of the ship in smooth water. The third term

$$\rho\,\frac{\partial\phi}{\partial t} = g\,\rho\,r\,e^{kz}\cos(p\,t + k\,x) \quad . \quad . \quad . \quad (15)$$

is the additional pressure due to the undisturbed wave system. The resultant forces and couples are obtained, to this approximation, by integrating this pressure, and its moment, over the immersed surface of the ship in its equilibrium position. With the same simplified model, we have for the additional buoyancy H,

$$H = g\,\rho\,r\,e^{-kd}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} B\left(1 - \frac{4x^2}{L^2}\right)^{\frac{1}{2}}\cos(p\,t + k\,x)\,dx$$

$$= H_0\cos p\,t,$$

where

$$H_0 = \tfrac{1}{2}\,g\,\rho\,r\,B\,\lambda\,e^{-2\pi d/\lambda}\,J_1\,(\pi\,L/\lambda) \quad . \quad . \quad (16)$$

There is also a resultant horizontal force from the pressures on the vertical sides; measured in the negative direction of O x, from bow to stern, it is

$$F_0 = \int_{-d}^{0} dz\int_0^{2\pi}\tfrac{1}{2}\,g\,\rho\,r\,B\,e^{kz}\cos(p\,t + \tfrac{1}{2}\,k\,L\cos\theta)\cos\theta\,d\theta$$

$$= \tfrac{1}{2}\,g\,\rho\,r\,B\,\lambda\,(1 - e^{-2\pi d/\lambda})\,J_1\,(\pi\,L/\lambda)\sin p\,t \quad . \quad (17)$$

This force might be used as a similar first approximation in regard to the surging motion of the ship. By comparison

with (16), it can be seen that in general it is only a fraction of the corresponding vertical force.

In evaluating the pitching moment we take moments about the transverse axis O y, assuming for simplicity that the centre of gravity G of the ship coincides with O. The pressures on the vertical sides will contribute to the total moment; but this part will be of the order of the horizontal force F_0 multiplied by some fraction of the draught, and it can be seen to be negligible compared with the moment of the pressures on the flat bottom. We have then for the pitching moment

$$P = g \rho r B e^{-kd} \int_{-\frac{1}{2}L}^{\frac{1}{2}L} x \left(1 - \frac{4x^2}{L^2}\right)^{\frac{1}{2}} \cos(pt + kx)\, dx$$

$$= -P_0 \sin pt,$$

where

$$P_0 = \tfrac{1}{4} g \rho r B L \lambda e^{-2\pi d/\lambda} J_2(\pi L/\lambda) \quad . \quad . \quad (18)$$

For a ship advancing through the waves, we have the same expressions, so far as this approximation goes, with $2\pi/p$ the relative period of encounter; thus if λ is the wave-length and v the corresponding wave velocity, and V is the speed of the ship, then $2\pi/p = \lambda/(v + V)$.

On this theory, the equations for heaving and pitching on waves are, for this symmetrical model

$$M\ddot{\zeta} + N_1\dot{\zeta} + g\rho S\zeta = H_0 \cos pt \quad . \quad . \quad (19)$$

$$I\ddot{\theta} + N_2\dot{\theta} + W m\theta = -P_0 \sin pt \quad . \quad . \quad (20)$$

The forced oscillations are

$$\zeta = \zeta_0 \cos(pt - \beta_1);\ \theta = -\theta_0 \sin(pt - \beta_2)$$

with

$$\zeta_0 = H_0/M\{(p_1^2 - p^2)^2 + k_1^2 p^2\}^{\frac{1}{2}};\ k_1 = N_1/M;$$

$$\theta_0 = P_0/I\{(p_2^2 - p^2)^2 + k_2^2 p^2\}^{\frac{1}{2}};\ k_2 = N_2/I;$$

$$\tan\beta_1 = k_1 p/(p_1^2 - p^2);\ \tan\beta_2 = k_2 p/(p_2^2 - p^2) \quad . \quad (21)$$

the natural periods of unresisted heaving and pitching being $2\pi/p_1$ and $2\pi/p_2$ respectively.

(3) *Resistance in Waves.*—Let (l, m, n) be the direction-cosines of the outward-drawn normal at any point of the immersed surface of the ship. Then with the pressure in the undisturbed wave motion given as in (14), the resultant horizontal force backwards

$$F = \iint p\, l\, dS = -g\rho k r \iiint e^{kz} \sin(pt + kx)\, dV \quad (22)$$

the latter integral being taken throughout the immersed volume of the ship at any instant.

If we calculate this for the immersed volume V_0 when the ship is held in its equilibrium position we obtain a purely periodic force F_0, such as was found in (17). Let the ship be in a slightly displaced position due to heaving and pitching, with the centre of gravity G raised a distance ζ and with a pitch θ about a transverse axis through G; we shall suppose G to be on the axis O z at a height c above O. Then, to the first order in ζ and θ, it can be shown (R.16) that the horizontal force backwards in the displaced position is

$$F = F_0 - g\rho k r \zeta \iint e^{kz} \sin(pt + kx)\, n\, dS$$

$$- g\rho k r \theta \iint e^{kz} \sin(pt + kx) \{n x - l(z - c)\}\, dS \quad (23)$$

where the integrals are taken over the equilibrium position of the immersed surface.

The additional buoyancy and pitching moment, which were calculated for a special case in (16) and (18), are given in this more general form by

$$H = -g\rho r \iint e^{kz} \cos(pt + kx)\, n\, dS$$

$$P = g\rho r \iint e^{kz} \cos(pt + kx) \{l(z - c) - n x\}\, dS \quad (24)$$

Hence we may write the backward force as

$$F = F_0 - \frac{k}{p}\zeta\frac{\partial H}{\partial t} - \frac{k}{p}\theta\frac{\partial P}{\partial t} \quad . \quad . \quad . \quad (25)$$

When calculated for any form of ship, H and P are in general of the form $H_0 \sin(pt + \alpha_1)$ and $P_0 \sin(pt + \alpha_2)$ respectively. The corresponding forced oscillations of heaving and pitching are then given by equations such as

$$\zeta = \mu_1 H_0 \sin(pt + \alpha_1 - \beta_1)$$

$$\theta = \mu_2 P_0 \sin(pt + \alpha_2 - \beta_2) \quad . \quad . \quad . \quad (26)$$

μ_1 and μ_2 being positive factors.

Putting these expressions in (25) and taking mean values of the quadratic terms, we obtain a mean backward force on the ship

$$R = \tfrac{1}{2} k \mu_1 H_0^2 \sin\beta_1 + \tfrac{1}{2} k \mu_2 P_0^2 \sin\beta_2 \quad . \quad . \quad (27)$$

This is an essentially positive expression, so that this force is always a resistance.

With ζ_0 and θ_0 the amplitudes of forced heaving and pitching respectively, this expression is equivalent to

$$R = (\pi/\lambda) H_0 \zeta_0 \sin\beta_1 + (\pi/\lambda) P_0 \theta_0 \sin\beta_2 \quad . \quad (28)$$

where β_1 and β_2 are the phase lags of the forced heaving and pitching behind the buoyancy and pitching moment respectively.

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SOME CALCULATIONS OF SHIP TRIM AT HIGH SPEEDS

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SUMMARY

Although much work has been done on the theory of wave resistance, that is on the horizontal resultant of the pressure system round a ship, there do not seem to have been any calculations of the resultant moment of the pressures about the transverse axis.

The present note records some work on this problem and a comparison of the results with measured trim in experimental models. Assuming the usual approximate theory of the pressure system, the effective part of the pressure for a symmetrical model is put into a suitable form and an expression is obtained for the moment for a certain series of models, used at Teddington, for which experimental results are available. Numerical calculations have been made for three models of this series over a considerable range of speed and curves are given showing the comparison between calculated and measured trim. The agreement is reasonably good, especially at the higher speeds, and in general the order of agreement is much the same as between calculated and measured wave resistance.

1. The pressure changes established by the forward motion of a ship may be considered in two parts: (i) those associated with the so-called local disturbance, (ii) those due to the wave motion trailing aft from the ship. In the usual approximate theory of wave resistance, neglecting viscosity, the pressures from (i) give no resultant horizontal force on the ship as a whole and we only need to calculate the resultant of these from (ii). If we wished to examine the sinkage of the ship, we should require the vertical resultant of the total pressure system and such calculations would be too laborious in general; though we may estimate the effect at low speeds by ignoring the surface disturbance of the water [1]. On the other hand, if we limit consideration to a model which is symmetrical fore and aft, the moment of the pressure system about the transverse axis will only involve the pressures from the wave system (ii). As this calculation does not seem to have been carried out hitherto, it was thought of interest to see how the results so obtained compare with the measured trim of experimental models.

The part of the pressure system which is effective for this purpose is first put into a suitable form, and an expression is obtained for the moment for a certain type of model whose form is given by an equation involving one parameter. This moment is then turned into an equivalent angle of trim for the ship, using the ordinary righting moment as if in still water. Finally, numerical calculations are made for three models of this series, with different values of the parameter, for which experimental results are available. Curves are given showing the calculated and measured trim for these three models.

2. We take the origin O in the undisturbed free surface of the water, Ox in the direction of motion, Oz vertically upwards and Oy transversely, v being the velocity. If there is a source of strength m at the point $(h, o, -f)$, the velocity potential is given by [2]

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} - \frac{\kappa_0 m}{\pi}\int_{-\pi}^{\pi} \sec^2\theta d\theta \int_0^{\infty} \frac{e^{-\kappa(f-z)+i\kappa\varpi}}{\kappa-\kappa_0\sec^2\theta + i\mu\sec\theta}\, d\kappa, \quad (1)$$

with

$$r_1^2 = (x-h)^2 + y^2 + (z+f)^2; \qquad r_2^2 = (x-h)^2 + y^2 + (z-f)^2;$$

$$\varpi = (x-h)\cos\theta + y\sin\theta \qquad \kappa_0 = g/v^2.$$

The pressure p, other than the hydrostatic pressure, is given by

$$p = -\rho v \frac{\partial\phi}{\partial x}. \quad (2)$$

We require the part of the pressure due to the waves trailing aft from the source. From (1) and (2), taking the limit for $\mu \to o$, we find this effective pressure at a point (x, o, z) due to the given source at $(h, o, -f)$ is

$$p = 8\rho\kappa_0^2 vm \int_0^{\pi/2} e^{-\kappa_0(f-z)\sec^2\theta} \cos\{\kappa_0(x-h)\sec\theta\} \sec^3\theta d\theta, \quad (3)$$

for $x - h < o$; and $p = o$ for $x - h > o$.

For a ship form given by $y = \pm F(x,z)$, we have the usual approximation of a source distribution over the section by the plane $y = o$, the source strength per unit area being

$$-\frac{v}{2\pi}\frac{\partial F}{\partial x}. \quad (4)$$

For a model of length $2l$, draft d, with O at the midship section, we obtain:—

$$p = -\frac{4}{\pi}\rho\kappa_0{}^2 v^2 \int_{-l}^{l} \frac{\partial F}{\partial h}\, dh \int_0^d df \int_0^{\pi/2} e^{-\kappa_0(f-z)\sec^2\theta}$$

$$\times \cos\{\kappa_0(x-h)\sec\theta\}\sec^3\theta\, d\theta. \qquad (5)$$

The horizontal component of this pressure integrated over the surface of the ship gives the wave resistance; it may be noted that, with the usual approximation, we evaluate the pressure not at the actual surface of the ship but over the plane $y = o$. We use this expression similarly for evaluating the moment of the pressure about the transverse axis Oy. Consider the total moment in two parts. First, for the horizontal component of the pressure, the moment will be of the order of the wave resistance multiplied by some fraction of the draft; it is found that this part is small compared with the moment of the vertical component and we neglect it meantime. However, when comparing calculated and experimental results we allow for this correction by estimating the moment of the total resistance of the model.

For the moment of the vertical component of the pressure we have

$$M = \iint xp\, dx\, dy\,, \qquad (6)$$

taken over the water plane section of the ship, with p given by (5).

3. We confine the calculations to a simple type of symmetrical model used at Teddington, for which experimental results are available, and for which the numerical calculations are not unduly laborious. This set of models is defined by

$$y = b\left(1 - \frac{x^2}{l^2}\right)\left(1 + a_2\frac{x^2}{l^2}\right)\left(1 - \frac{z^2}{d^2}\right) \qquad (7)$$

For this form, we have

$$p(x,z) = \frac{8}{\pi}\rho v\kappa_0{}^2 b \int_{-l}^{l} \left\{(1 - a_2)\frac{h}{l^2} + 2a_2\frac{h^3}{l^4}\right\} dh \times$$

$$\times \int_0^d \left(1 - \frac{f^2}{d^2}\right) df \int_0^{\pi/2} e^{-\kappa_0(f-z)\sec^2\theta} \times$$

$$\times \cos\{\kappa_0(x-h)\sec\theta\}\sec^3\theta\, d\theta\,, \qquad (8)$$

and

$$M = -\frac{4b}{d^2}\int_{-l}^{l}\int_{-d}^{0} xz\left(1 - \frac{x^2}{l^2}\right)\left(1 + a_2\frac{x^2}{l^2}\right)p(x,z)\,dx\,dz \tag{9}$$

Carrying out the integrations and, for convenience in computation, changing the variable from θ to u with $\sec\theta = \cosh u$, we obtain the result

$$M = \frac{64 g\rho b^2}{\pi\kappa_0^{\,5} d^2 l}\int_0^\infty X_1(\beta)\,X_2(\beta)\,X_3(\alpha)\,\mathrm{sech}^3 u\,du, \tag{10}$$

with $\beta = \kappa_0 d\cosh^2 u; \quad \alpha = \kappa_0 l\cosh u;$

$$X_1 = 1 - (1 + \beta)e^{-\beta}; \qquad X_2 = 1 - \frac{2}{\beta^2} + \left(\frac{2}{\beta} + \frac{2}{\beta^2}\right)e^{-\beta};$$

$$\begin{aligned}
X_3 = {} & \frac{2}{\alpha} + \frac{3}{\alpha^3} + \left(1 - \frac{6}{\alpha^2}\right)\sin 2\alpha + \left(\frac{4}{\alpha} - \frac{3}{\alpha^3}\right)\cos 2\alpha + \\
& + a_2\left\{\frac{4}{\alpha} - \frac{42}{\alpha^3} - \frac{96}{\alpha^5} + \left(2 - \frac{64}{\alpha^2} + \frac{192}{\alpha^4}\right)\sin 2\alpha + \right. \\
& \left. + \left(\frac{16}{\alpha} - \frac{150}{\alpha^3} + \frac{96}{\alpha^5}\right)\cos 2\alpha\right\} + \\
& + a_2^{\,2}\left\{\frac{2}{\alpha} + \frac{3}{\alpha^3} + \frac{96}{\alpha^5} + \frac{720}{\alpha^7} + \left(1 - \frac{74}{\alpha^2} + \frac{768}{\alpha^4} - \frac{1440}{\alpha^6}\right)\sin 2\alpha + \right. \\
& \left. + \left(\frac{12}{\alpha} - \frac{291}{\alpha^3} + \frac{1344}{\alpha^5} - \frac{720}{\alpha^3}\right)\cos 2\alpha\right\}.
\end{aligned}$$

The moment M given by (10) will, if positive, tend to give a trim of angle θ which is positive with bow up and stern down. In comparing with model results, we note that the model is towed, the point of attachment of the tow-line being at the water level in the midship section; if R is the total resistance, we have therefore a reverse moment Rd', where d' is some fraction of the draft d of the model. The effective positive moment is $M - Rd'$.

We have also the restoring moment due to the hydrostatic pressure and for this we take, as a sufficient approximation for the present purpose, $g\rho Ak^2\theta$, where Ak^2 is the moment of inertia of the area of the water plane section about Oy. For the models defined by (7),

$$Ak^2 = 4bl^3\left(\frac{2}{15} + \frac{2}{35} a_2\right). \tag{11}$$

With these assumptions we turn the calculated moment into an equivalent trim θ given by

$$\theta = \frac{M - Rd'}{g\rho Ak^2}, \tag{12}$$

calculating values from (10) and (11), and using the measured total resistance for R and an estimated value of d'; the actual value of d' is not important as in any case Rd' is found to be a small fraction of the value of M.

4. Numerical calculations were made in the first place for two Teddington models of this type, with extreme values of the parameter a_2: namely

Model 1805A, with $a_2 = -0.6$
Model 1846A, with $a_2 = 0.6$

For each model we have

length $= L = 2l = 16$ ft;
beam $= 2b = 1.5$ ft;
draft $= d = 1$ ft.

Further details of the models, and measured values of the trim are given by Wigley [3].

For these models d' was taken to be 5 inches. The integral in (10) was computed by quadrature, the value of the integrand being calculated for values of u differing by 0.1; it was not generally necessary to go beyond about 3.6 for the upper limit of u. This process was carried out for six values of the Froude speed ratio f in the range 0.32 to 0.54, f being equal to $v/\sqrt{(gL)}$. Finally the results were expressed as trim by the stern in inches for the 16-foot model, that is by 192θ the experimental results for these models being recorded in that form.

As an example of numerical values, at a speed ratio $f = 0.5$, the calculated trim for model 1805A is 6.45 inches, while the measured value was 6.0 inches; of the calculated value, the moment M of (10) gave 6.72 inches and the term $-Rd'$ reduced this by 0.27 inches. Similarly for Model 1846A at $f = 0.5$, the calculated trim is 4.82 inches, the measured value being 4.7 inches.

The results for the two models are shown in Fig. 1. The full curves are the measured values, and the broken curves show the values obtained from the present calculations.

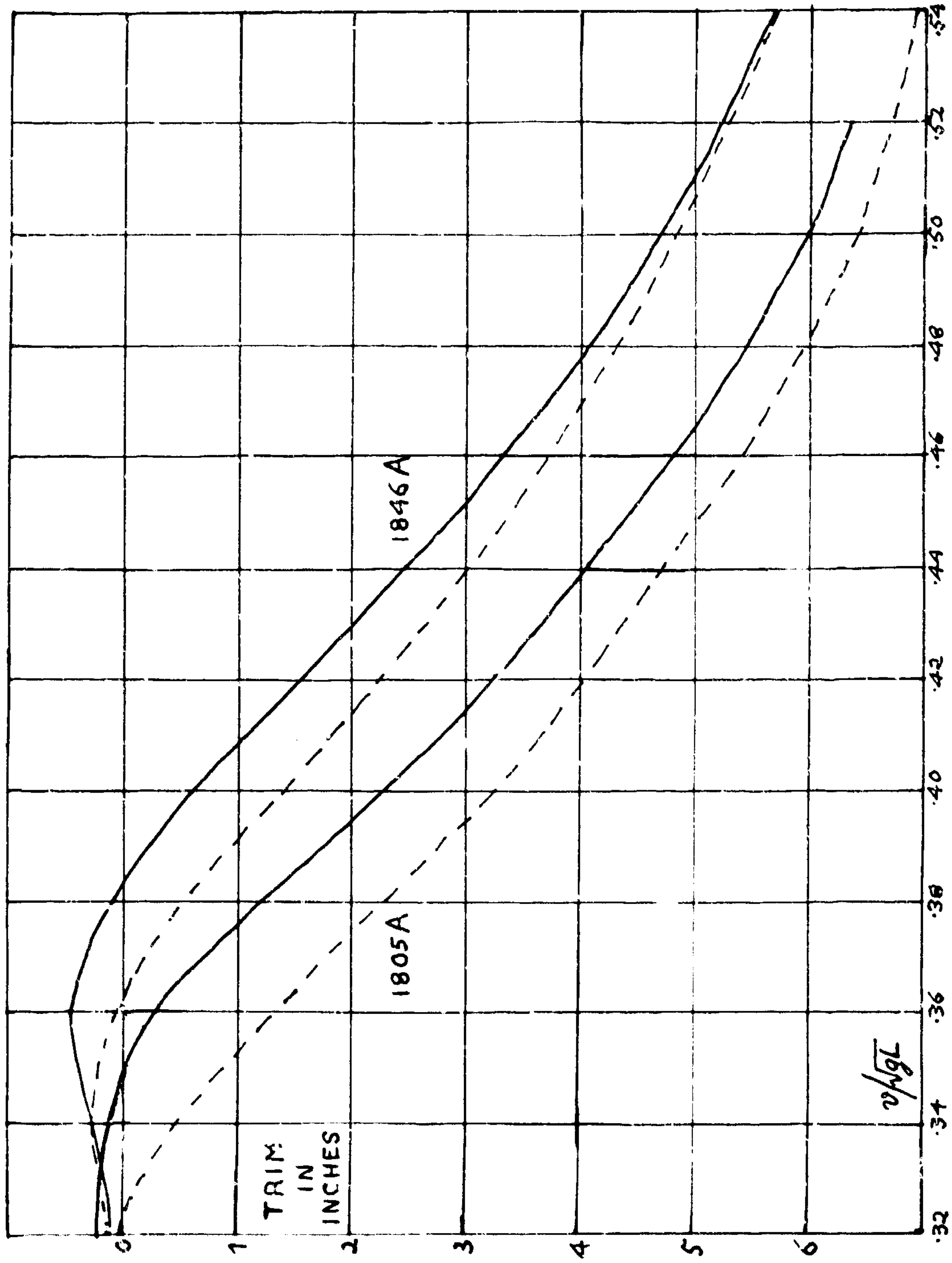
1846A
1805A
TRIM IN INCHES
$v/\sqrt{gL}$
0
1
2
3
4
5
6
.32
.34
.36
.38
.40
.42
.44
.46
.48
.50
.52
.54

Figure 1

A third example of this series was also examined, because it has a larger beam and only half the draft: namely Model 2038C, with $a_2 = -0.5$; length $= 2l = 16$ ft; beam $= 2b = 1.75$ ft; draft $= d = 0.5$ ft. Further details with the measured trim, may be found in Wigley's paper [4].

In this case for the small correction Rd', the value of d' was taken to be 2.5 inches. The calculated trim at $f = 0.5$ was found to be 4.75 inches, the measured value being 4.37 inches. The complete results are shown in Fig. 2, the full curve being the measured values and the broken curve those found by calculation.

Considering the three cases together, the general measure of agreement between calculated and experimental curves is perhaps as good as could be expected from a first-order theory with the various approximations involved and including the neglect of viscosity effects. The order of agreement is much the same as that between calculated and measured curves of wave resistance, the greater discrepancies in trim occurring at speeds at which there are corresponding differences between calculated and measured resistances. It may be said that the present calculations afford further confirmation of the approximate theory of wave resistance.

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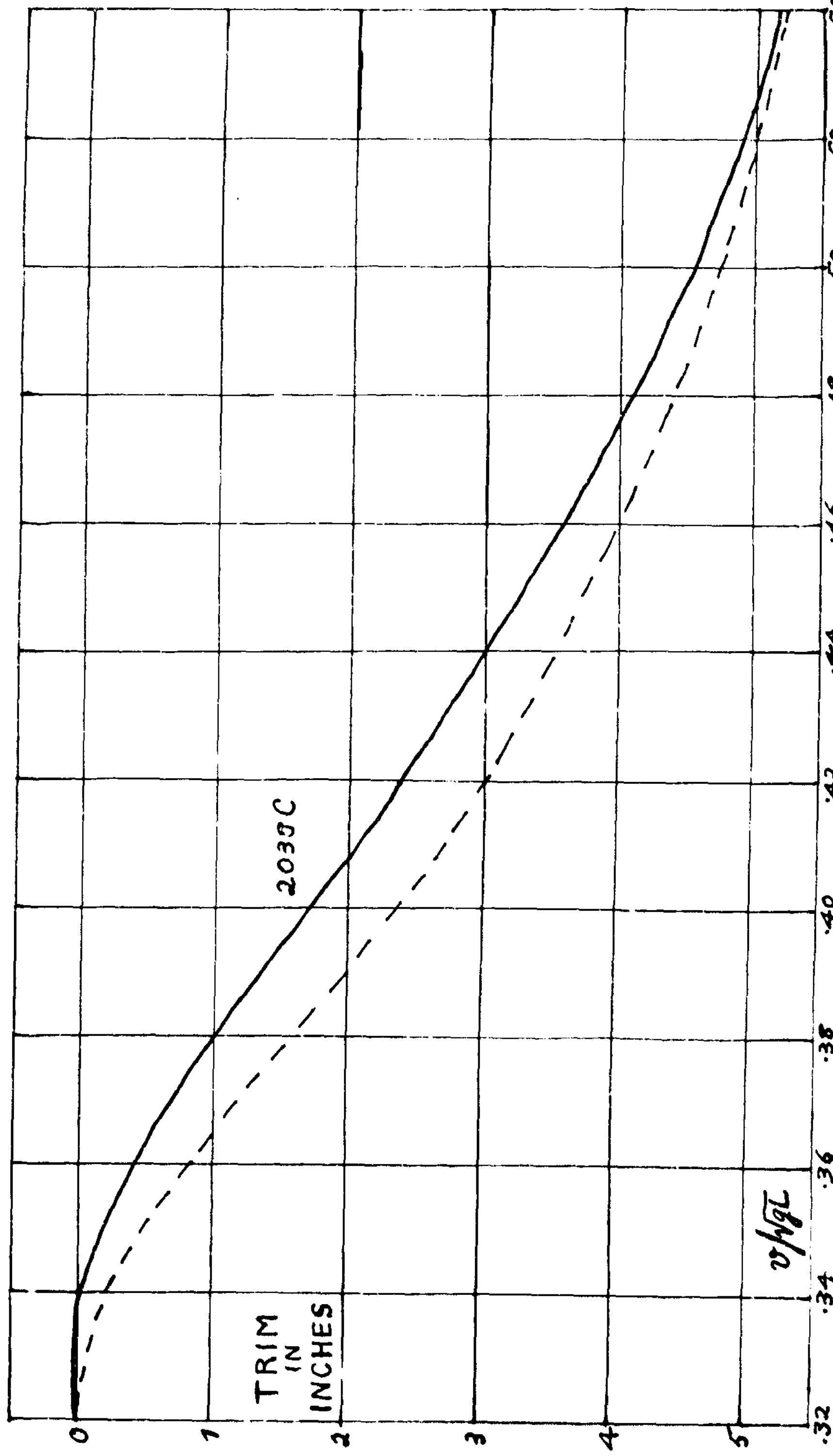
4. W.C.S. Wigley, *Trans, Inst. Nav. Arch.* 84, p. 52 (1942).


TRIM
IN
INCHES
2039C
$v/\sqrt{gL}$
0
1
2
3
4
5
.32
.34
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.52
.54

Figure 2

CALCULATIONS ILLUSTRATING THE EFFECT OF BOUNDARY LAYER ON WAVE RESISTANCE

By Professor T. H. HAVELOCK, M.A., D.Sc., F.R.S., Honorary Member (Associate Member of Council)

Summary

The main object of the paper is to examine the possible effect of the boundary layer in producing a virtual modification of the lines of the ship near the stern. This is regarded as a deflection of the streamlines due to increased displacement thickness of the boundary layer in this region. By superposing a source distribution to produce this additional deflection, expressions can be obtained for the modified wave resistance. No attempt is made to attack the problem directly for actual ship forms. Instead, an indirect method is taken of considering some ideal simple forms and assuming small modifications of the lines near the stern such as might reasonably be ascribed to boundary layer effects. It is shown that such variations suffice to eliminate the humps and hollows on resistance curves at low speeds while making relatively much less difference at high speeds, a result which would improve the general comparison between calculated and measured wave resistances. The paper also includes some remarks on experiments with plank-like forms which are not wholly submerged, and an attempt is made to assess numerically the wave making resistance in such experiments on skin friction.

Introduction

The theory of wave resistance in a frictionless liquid leads to a resistance curve which oscillates rapidly and excessively at low speeds, and such oscillations do not occur in resistance curves derived from experimental results. This is commonly ascribed to the wave making at low speeds being mainly due to the bow of the ship; and an obvious explanation is that the effect of viscosity has been to render the stern relatively ineffective in wave production at low speeds. Some years ago[1] the author considered the matter from the point of view that the effect of the friction belt surrounding the ship is equivalent to smoothing out the lines in the rear portion and some calculations were made to show that this would lead to a diminution of interference effects at low speeds; however, the calculations were too complicated to pursue in any detail at that time. Later[2] the direct assumption of a reduction factor for the rear half of the model was made; the assumption was as simple as possible so as to make calculations practicable, the wave-making properties of the whole of the rear half being reduced by an arbitrary factor less than unity. Subsequently this idea of a reduction factor was largely extended and examined in detail by Wigley.[3] In particular, Wigley compared theoretical and experimental resistance curves for a large number of models, deducing the necessary reduction factor to give reasonable agreement and obtaining an empirical formula for the variation of the factor with the speed. In this work also the factor was applied to the whole of the rear half of the model and it was found to vary in value from zero at the lowest speeds, where only the front half is effective, to unity at the highest speeds, where front and rear are equally effective. This extension and analysis by Wigley is very useful in giving a practicable way of modifying theoretical resistance curves, but, admittedly, it leaves much to be desired from a theoretical point of view. In particular, the variation of the factor from zero to unity seems rather paradoxical; no doubt viscous effects vary with the velocity, but not to such an extent as is implied by that range of values. I believe an explanation can be found in the fact that boundary layer effects on wave formation are appreciable over only a small length of the model near the stern; just as one has a similar comparison between actual normal pressures and those calculated for a frictionless liquid. It is well known that for a frictionless liquid the wave-making effect of bow and stern angles is predominant at low speeds, while at high speeds this is not the case. Hence if the modification of the form is confined to a region near the stern, and even if that modification does not vary much with the speed, it will automatically have greater effect at the lower speeds than at the higher. The present paper is an attempt to find out how far this is the case.

The general point of view so far as the friction belt is concerned has been well expressed by Baker[4] in the remark: "In the after body two things take place, first the contraction of the virtual body, round which the free flow is taking place, which includes the slow-moving portion of the friction belt—a rather indefinite extension of the real form—causes an expansion of all the stream tubes and of the frictional belt, and second expanding stream lines are never very stable and do not adhere to the form from midships to stern post." It must be admitted that this "rather indefinite extension" of the form still remains undefined. In principle, if we know the thickness of the boundary layer and can deduce its displacement thickness, we know by how much the streamlines of the outer flow are deflected. We can then, in theory, superpose on the original form a source distribution which would produce the required extra deflection and hence calculate the modified wave resistance. It may be said at once that the necessary data are not available, and in any case the calculation would be almost impracticable. The scope of the present paper is much less ambitious, and the work may be described

1

as an illustration of the possible effect of boundary layer on wave resistance. The problem is attacked indirectly by taking a simple form and making small modifications of the lines near the stern so as to obtain the required kind of change in the calculated resistance curve; one may then consider whether such modifications can reasonably be ascribed to boundary layer displacement. We consider first the ideal case of a thin plank, with some incidental remarks on wave-making in experiments with planks. Then we consider a form with simple parabolic lines and with vertical sides: in the first place of infinite draught, and then of finite draught. Finally calculations are made for a form which is unsymmetrical fore and aft, in order to show the difference in resistance between motion with bow leading and motion with stern leading.

Wave Resistance of Planks

We begin with the ideal case of a plank of negligible thickness. Assuming the boundary layer to be turbulent, we take for its thickness δ at a distance x from the leading edge the expression

$$\delta = 0 \cdot 37 \, (v\,x/\nu)^{-\frac{1}{5}} \, x \quad . \quad . \quad . \quad . \quad (1)$$

where v is the velocity. In the present problem it is the displacement thickness δ_1 with which we are concerned, as this gives a measure of the outward deflection of the streamlines; in general, δ_1 is defined by

$$u_1 \, \delta_1 = \int (u_1 - u) \, d\,y \quad . \quad . \quad . \quad . \quad (2)$$

where u is the fluid velocity at a point in the boundary layer, u_1 the velocity at the outer limit of the layer, and the integral is taken along a normal through the layer.[13] Assuming the usual velocity distribution we have $\delta_1 = \frac{1}{8}\,\delta$. At the rear end of a plank of length L, the displacement thickness δ_1 has a value b given by

$$b = 0 \cdot 04625 \, \mathrm{R}^{-\frac{1}{5}} \, L \quad . \quad . \quad . \quad . \quad (3)$$

R being Reynolds number. Some values for a plank 16 ft. long are given in the following table for various values of the Froude number $f = v/\sqrt{(g\,\mathrm{L})}$; taking $\nu = 1 \cdot 228 \times 10^{-5}$, and with b in inches, we have

f	0·2	0·3	0·4	0·5
b	0·393	0·362	0·342	0·327

We suppose the plank to be immersed in water and cutting the free surface. We might devise a source distribution which would give this displacement boundary as a streamline. Knowing the surface waves produced by the source distribution and hence the energy required to maintain the system, we can deduce the corresponding wave resistance. Of course, for any body with form the wave resistance is associated with the normal pressures, and the skin friction with the tangential forces; there must always be some interaction between these, but the usual practice of treating them as entirely independent is a valid approximation in most cases. It is obvious that in the present hypothetical case any wave resistance must be associated with a change in the tangential frictional forces with, no doubt, a consequent disturbance of the conditions in the boundary layer. However, leaving aside this interaction for the time being, we may attempt to find some numerical value for a possible wave resistance. It is clear, from the values of b given above, that on any assumption it will be very small compared with the usual skin friction: but a rough estimate may be made.

We quote now expressions for the wave resistance of a given source distribution.[5] We take the origin O in the free surface, O x in the direction of motion, O z vertically downwards, and O y transversely. If the source distribution is in the $z\,x$-plane and is of amount σ per unit area at any point, the corresponding wave resistance is

$$\mathrm{R} = 16\,\pi\,\kappa_0^2\,\rho \int_0^{\pi/2} (\mathrm{P}^2 + \mathrm{Q}^2) \sec^3\theta \, d\,\theta \quad . \quad (4)$$

$$\text{with} \quad \mathrm{P} + i\,\mathrm{Q} = \iint \sigma \, e^{-\kappa_0 z \sec^2\theta + i \kappa_0 x \sec\theta} \, d\,x \, d\,z \quad . \quad (5)$$

where $\kappa_0 = g/v^2$, and the latter integration is taken over the distribution.

We shall be concerned with streamline forms which are narrow compared with their length, and if the form is given by an equation for y as a function of x and z, we use the approximation

$$\sigma = -\frac{v}{2\,\pi} \frac{\partial\,y}{\partial\,x} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Further, in the present paper, all the forms have vertical sides; so that, if d is the draught, we have

$$\mathrm{R} = \frac{4\,\rho\,v^2}{\pi} \int_0^{\pi/2} (1 - e^{-\kappa_0 d \sec^2\theta})^2 \, (\mathrm{I}^2 + \mathrm{J}^2) \cos\theta \, d\,\theta \quad . \quad (7)$$

$$\text{with} \qquad \mathrm{I} + i\,\mathrm{J} = \int \frac{\partial\,y}{\partial\,x} \, e^{i \kappa_0 x \sec\theta} \, d\,x \quad . \quad . \quad . \quad (8)$$

Returning to the immediate problem, we simplify it by replacing the displaced streamline by a simple parabolic curve which starts from the front edge of the plank and leaves the rear end parallel to the plank and at a distance b from it; the resulting integrals are familiar in this work and can be readily computed and the approximation will serve for the immediate purpose. Taking the origin at the rear end of the plank, we use (7) and (8) with

$$y = b\,(1 - x^2/\mathrm{L}^2) \quad . \quad . \quad . \quad . \quad (9)$$

and we obtain

$$\frac{\mathrm{R}}{b^2 v^2} = \frac{16\,\rho}{\pi\,\gamma_0^2} \int_0^{\pi/2} (1 - e^{-\beta})^2 \left(1 + \frac{2}{\gamma^2} - \frac{2}{\gamma} \sin\gamma - \frac{2}{\gamma^2} \cos\gamma\right) \cos^3\theta \, d\,\theta \qquad (10)$$

with

$$\gamma_0 = g\,\mathrm{L}/v^2; \; \beta_0 = g\,d/v^2; \; \gamma = \gamma_0 \sec\theta; \; \beta = \beta_0 \sec^2\theta.$$

Computation was made for a plank 16 ft. long, with a draught of 1 ft. at a speed of 8 ft. per sec., or a value of $f = 0{\cdot}3535$. Further, b as given by (3) was taken to be 0·03 ft. The result was R = 0·003 lb., compared with the usual skin friction of about 5·86 lb. The point of this calculation is simply to confirm that the effect of the plank boundary layer may be taken as inappreciable. It will be still less relatively when we consider a form of finite beam with any appreciable wave resistance. When we deal with such forms we shall therefore simplify the work by neglecting that part of the boundary layer which is the same as that for a plank, and shall consider only the region near the stern where the boundary layer becomes appreciably and rapidly thicker on account of the curvature of the form.

Before proceeding, a few remarks may be made on skin friction experiments with planks. Actual planks have thickness and form, and if the upper edge is above the free surface there will be wave disturbance due to the form, modified to some extent by the boundary layer. Reference has often been made to the possibility of wave resistance being included in some measurements of skin friction, but usually only in the form of a caution; there do not seem to have been any attempts to give a numerical estimate of its value. Perring[6] refers to the possibility of having to make allowance for wave-making in experiments with plank-like forms, and Schoenherr,[7] in referring to his experiments with 3-ft. planks, remarks that the speed should not exceed about 2·7 ft./sec. on account of appreciable wave-making; it may be noted that this is a Froude number f of about 0·27, but other experiments with partially submerged planks have been made up to $f = 0{\cdot}4$ or even higher.

Knowing the form of the plank it would be possible to calculate the wave resistance from the usual formulae, but such results would be of doubtful value at low speeds because of the viscous effects which are now under discussion. However, wave resistance theory suggests another line of attack. According to the formulae, for models with the same mathematical lines and with constant length and draught, the wave resistance varies as the square of the beam. This relation was examined by Wigley[8] for a series of three models satisfying these conditions. The residuary resistance, deduced from the total resistance by the usual method, did not quite obey this law; but the divergence was attributed to the neglect of form effect in estimating the skin friction, and small increases in this part of the resistance would give a wave resistance approximately obeying the theoretical relation. It may be remarked in passing that form effect is not easy to estimate for these narrow models because it is not sufficiently greater than the possible experimental errors in measuring resistance and velocity at low speeds, where in addition there may be the complication of laminar flow. For a general discussion of the relevant data for form effect reference may be made to Todd.[9] For our present purpose we choose Model 1970B, an experimental model used at the N.P.L. by Wigley.[10]

The model lines are given by

$$y = b\left(1 - \frac{x^2}{l^2}\right)\left(1 + a_2\frac{x^2}{l^2} + a_4\frac{x^4}{l^4}\right)\left(1 - \frac{z^2}{d^2}\right) \quad . \quad (11)$$

with $a_2 = 0{\cdot}4375$; $a_4 = -0{\cdot}4375$; length $= 2l = 16$ ft.; beam $= 2b = 1{\cdot}5$ ft.; draught $= d = 1$ ft.

The skin friction has now been calculated from the standard plank formula corrected for temperature.[11] Form effect has been allowed for by adding a constant amount 0·05 to the corresponding Ⓒ values, which is equivalent to increasing the skin friction by an amount ranging from 5% to 6%. The skin friction so increased was subtracted from the total measured resistance, and the residue was taken to be pure wave resistance. We now reduce these values according to the square of the beam for a plank-like form, with lines given by (11), length 16 ft., draught 1 ft., and beam 3 in.; this value of the beam gives an angle of entrance (to the middle line) of 1·75°. The wave resistance for this plank-like form, so estimated, is given as R_w in the following Table: R_f is the skin friction derived from the standard plank formula.

f	v, ft/sec	R_w, lb	R_f, lb
0·20	4·54	0·0092	2·057
0·22	4·99	0·0141	2·448
0·24	5·45	0·0295	2·869
0·26	5·90	0·0352	3·320
0·28	6·36	0·0601	3·890
0·30	6·81	0·1063	4·311
0·32	7·26	0·1376	5·079
0·34	7·72	0·1405	5·417
0·36	8·17	0·1711	6·013
0·38	8·63	0·2139	6·637
0·40	9·08	0·2589	7·288

It appears that the wave resistance is about 1 per cent of the total resistance at $f = 0{\cdot}24$ and rises to about 3 per cent at $f = 0{\cdot}4$. Direct comparison with planks used in skin friction experiments is not intended, because these have different forms and, in particular, their smaller area of vertical cross sections would diminish the wave-making. Further, it is clear, from the various steps in the above calculation, that the results cannot be more than a rough approximation; nevertheless they will serve as an indication of the possible numerical magnitude of wave-making resistance in plank experiments.

Parabolic Form of Infinite Draught

Returning to the main problem, we consider a model with parabolic lines and make small modifications near the stern. We suppose at first that the draught is infinite, as the integrals can then be expressed in terms of functions for which tabulated values are available, and the general character of the results is not much affected by the draught.

Putting $y = b\,\eta$ and $x = l\xi$, and the model having vertical sides, the form is given by

$$\eta = 1 - \xi^2 \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Hence from (7) and (8)

$$R = \frac{4\rho b^2 v^2}{\pi} \int_0^{\pi/2} (I^2 + J^2) \cos\theta \, d\theta \quad . \quad . \quad (13)$$

with
$$I + iJ = -2 \int_{-1}^{1} \xi e^{i\gamma\xi} d\xi \quad . \quad . \quad . \quad (14)$$

where
$$\gamma = \gamma_0 \sec\theta;\ \gamma_0 = \kappa_0 l = g\, l/v^2$$

Using P functions, which are defined for integral values of n by

$$\left.\begin{aligned} P_{2n}(p) &= (-1)^n \int_0^{\pi/2} \cos^{2n}\theta \sin(p \sec\theta)\, d\theta \\ P_{2n+1}(p) &= (-1)^{n+1} \int_0^{\pi/2} \cos^{2n+1}\theta \cos(p\sec\theta)\, d\theta \end{aligned}\right\} (15)$$

we obtain

$$\frac{R}{b^2 v^2} = \frac{32\rho}{\pi\gamma_0^2}\left[\frac{2}{3} + \frac{8}{15}\frac{1}{\gamma_0^2} + P_3(2\gamma_0) - \frac{2}{\gamma_0} P_4(2\gamma_0) + \frac{1}{\gamma_0^2} P_5(2\gamma_0)\right] \quad . \quad (16)$$

This has been graphed on a base f in curve A of Fig. 1, corresponding to the form section A in the same diagram.

Suppose this form has, say, a length of 16 ft. and a beam of 1·5 ft., and consider the virtual modifications which might be ascribed to boundary layer effect. Let B M S be one side of the contour of the model. The wave resistance formulae are, in fact, derived by following the streamline which starts from the bow B, follows the contour B M S to the stern S, and then goes off along the central line. Suppose we know the displacement thickness of the boundary layer at each point and set it off to form a new curve B M′ S′; we propose to take this as the virtual streamline form and to apply wave resistance theory to this line instead of the original curve B M S. This new line starts from the bow B, deviates slightly from the model except possibly near the stern, and we shall suppose that it becomes parallel to the central line at a point S′ somewhat to the rear of the stern S and possibly at some small distance from the central line. In default of sufficient information about the boundary layer in such cases, we shall make some arbitrary assumptions and see what effect is produced on the wave resistance.

We shall neglect the displacement thickness calculated as if for a plank, as we have seen already that this has no appreciable effect; this simplifies the work considerably, as it enables us to follow the actual form from the bow to some point near the stern. We suppose that the

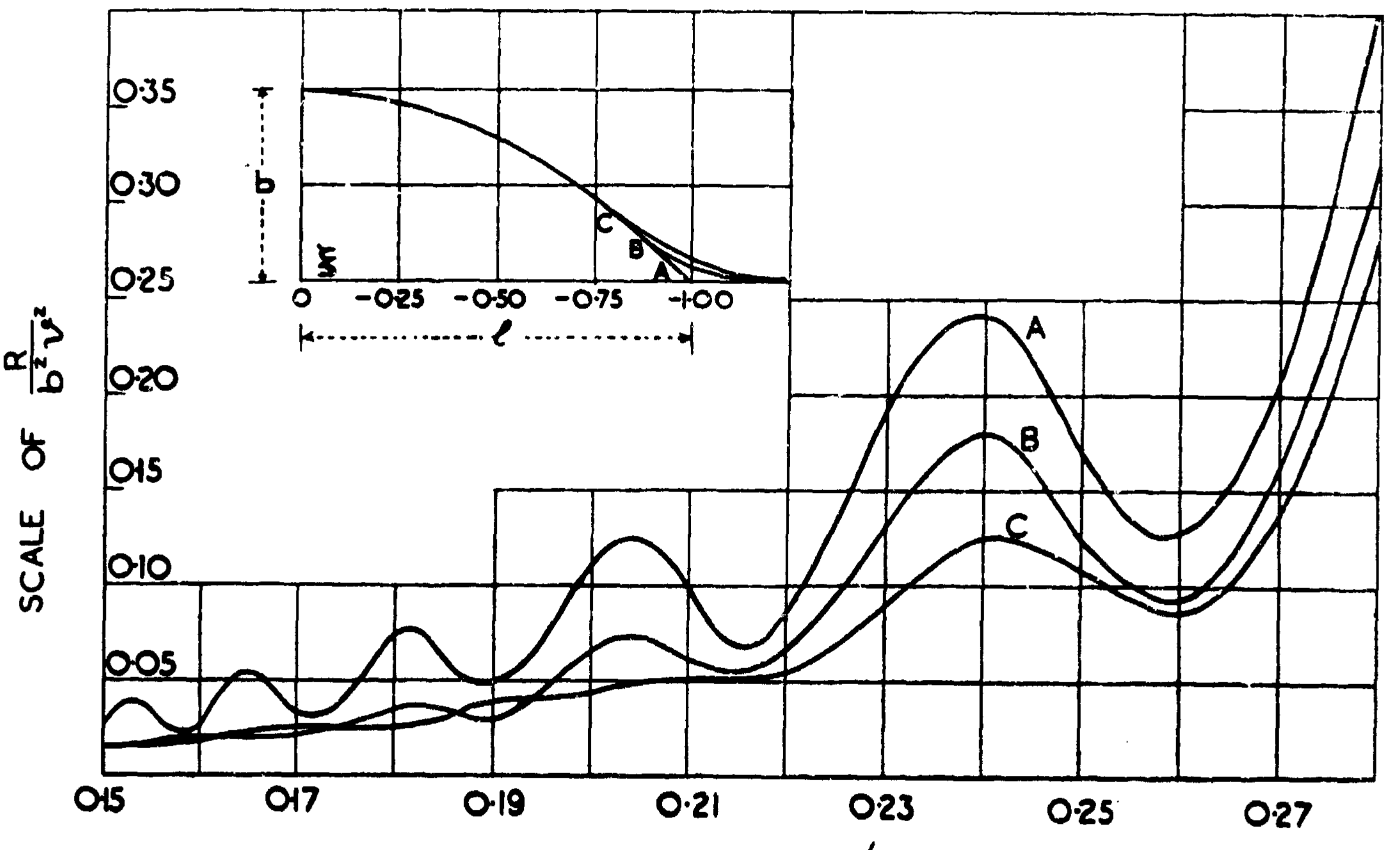

Figure 1

4

new streamline then leaves the form tangentially, and gradually becomes parallel to the central line at some small distance to the rear of the stern. It may be noted that this departure of the virtual streamline from the form does not necessarily mean separation of flow in the usual sense; the latter phenomenon might be represented on this scheme by the new line leaving at an angle to the form. We shall take two examples and in both we shall suppose the new curve to reach the central line at its rear end; we are then dealing effectively with closed forms and this simplifies the work, though it could be extended to include a permanent wake to the rear.

First, considering a 16 ft. model, we take the point of departure to be 1 ft. before the stern and suppose the new line to close in at a point 2 ft. behind the stern. If this new curve is given by

$$\eta = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 \quad . \quad . \quad (17)$$

the conditions are

$$\left.\begin{aligned} &\eta = \frac{15}{64};\ \frac{d\eta}{d\xi} = \frac{7}{4};\ \text{for } \xi = -\frac{7}{8} \\ &\eta = 0;\ \frac{d\eta}{d\xi} = 0;\ \text{for } \xi = -\frac{5}{4} \end{aligned}\right\} \quad . \quad . \quad (18)$$

These determine the coefficients in (17) and we get

$$\eta = \frac{1075}{144} + \frac{35}{2}\xi + \frac{41}{3}\xi^2 + \frac{32}{9}\xi^3 \quad . \quad . \quad (19)$$

This curve is shown starting from the point B on the section in Fig. 1. It passes the stern at a transverse distance $5\,b/72$ from it, and represents little more than smoothing out the stern angle of the model.

We now have instead of (14)

$$\mathrm{I} + i\,\mathrm{J} = -2\int_{-\frac{7}{8}}^{1} \xi\, e^{i\gamma\xi}\, d\xi + \int_{-\frac{5}{4}}^{-\frac{7}{8}} \left(\frac{35}{2} + \frac{82}{3}\xi + \frac{32}{3}\xi^2\right) e^{i\gamma\xi}\, d\xi \quad . \quad (20)$$

From (13) this gives the result

$$\begin{aligned} \frac{\mathrm{R}}{b^2 v^2} = \frac{16\rho}{\pi\gamma_0^2}\Big[&\frac{2}{3} + \frac{2128}{135}\frac{1}{\gamma_0^2} + \frac{32768}{315}\frac{1}{\gamma_0^4} - \frac{32}{3\gamma_0}P_4(p_1) \\ &- \frac{32}{3\gamma_0^2}P_5(p_1) + \frac{64}{3\gamma_0^3}P_6(p_1) + \frac{2}{3\gamma_0}P_4(p_2) \\ &+ \frac{62}{3\gamma_0^2}P_5(p_2) - \frac{64}{3\gamma_0^3}P_6(p_2) + \frac{32}{9\gamma_0^2}P_5(p_3) \\ &+ \frac{320}{3\gamma_0^3}P_6(p_3) - \frac{2048}{9\gamma_0^4}P_7(p_3)\Big] \quad . \quad . \quad . \quad (21) \end{aligned}$$

where $p_1 = 15\,\gamma_0/8;\ p_2 = 9\,\gamma_0/4;\ p_3 = 3\,\gamma_0/8$

This is graphed in curve B of Fig. 1. It shows how this small modification practically eliminates the bumps and hollows at very low speeds and reduces them considerably up to about $f = 0{\cdot}24$.

To make a rather larger change, we suppose next that the point of departure is 2 ft. before the stern, the line closing in as before at 2 ft. behind the stern. The coefficients in (17) are now determined from

$$\left.\begin{aligned} &\eta = \frac{7}{16};\ \frac{d\eta}{d\xi} = \frac{3}{2};\ \text{for } \xi = -\frac{3}{4} \\ &\eta = 0;\ \frac{d\eta}{d\xi} = 0;\ \text{for } \xi = -\frac{5}{4} \end{aligned}\right\} \quad . \quad (22)$$

These give the curve

$$\eta = \frac{25}{16} + \frac{15}{16}\xi - \frac{3}{2}\xi^2 - \xi^3 \quad . \quad . \quad . \quad (23)$$

The curve is shown starting from C on the section in Fig. 1. It passes the stern of the model at a transverse distance $\frac{1}{8}\,b$ from it. In this case, we have

$$\mathrm{I} + i\,\mathrm{J} = -2\int_{-\frac{3}{4}}^{1} \xi\, e^{i\gamma\xi}\, d\xi + \int_{-\frac{5}{4}}^{-\frac{3}{4}} \left(\frac{15}{16} - 3\xi - 3\xi^2\right) e^{i\gamma\xi}\, d\xi \quad . \quad (24)$$

and hence we obtain

$$\begin{aligned} \frac{\mathrm{R}}{b^2 v^2} = \frac{16\rho}{\pi\gamma_0^2}\Big[&\frac{2}{3} + \frac{73}{15}\frac{1}{\gamma_0^2} + \frac{288}{35}\frac{1}{\gamma_0^4} - \frac{7}{2\gamma_0}P_4(p_1) \\ &+ \frac{19}{2\gamma_0^2}P_5(p_1) - \frac{6}{\gamma_0^3}P_6(p_1) + \frac{9}{2\gamma_0}P_4(p_2) \\ &- \frac{21}{2\gamma_0^2}P_5(p_2) + \frac{6}{\gamma_0^3}P_6(p_2) + \frac{63}{8\gamma_0^2}P_5(p_3) \\ &+ \frac{3}{\gamma_0^3}P_6(p_3) - \frac{18}{\gamma_0^4}P_7(p_3)\Big] \quad . \quad . \quad . \quad . \quad (25) \end{aligned}$$

where $p_1 = 7\,\gamma_0/4;\ p_2 = 9\,\gamma_0/4;\ p_3 = \gamma_0/2$

This is graphed in curve C of Fig. 1. Here the difference from curve A for the original model is very marked, and the modification is probably more than is needed so far as low and medium speeds are concerned. It remains to be seen what difference is made at high speeds, but a model of infinite draught is not suitable owing to the exaggerated values obtained at high speeds.

Parabolic Model of Finite Draught

We turn now to a model of the same form, with vertical sides, and of draught d. We shall take the draught d to be one-twentieth of the length $2\,l$, because this ratio was used in some previous work[12] and the results given there can be used to check the present work.

For the model itself, we have from (7) and (8)

5

$$\frac{R}{b^2 v^2} = \frac{32\rho}{\pi\gamma_0^2}\int_0^{\pi/2}(1 - e^{-\beta})^2\left[1 + \cos 2\gamma - \frac{2}{\gamma}\sin 2\gamma + \frac{1}{\gamma^2}(1 - \cos 2\gamma)\right]\cos^3\theta\, d\theta \quad . \quad (26)$$

with

$$\gamma_0 = g\, l/v^2;\ \beta_0 = g\, d/v^2;\ \gamma = \gamma_0 \sec\theta;\ \beta = \beta_0 \sec^2\theta$$

In this, and similar integrals, it has been found more convenient for computation to change the variable from θ to u given by $\cos\theta = \operatorname{sech} u$. The integral was then evaluated by direct quadrature, together with an asymptotic expansion for low speeds when the parameter γ_0 is large. The curve is shown as D in Fig. 2.

line at 1 ft. to the rear, assuming a model length of 16 ft. Hence the conditions are

$$\left.\begin{aligned} &\eta = \frac{15}{64};\ \frac{d\eta}{d\xi} = \frac{7}{4};\ \text{for } \xi = -\frac{7}{8} \\ &\frac{d\eta}{d\xi} = 0;\ \text{for } \xi = -\frac{9}{8} \end{aligned}\right\} \quad . \quad (28)$$

These give the curve

$$\eta = \frac{569}{128} + \frac{63}{8}\xi + \frac{7}{2}\xi^2 \quad . \quad . \quad . \quad (29)$$

which is shown starting from the point E of the section in Fig. 2. It passes the stern at a transverse distance

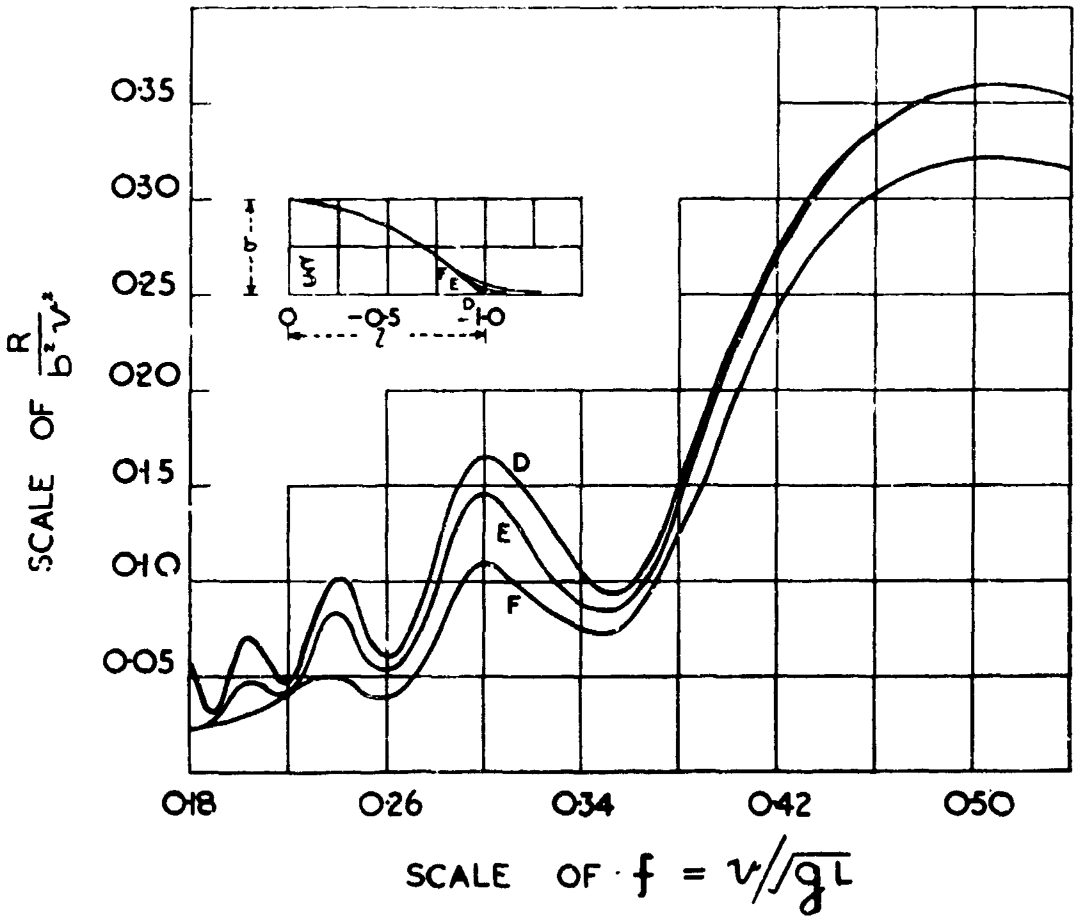

Fig. 2

We shall not use the same modifications as in the previous section, simply because the cubic curve adds so much to the numerical work. It is sufficient for the present to assume a simple parabolic curve

$$\eta = a_0 + a_1\xi + a_2\xi^2 \quad . \quad . \quad . \quad (27)$$

for the new part of the streamline near the stern. First we suppose the curve leaves the form tangentially at 1 ft. before the stern and becomes parallel to the central

$9\,b/128$ and it finishes at a transverse distance $b/64$. We have now

$$I + iJ = -2\int_{-\frac{7}{8}}^{1}\xi e^{i\gamma\xi}\, d\xi + \int_{-\frac{9}{8}}^{-\frac{7}{8}}\left(\frac{63}{8} + 7\xi\right)e^{i\gamma\xi}\, d\xi \quad (30)$$

and this leads to

$$\frac{R}{b^2 v^2} = \frac{16\,\rho}{\pi\,\gamma_0^2}\int_0^{\pi/2}(1-e^{-\beta})^2\left\{1+\frac{1}{\gamma}\left[7\sin\left(\frac{17}{8}\gamma\right)-9\sin\left(\frac{15}{8}\gamma\right)\right]+\frac{1}{\gamma^2}\left[\frac{67}{2}+7\cos\left(\frac{17}{8}\gamma\right)-9\cos\left(\frac{15}{8}\gamma\right)-\frac{63}{2}\cos\left(\frac{1}{4}\gamma\right)\right]\right\}\cos^3\theta\,d\theta \quad . \; . \; . \; . \; . \; (31)$$

Values computed from this are shown in curve E of Fig. 2.

In the next place we take the new curve to extend similarly from 2 ft. before the stern to 2 ft. behind the stern. The conditions are thus

$$\left.\begin{array}{l} \eta=\dfrac{7}{16};\ \dfrac{d\eta}{d\xi}=\dfrac{3}{2};\ \text{for } \xi=-\dfrac{3}{4} \\ \dfrac{d\eta}{d\xi}=0;\ \text{for } \xi=-\dfrac{5}{4} \end{array}\right\} \quad . \; . \; (32)$$

From these we get

$$\eta=\frac{77}{32}+\frac{15}{4}\xi+\frac{3}{2}\xi^2 \quad . \; . \; . \; (33)$$

the curve shown starting from the point F on the section. It passes the stern at a transverse distance $5\,b/32$, and finishes leaving a narrow wake of half-width $b/16$.

In this case

$$I+iJ=-2\int_{-\frac{3}{4}}^{1}\xi\,e^{i\gamma\xi}\,d\xi+\int_{-\frac{5}{4}}^{-\frac{3}{4}}\left(\frac{15}{4}+3\,\xi\right)e^{i\gamma\xi}\,d\xi \quad . \; . \; . \; . \; (34)$$

and for the resistance we obtain

$$\frac{R}{b^2 v^2} = \frac{16\,\rho}{\pi\,\gamma_0^2}\int_0^{\pi/2}(1-e^{-\beta})^2\left\{1+\frac{1}{\gamma}\left[3\sin\left(\frac{9}{4}\gamma\right)-5\sin\left(\frac{7}{4}\gamma\right)\right]+\frac{1}{\gamma^2}\left[\frac{19}{2}+3\cos\left(\frac{9}{4}\gamma\right)-5\cos\left(\frac{7}{4}\gamma\right)-\frac{15}{2}\cos\left(\frac{1}{2}\gamma\right)\right]\right\}\cos^3\theta\,d\theta \quad . \; . \; (35)$$

This resistance is graphed as Curve F in Fig. 2.

It is submitted that inspection of the curves in Fig. 1 and Fig. 2 supports the general conclusion that these small modifications give the required kind of change in the resistance curve for the original model, namely elimination of the humps and hollows at low speeds with a much smaller relative effect at high speeds; further, it is considered that the modifications of form are such as might be caused by displacement of the streamlines by boundary layer effects near the stern. No doubt the results could be improved by further detail: for instance, by change of boundary layer with velocity which might possibly correspond to a change in the point of departure of the new line, or by assuming a greater virtual extension of the form to the rear with a permanent wake. However, the simple cases given now are sufficient to illustrate the point of view.

Unsymmetrical Model

We consider finally a model which is unsymmetrical fore and aft. The difference in resistance according to the direction of motion may have various contributing factors; for example, wave reflection might be important if there were considerable difference between bow and stern angles. However, the main effect may be taken as due to boundary layer modifications. We choose, as the simplest case, a model with vertical sides and with draught equal to one-twentieth of the length, and with the lines given by

$$\eta=(1-\xi^2)(1+\tfrac{1}{3}\xi) \quad . \; . \; . \; (36)$$

This gives a bow angle twice the stern angle. When going bow first, we take the new streamline near the stern to be given by (27) with the conditions

$$\left.\begin{array}{l} \eta=\dfrac{85}{512};\ \dfrac{d\eta}{d\xi}=\dfrac{253}{192};\ \text{for } \xi=-\dfrac{7}{8} \\ \dfrac{d\eta}{d\xi}=0;\ \text{for } \xi=-\dfrac{9}{8} \end{array}\right\} \quad . \; (37)$$

from which we obtain the curve

$$\eta=\frac{20501}{6144}+\frac{2277}{384}\xi+\frac{253}{96}\xi^2 \quad . \; . \; (38)$$

With the fine end leading, the model is given by

$$\eta=(1-\xi^2)(1-\tfrac{1}{3}\xi) \quad . \; . \; . \; (39)$$

To lighten the numerical work, we assume the two ends of the new streamline to be in the same relative positions as when going bow first. The conditions are now

$$\left.\begin{array}{l} \eta=\dfrac{155}{512};\ \dfrac{d\eta}{d\xi}=\dfrac{419}{192};\ \text{for } \xi=-\dfrac{7}{8} \\ \dfrac{d\eta}{d\xi}=0;\ \text{for } \xi=-\dfrac{9}{8} \end{array}\right\} \quad . \; (40)$$

and hence the curve

$$\eta=\frac{34123}{6144}+\frac{3771}{384}\xi+\frac{419}{96}\xi^2 \quad . \; . \; (41)$$

The form of the model with the modifications (38) and (41) is shown in the section in Fig. 3.

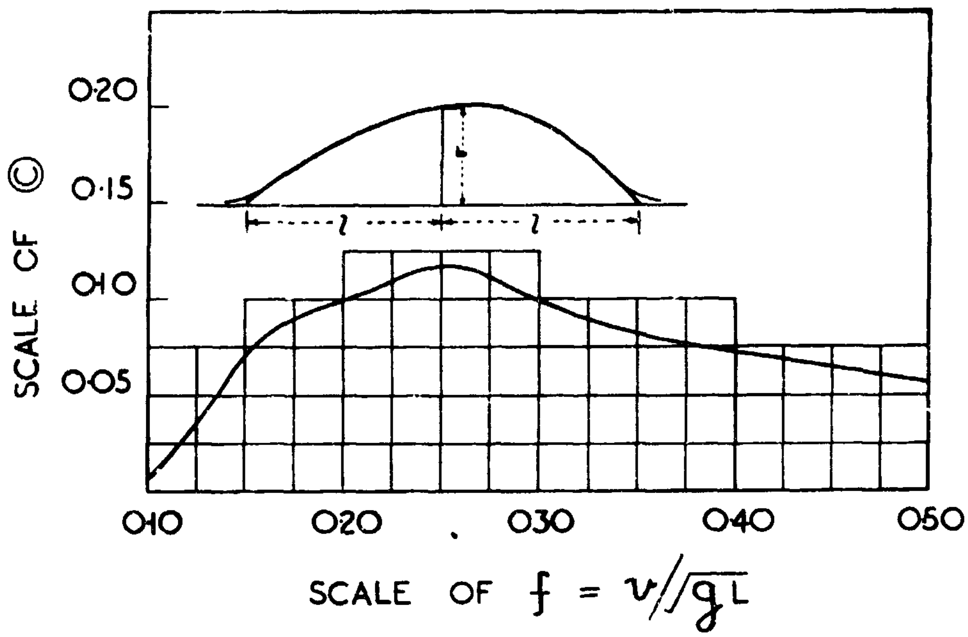

FIG. 3

With the full end leading we have

$$\mathrm{I} + i\mathrm{J} = \int_{-\frac{7}{8}}^{1}\left(\frac{1}{3} - 2\xi - \xi^2\right)e^{i\gamma\xi}\,d\xi + \int_{-\frac{9}{8}}^{-\frac{7}{8}}\left(\frac{2277}{384} + \frac{253}{48}\xi\right)e^{i\gamma\xi}\,d\xi \quad . \quad (42)$$

and with the fine end leading we have

$$\mathrm{I} + i\mathrm{J} = \int_{-\frac{7}{8}}^{1}\left(-\frac{1}{3} - 2\xi + \xi^2\right)e^{i\gamma\xi}\,d\xi + \int_{-\frac{9}{8}}^{-\frac{7}{8}}\left(\frac{3771}{384} + \frac{419}{48}\xi\right)e^{i\gamma\xi}\,d\xi \quad . \quad (43)$$

We shall examine only the difference in resistance between the two cases. Forming $\mathrm{I}^2 + \mathrm{J}^2$ from (42) and (43) and using the difference we obtain eventually

$$\frac{\mathrm{R\,(diff.)}}{b^2 v^2} = \frac{4\rho}{\pi\gamma_0^2}\int_0^{\pi/2}(1 - e^{-\beta})^2\left[\frac{16}{3} - \frac{521}{3}\frac{1}{\gamma^2} + \left(\frac{23}{6\gamma} + \frac{56}{\gamma^3}\right)\sin\left(\frac{15}{8}\gamma\right) - \frac{169}{6\gamma^2}\cos\left(\frac{15}{8}\gamma\right) + \left(\frac{29}{6\gamma} - \frac{56}{\gamma^3}\right)\sin\left(\frac{17}{8}\gamma\right) + \frac{253}{6\gamma^2}\cos\left(\frac{17}{8}\gamma\right) + \frac{479}{3\gamma^2}\cos\left(\frac{1}{4}\gamma\right) + \frac{56}{\gamma^3}\sin\left(\frac{1}{4}\gamma\right)\right]\cos^3\theta\,d\theta \quad . \quad . \quad . \quad . \quad (44)$$

These integrals have been computed and the curve is shown in Fig. 3; the ordinates in this curve are the corresponding difference in © values for this model.

This curve may be compared with results from experiments with unsymmetrical models, for instance in work at the N.P.L. by Wigley.[10] The curve has the same general character and the right order of magnitude, except that in the experimental models the difference in resistance diminishes to zero at about $f = 0{\cdot}5$. The agreement could probably have been improved in the calculations by taking the point of departure of the new curve at different points for the full end and the fine end. However, there is little to guide one meantime in making these assumptions, and moreover, in dealing with actual ship models boundary layer structure is still more difficult to unravel in its dependence upon form.

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THE WAVE RESISTANCE OF A CYLINDER STARTED FROM REST

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SUMMARY

A method of obtaining expressions for wave resistance in accelerated motion is given, but the particular problem examined is the motion due to a circular cylinder submerged at a given depth below the free surface, the cylinder being suddenly started from rest and made to move with uniform velocity. The surface elevation at any time is discussed, and expressions obtained for the finite wave resistance at any time after the start. Numerical calculations have been made for three different speeds, and curves are given showing how the resistance rises initially and oscillates about the steady value for each speed.

1. Introduction

CALCULATIONS of wave resistance have hitherto been made only for a body moving with constant velocity, the problem being treated directly as one of a steady state when referred to axes moving with the body. The case of non-uniform motion is of interest in itself, and also has possible applications. For instance, in measuring the resistance of ship models, the question arises how long it is before the effect of the starting conditions becomes inappreciable. As a matter of fact, measured resistance curves always show oscillations about the steady value for a given speed, but these are no doubt mainly due to the natural period of the measuring apparatus; however, it would be of interest to have some examination of the approach to the steady resistance after the initial stage of accelerated motion.

Expressions for wave resistance in accelerated motion have been given by Sretensky (**1**), who obtained them by transforming the hydrodynamical equations to a form suitable for axes moving with acceleration, but the formulae are too complicated for numerical calculations in general; Sretensky has, it is understood, made some calculations more recently for a particular law of acceleration, but the results are not available.

In some early work (**2**), instead of assuming the steady state as established, I used an alternative method for uniform motion. This may be described as finding the disturbance due to an infinitesimal step in the motion of the body and then integrating. It was pointed out at the time that the method could be applied to motion with variable velocity. It is shown now that this method leads directly to expressions equivalent to those obtained otherwise by Sretensky. However, the present note deals mainly with one particular problem, namely, a circular cylinder submerged

in water at a given depth, suddenly started from rest with a given velocity and maintained at that speed. It has been found possible to make numerical calculations in this case, and the results illustrate various points of interest.

2. Circular cylinder

Take the origin O at the centre of the circular section, of radius a, at a depth f below the free surface, with Ox horizontal and Oy vertically upwards. If the cylinder is given a small horizontal displacement $c\,\delta\tau$, from rest to rest, the velocity potential of the subsequent fluid motion is given by

$$\phi = 2ca^2g^{\frac{1}{2}}\,\delta\tau \int\limits_0^\infty e^{-\kappa(2f-y)}\sin(\kappa x)\sin(g^{\frac{1}{2}}t\kappa^{\frac{1}{2}})\kappa^{\frac{1}{2}}\,d\kappa. \tag{1}$$

This is equation (12) of the paper already quoted (**2**), obtained there by a Fourier integral method; it can also be derived in the manner given later by Lamb (**3**) for the three-dimensional case.

The velocity potential for continuous motion with variable velocity can be found by a direct integration. We consider first the simple case when the cylinder is suddenly started at time $t = 0$ and made to move with uniform velocity c. We obtain, noting that the origin is at the centre of the moving cylinder,

$$\phi = \frac{ca^2x}{x^2+y^2} - \frac{ca^2x}{x^2+(2f-y)^2} + $$
$$+2ca^2g^{\frac{1}{2}}\int\limits_0^t d\tau \int\limits_0^\infty e^{-\kappa(2f-y)}\sin\{\kappa(x+ct-c\tau)\}\sin\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa^{\frac{1}{2}}\,d\kappa. \tag{2}$$

Deriving the surface elevation η from the relation

$$\frac{\partial\eta}{\partial t} = -\frac{\partial\phi}{\partial y} \quad (y = f), \tag{3}$$

we obtain

$$\eta = 2ca^2\int\limits_0^t d\tau \int\limits_0^\infty e^{-\kappa f}\sin\{\kappa(x+ct-c\tau)\}\cos\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa\,d\kappa. \tag{4}$$

Hence we have

$$\eta = \frac{2a^2f}{x^2+f^2} + 2\kappa_0 a^2 \int\limits_0^\infty \frac{\cos\kappa x}{\kappa-\kappa_0}e^{-\kappa f}\,d\kappa -$$
$$- ca^2\int\limits_0^\infty \left\{\frac{\cos(\kappa x+\kappa ct+g^{\frac{1}{2}}t\kappa^{\frac{1}{2}})}{\kappa c+g^{\frac{1}{2}}\kappa^{\frac{1}{2}}} + \frac{\cos(\kappa x+\kappa ct-g^{\frac{1}{2}}t\kappa^{\frac{1}{2}})}{\kappa c-g^{\frac{1}{2}}\kappa^{\frac{1}{2}}}\right\}\kappa e^{-\kappa f}\,d\kappa, \tag{5}$$

where $\kappa_0 = g/c^2$, and the principal values of the integrals are to be taken.

The first two terms in (5) give

$$\eta_1 = \frac{2a^2f}{x^2+f^2} - 2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x - $$
$$- 2\kappa_0 a^2 \int_0^\infty \frac{\kappa_0 \sin\kappa f - \kappa\cos\kappa f}{\kappa^2+\kappa_0^2} e^{-\kappa x}\, d\kappa \quad (x > 0),$$

$$\eta_1 = \frac{2a^2f}{x^2+f^2} + 2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x - $$
$$- 2\kappa_0 a^2 \int_0^\infty \frac{\kappa_0 \sin\kappa f - \kappa\cos\kappa f}{\kappa^2+\kappa_0^2} e^{\kappa x}\, d\kappa \quad (x < 0). \qquad (6)$$

The expressions for η_1 represent a steady state relative to the moving cylinder and symmetrical fore and aft of it. With $x+ct = \xi$ = distance from a fixed origin at the starting-point, the last two terms in (5) represent the surface elevation at any time due to an initial displacement and velocity which is the negative of that given by (6); this must be the case in the present problem and it can be directly verified. With a change of variable, and with $u_0^2 = \kappa_0$, the last two terms in (5) are given by the real part of

$$\eta_2 = -2a^2 \int_{-\infty}^{\infty} \frac{e^{i(\xi u^2 - g^{\frac{1}{2}}tu)}}{u-u_0} u^2 e^{-fu^2}\, du - 4a^2 \int_0^\infty \frac{e^{i(\xi u^2 + g^{\frac{1}{2}}tu)}}{u+u_0} u^2 e^{-fu^2}\, du. \qquad (7)$$

The limiting value of η_2 as t becomes infinite is derived from the principal value of the first integral in (7); taking the real part, we find that

$$\eta_2 \to 2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x \quad \text{as} \quad t \to +\infty. \qquad (8)$$

Turning to (6), we see that ultimately (8) cancels out the regular waves in advance of the cylinder and doubles the amplitude of those in the rear. Without examining the surface elevation in detail, we may specify more closely the part which at any time consists of a regular train of waves accompanying the moving cylinder. It is clear from the form of the integrals in (7) that the only contribution to such a train comes from the first integral, or from

$$-2a^2u_0^2 \int_{-\infty}^{\infty} \frac{e^{i(\xi u^2 - g^{\frac{1}{2}}tu)}}{u-u_0} e^{-fu^2}\, du. \qquad (9)$$

and is due to the pole at $u = u_0$. Regarding u as a complex variable, the path is along the real axis indented at $u = u_0$. There is a saddle-point at $u = g^{\frac{1}{2}}t/2\xi$. First suppose $\xi > 0$. The path of integration may be rotated round the saddle-point to the line of steepest descent, namely, the line $u = g^{\frac{1}{2}}t/2\xi + re^{\frac{1}{4}i\pi}$, the contribution of the circular arcs required to complete the closed contour being zero in the limit. We have also to take account

of the indentation at u_0 according as $u_0 >$ or $< g^{\frac12}t/2\xi$, that is, according as $\xi >$ or $< \frac{1}{2}ct$. In this manner, it is found that as far as the regular waves are concerned, (9) gives

$$2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x \quad (\xi > \tfrac{1}{2}ct),$$
$$-2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x \quad (\xi < \tfrac{1}{2}ct). \tag{10}$$

Similarly if $\xi < 0$ the line of steepest descent is the line $u = g^{\frac12}t/2\xi + re^{\frac34 i\pi}$ and the corresponding contribution is $-2\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x$.

Summing up this outline of an analysis, the surface elevation at any time is made up of three parts: (i) the local symmetrical disturbance travelling with the cylinder given by the first and third terms in (6); (ii) a regular train of waves $4\pi\kappa_0 a^2 e^{-\kappa_0 f}\sin\kappa_0 x$ behind the cylinder extending from $x = 0$ to $x = -\frac{1}{2}ct$; (iii) the part given by the remaining integrals, representing a disturbance which spreads out in both directions and diminishes in magnitude as time goes on.

The second part agrees with the general description using the idea of group velocity. The third part has not been examined in detail, but an asymptotic expansion suitable for large values of ξ and t may be found from the transformed integrals indicated in the previous discussion. For large positive values of ξ and $(g^{\frac12}t/2\xi^{\frac12}) - u_0\xi^{\frac12}$, the first term in such an expansion is

$$\frac{\pi^{1/2}g^{1/2}\,a^2ct^2}{2\xi^{3/2}(\xi - \frac{1}{2}ct)}\,e^{-gt^{2/4}\xi^2}\cos\left(\frac{1}{4}\pi - gt^{2/4}\xi\right). \tag{11}$$

For $\xi = ct$, that is, at a point over the centre of the moving cylinder, this reduces to

$$a^2\left(\frac{\pi\kappa_0}{ct}\right)^{\frac12} e^{-\frac12\kappa_0 f}\cos\tfrac{1}{4}(\pi - \kappa_0 ct), \tag{12}$$

a result which can be obtained directly from the integrals in (7) by using the method of stationary phase. After a sufficient time, (12) gives approximately the departure of the motion over the cylinder from the quasi-steady state consisting of the local symmetrical disturbance and the regular train of waves to the rear. If λ_0 $(= 2\pi/\kappa_0)$ is the wave-length in the regular train, the wave-length of the disturbance near the cylinder is $4\lambda_0$, the wave-length for which c is the group velocity. The usual direct solution for motion with uniform velocity leads to the surface elevation (6) with regular waves in advance as well as to the rear. The so-called practical solution is then obtained by superposing a free infinite wave train cancelling out those in advance and doubling the amplitude to the rear. Another well-known method of obtaining this practical solution directly is to use the frictional coefficient introduced by Rayleigh. In the present analysis we have not used this frictional coefficient, the values of the integrals being interpreted

as principal values wherever necessary. The chief point of the discussion is that there is no ambiguity when the motion starts from rest. The motion which is gradually established as time goes on is the practical solution for the steady state, with regular waves only to the rear of the cylinder; this result is in fact associated with the group velocity being less than the wave velocity.

3. The Wave Resistance (Revised, 1959).*

The velocity potential (2) is sufficient for the surface elevation to the usual approximation; but, in order to calculate the forces on the cylinder from the fluid pressure, it is necessary to add a further approximation so as to satisfy the correct boundary condition on the surface of the cylinder with $z = x + iy$ and $V = (g/\kappa)^{1/2}$, the complex potential of which (2) is the real part is given by

$$w = \frac{ca^2}{2} - \frac{ca^2}{z - 2if} + ca^2 g\int_0^t d\tau\int_0^\infty$$

$$\left\{e^{-i\kappa(c-V)(t-\tau)} - e^{-(c+V)(t-\tau)}\right\} \times e^{-i\kappa z - 2\kappa f}\kappa^{\frac{1}{2}}\, d\kappa \,. \qquad (13)$$

We may expand this in the neighbourhood of the cylinder in the form

$$\frac{ca^2}{2} + \Sigma_n A_n z^n \,. \qquad (14)$$

Hence the required form for the complex potential is

$$w = \frac{ca^2}{2} + \sum_n A_n z^n + \sum_n \frac{a^{2n} A_n^*}{z^n} \qquad (15)$$

valid near the cylinder, the asterisk denoting the conjugate complex quality.

If X and Y are the horizontal and vertical forces on the cylinder, we have from

$$X - iY = \frac{1}{2}\rho i\int\left(\frac{dw}{dz}\right)^2 dz - \rho i\frac{\partial}{\partial t}\int w^* dz^* \qquad (16)$$

the integrals being taken round a small contour surrounding the origin

From (14) and (15) we get, to the first order in the co-efficients A_n

$$X - iy = 4\pi\rho c a^2 A_2 - 2\pi\rho a^2 \frac{\partial}{\partial t} A_1 \qquad (17)$$

*EDITOR'S NOTE: In preparing this 1959 revision of Section 3, pages 329, 331, and 332 of the original paper were modified and page 330 was deleted completely.

with

$$A_1 = -\frac{ca^2}{4f^2} - \gamma c a^2 g^{1/2} \int_0^t d\tau \int_0^\infty \left\{e^{-i\kappa(c-V)(t-\tau)} - e^{-i\kappa(c+V)(t-\tau)}\right\} e^{-2\kappa f} \kappa^{3/2} \, d\kappa$$

$$A_2 = \frac{ica^2}{8f^3} - \tfrac{1}{2} c a^2 g^{1/2} \int_0^t d\tau \int_0^\infty \left\{e^{-i\kappa(c-V)(t-\tau)} - e^{-i\kappa(c+V)(t-\tau)}\right\} e^{-2\kappa f} \kappa^{5/2} \, d\kappa \tag{18}$$

Taking the real part of (16) and integrating with respect to τ, we obtain for the wave resistance

$$R = 2\pi g \rho c a^4 \int_0^\infty \left\{\frac{\sin \kappa(c - v)t}{c - v} + \frac{\sin \kappa(c + V)t}{c + v}\right\} e^{-2\kappa f} \kappa d\kappa . \tag{19}$$

Putting $\kappa = \kappa_0 u^2 = gu^2/c^2$ this becomes

$$R = 4\pi g \rho \kappa_0{}^2 a^4 \int_0^\infty \left[\frac{\sin\{\alpha u(u-1)\}}{u - 1} + \frac{\sin\{\alpha u(u+1)\}}{u + 1}\right] e^{-\beta u^2} u^4 du \tag{20}$$

with $\alpha = \kappa_0 ct$, $\beta = 2\kappa_0 f$.

For suitable values of the parameters α, β the integrals in (20) may be computed by direct quadrature, or from convergent and asymptotic expansions which may readily be deduced. In particular, the limiting value as t becomes infinite follows directly from the first term in the integrand and is

$$R = 4\pi^2 g \rho \kappa_0{}^2 a^4 e^{-2\kappa_0 f} \tag{21}$$

the wave resistance for uniform motion. The next approximation for t large is of order $t^{-1/2}$ and can be obtained from the same integral by the method of stationary phase. This gives, as $t \to \infty$

$$R \to 4\pi^2 g \rho \kappa_0{}^2 a^4 e^{-2\kappa_0 f} + \tfrac{1}{2}\pi g \rho \kappa_0{}^2 a^4 \left(\frac{\pi}{\kappa_0 ct}\right)^{\frac{1}{2}} - e^{-\frac{1}{2}\kappa_0 f} \sin\left(\tfrac{1}{4}\kappa_0 ct - \tfrac{\pi}{4}\right) \tag{22}$$

Thus ultimately the resistance oscillates about the steady value, the amplitude of the oscillations diminishing slowly with the distance travelled and the period being roughly $4\lambda_0$, corresponding to the perturbation of the wave motion given in (12).

4. Three-dimensional motion

Turning to the general problem, we take the origin O in the free surface, with Oz vertically upwards. Suppose a point source of strength m is suddenly created at the point $(0, 0, -f)$ and maintained for a short time $\delta\tau$. To satisfy the condition at the free surface for the initial motion, we take

$$\phi_0 = \frac{m}{r_1} - \frac{m}{r_2}, \tag{23}$$

with $r_1^2 = x^2+y^2+(z+f)^2$; $r_2^2 = x^2+y^2+(z-f)^2$.

The initial surface velocity found from (23) acting for a time $\delta\tau$ gives a surface elevation which can be put in the form

$$\zeta_0 = \frac{m}{\pi}\delta\tau \int\limits_{-\pi}^{\pi} d\theta \int\limits_0^\infty e^{-\kappa f}\cos(\kappa\varpi)\kappa\, d\kappa, \tag{24}$$

with $\varpi = x\cos\theta + y\sin\theta$.

The velocity potential of the fluid motion at any subsequent time t due to this initial displacement without velocity is

$$\phi = \frac{m}{\pi} g^{\frac{1}{2}}\delta\tau \int\limits_{-\pi}^{\pi} d\theta \int\limits_0^\infty e^{-\kappa f+\kappa z}\cos(\kappa\varpi)\sin(g^{\frac{1}{2}}t\kappa^{\frac{1}{2}})\kappa^{\frac{1}{2}}\, d\kappa. \tag{25}$$

Consider now a source moving parallel to Ox at constant depth f, the strength m being a function of the time. Let x be measured from a moving origin vertically over the source, ξ from a fixed origin at the starting-point; and let s_t be the ξ-coordinate of the source at any time t. Then we obtain, from (25),

$$\phi = \frac{m(t)}{r_1} - \frac{m(t)}{r_2} + \frac{g^{\frac{1}{2}}}{\pi}\int\limits_0^t m(\tau)\, d\tau \int\limits_{-\pi}^{\pi} d\theta \int\limits_0^\infty e^{-\kappa f+\kappa z}\cos(\kappa\varpi')\sin\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa^{\frac{1}{2}}\, d\kappa, \tag{26}$$

with $\varpi' = (\xi - s_\tau)\cos\theta + y\sin\theta$.

We may generalize this result for a solid body moving through the liquid. If the solid moves through an infinite liquid with unit velocity, we may

take the fluid motion to be that due to a certain distribution of sources and sinks over its surface and of amount σ per unit area at each point. We assume this distribution in the present problem in order to obtain the wave motion to the first approximation. Thus in (26) we replace m by $\sigma c(t)$, where c is the velocity at time t; if $(h, k, -f)$ is a point on the surface of the body we also put $x-h$ for x, $y-k$ for y, and

$$\varpi' = (\xi-h-s_\tau)\cos\theta+(y-k)\sin\theta,$$

and the required velocity potential is obtained by integrating over the surface of the body.

We shall not carry the general problem further meantime, but consider the case of a slender ship form. Here the usual approximation is to take $\sigma = -(\partial y/\partial h)/2\pi$, where the surface of the form is given as an equation for y in terms of h and f; further, the source distribution is taken to be in the longitudinal section of the form by the plane $y = 0$. We obtain, in this case,

$$\phi = -\frac{c(t)}{2\pi}\iint\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\frac{\partial y}{\partial h}\,dhdf-$$
$$-\frac{g^{\frac{1}{2}}}{2\pi^2}\iint\frac{\partial y}{\partial h}\,dhdf\int\limits_0^t c(\tau)\,d\tau\int\limits_{-\pi}^{\pi}d\theta\int\limits_0^\infty e^{-\kappa f+\kappa z}\cos(\kappa\varpi')\sin\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa^{\frac{1}{2}}\,d\kappa, \tag{27}$$

with $\varpi' = (\xi-h-s_\tau)\cos\theta+y\sin\theta$. This result is equivalent to that obtained by Sretensky by a different method.

The pressure at any point is given by $\rho\,\partial\phi/\partial t$, neglecting the square of the fluid velocity; and the resistance is found from

$$R = -2\iint p(h', 0, -f')\frac{\partial y}{\partial h'}\,dh'df', \tag{28}$$

taken over the longitudinal vertical section. Hence, from (27) and (28), we find

$$R = \frac{\rho\dot{c}}{2\pi}\iint\frac{\partial y}{\partial h'}\,dh'df'\times$$
$$\times\iint[\{(h'-h)^2+(f'+f)^2\}^{-\frac{1}{2}}-\{(h'-h)^2+(f'+f)^2\}^{\frac{1}{2}}]\frac{dy}{dh}\,dhdf+$$
$$+\frac{g\rho}{\pi^2}\iint\frac{\partial y}{\partial h'}\,dh'df'\iint\frac{\partial y}{\partial h}\,dhdf\int\limits_0^t c(\tau)\,d\tau\int\limits_{-\pi}^{\pi}d\theta\times$$
$$\times\int\limits_0^\infty e^{-\kappa(f+f')}\cos(\kappa\varpi')\cos\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa\,d\kappa, \tag{29}$$

with $\varpi' = (h'-h+s_t-s_\tau)\cos\theta$.

The coefficient of $\dot{c}$ is an effective mass for this particular problem, taking account of the free surface and assuming no wave formation and noting that the square of the fluid velocity has been neglected.

As a special case, suppose the model to be started from rest with a velocity c which is then maintained constant. The finite resistance at any time after the start is given by the second term of (29), with c a constant and

$$\varpi' = \{h'-h+c(t-\tau)\}\cos\theta.$$

The result can be reduced to the form

$$R = \frac{g\rho c}{\pi^2}\int\limits_0^t d\tau \int\limits_{-\pi}^{\pi} d\theta \int\limits_0^\infty (I^2+J^2)\cos\{\kappa c(t-\tau)\cos\theta\}\cos\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\kappa\, d\kappa, \tag{30}$$

with

$$I+iJ = \iint \frac{\partial y}{\partial h}\, e^{-\kappa f+i\kappa h\cos\theta}\, dh df. \tag{31}$$

Integrating with respect to τ, this gives

$$R = \frac{2g\rho c}{\pi^2}\int\limits_0^{\frac{1}{2}\pi} d\theta \int\limits_0^\infty (I^2+J^2)\left[\frac{\sin\{(\kappa c\cos\theta-g^{\frac{1}{2}}\kappa^{\frac{1}{2}})t\}}{\kappa c\cos\theta-g^{\frac{1}{2}}\kappa^{\frac{1}{2}}}+\frac{\sin\{(\kappa c\cos\theta+g^{\frac{1}{2}}\kappa^{\frac{1}{2}})t\}}{\kappa c\cos\theta+g^{\frac{1}{2}}\kappa^{\frac{1}{2}}}\right]\kappa\, d\kappa. \tag{32}$$

It can be verified readily that the limiting value to which this tends as t becomes infinite is

$$R = \frac{4\kappa_0 g\rho}{\pi}\int\limits_0^{\frac{1}{2}\pi} (I_0^2+J_0^2)\sec^3\theta\, d\theta, \tag{33}$$

where I_0+iJ_0 is given by (31) with κ replaced by $\kappa_0\sec^2\theta$.

This result (33) is the known expression for the steady resistance at constant speed.

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THE RESISTANCE OF A SUBMERGED CYLINDER IN ACCELERATED MOTION

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SUMMARY

The problem considered is the resistance to motion of a circular cylinder at a constant depth below the free surface, in particular when the motion starts from rest and has uniform acceleration. The resistance is expressed as the sum of two terms; one corresponds to the wave resistance for uniform velocity, and the other may be taken as giving an effective inertia coefficient, the variation of which during the motion is of special interest. The expressions are carried to the second order of approximation and have been reduced to forms suitable for numerical computation. Curves are given showing the variation of both parts of the resistance during the motion, for various values of the acceleration.

1. FOR the steady motion of a submerged body with velocity c parallel to Ox, the condition at the free surface of the water is

$$c^2\partial^2\phi/\partial x^2+g\,\partial\phi/\partial y = 0,$$

where ϕ is the velocity potential, Ox is horizontal, and Oy upwards. For small values of c this becomes formally equivalent to $\partial\phi/\partial y = 0$, while for large velocities the corresponding limit may be taken as $\phi = 0$. The same effect may be seen if we consider the expressions for, say, a moving point source at a given depth below the surface; it is easily seen that in the limit the image system becomes a point source for small velocities, while it approximates to a sink for large velocities. Some discussion has arisen as to the appropriate surface condition to use when estimating the effective inertia of submerged or floating bodies; but any argument based on steady motion assumes a state which has been uniform for a long time, and cannot be applied directly to accelerated motion or motion started at a given instant. In a previous paper (1) expressions were given for resistance in accelerated motion, but no case has hitherto been worked out. It can be seen from equation (30) of that paper that, if we proceed only as far as the first approximation, the total resistance separates into two parts, the wave resistance and the inertia resistance; further, the latter part is, to that approximation, the same as for motion under a free surface neglecting gravity and thus corresponding to the surface condition $\phi = 0$. To obtain a more accurate result it is necessary to proceed further in the approximation to the solution. In the present paper we consider the problem of the circular cylinder moving at constant depth below the surface, examining, in particular, motion with uniform acceleration starting

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from rest; the solution is carried to the second order of approximation. It has been found possible in this case to reduce the expressions to forms which are not too difficult for numerical computation, and curves have been drawn to show the influence of the acceleration upon the resistance and upon the effective inertia coefficient.

2. We shall construct the expressions by the method used in the previous paper. With the origin O at a depth f below the free surface, Ox horizontal and Oy upwards, suppose a singularity of order n created at the origin at time $t = 0$, maintained for a short time $\delta\tau$ and then annihilated. To satisfy the condition at the free surface during this impulsive motion, we have for the complex potential function

$$w_0 = P_n z^{-n} - P_n^*(z-2if)^{-n}, \tag{1}$$

where P_n may be complex, and P_n^* denotes the conjugate complex quantity.

To obtain the surface elevation in a convenient form we write (1) as

$$w_0 = \frac{(-i)^n}{(n-1)!} P_n \int_0^\infty \kappa^{n-1} e^{i\kappa z}\, d\kappa - \frac{(i)^n}{(n-1)!} P_n^* \int_0^\infty \kappa^{n-1} e^{-i\kappa z - 2\kappa f}\, d\kappa. \tag{2}$$

The result of the initial vertical velocity acting for a time $\delta\tau$ is to leave the free surface with an elevation η given by

$$\eta = \mathrm{Re}\frac{2(i)^n P_n^*}{(n-1)!}\delta\tau \int_0^\infty \kappa^n e^{-i\kappa x - \kappa f}\, d\kappa, \tag{3}$$

where Re denotes the real part.

The potential function for the subsequent fluid motion due to this initial surface elevation, without velocity, is

$$w = \frac{2g^{\frac{1}{2}} i^n}{(n-1)!} P_n^* \,\delta\tau \int_0^\infty \kappa^{n-\frac{1}{2}} e^{-i\kappa z - 2\kappa f} \sin(g^{\frac{1}{2}} t \kappa^{\frac{1}{2}})\, d\kappa. \tag{4}$$

We now consider this to be a continuous process occurring as the origin moves parallel to Ox with a velocity c, with c and P_n functions of the time. Let s_t be the distance travelled by the origin from the starting-point; then in (4) we replace t by $t-\tau$, and z by $z-(s_t-s_\tau)$, so that z is now referred to the moving origin. Integrating from the start up to the instant t, we obtain for the complex potential

$$w = \frac{P_n(t)}{z^n} - \frac{P_n^*(t)}{(z-2if)^n} - \frac{i^{n+1} g^{\frac{1}{2}}}{(n-1)!} \int_0^t P_n^*(\tau)\, d\tau \int_0^\infty L e^{-i\kappa z - 2\kappa f} \kappa^{n-\frac{1}{2}}\, d\kappa; \tag{5}$$

with
$$L = e^{-i\{\kappa(s_t - s_\tau) - g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}} - e^{-i\{\kappa(s_t - s_\tau) + g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}}. \tag{6}$$

This result may be confirmed by using the pressure condition at the free surface, when the axes are moving parallel to Ox with velocity c and acceleration $\dot{c}$. For these relative coordinates we have

$$\frac{p}{\rho} = p_0 + \frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x} - \tfrac{1}{2}q^2 - gy, \tag{7}$$

$$\eta = \frac{1}{g}\frac{\partial\phi}{\partial t} - \frac{c}{g}\frac{\partial\phi}{\partial x}. \tag{8}$$

The condition that p is constant on the free surface leads to the condition

$$\mathrm{Re}\left\{\frac{\partial^2 w}{\partial t^2} + c^2\frac{d^2w}{dz^2} + (ig - \dot{c})\frac{dw}{dz} - 2c\frac{\partial}{\partial t}\left(\frac{dw}{dz}\right)\right\} = 0; \qquad y = f. \tag{9}$$

It may be verified by direct substitution that (5) satisfies this condition.

3. Suppose a circular cylinder, of radius a, centre at the origin, is moving horizontally with velocity c. We assume that the potential can be expressed as an infinite series of terms like (5) for integral values of n; and the quantities P_n are to be determined from the boundary condition on the circle $|z| = a$. If we write

$$F(\kappa, t) = \sum_1^\infty (-i)^n P_n(t)\kappa^{n-1}/(n-1)!, \tag{10}$$

we have the general expression

$$w = \sum P_n(t)z^{-n} - \int_0^\infty F^*(\kappa, t)e^{-i\kappa z - 2\kappa f}\,d\kappa - \\ - ig^{\frac{1}{2}}\int_0^t F^*(\kappa, \tau)\,d\tau \int_0^\infty L e^{-i\kappa z - 2\kappa f}\kappa^{\frac{1}{2}}\,d\kappa. \tag{11}$$

We may expand the second and third terms in positive powers of z in the neighbourhood of the circular boundary, and we get w in the form

$$w = \sum (P_n z^{-n} + Q_n z^n), \tag{12}$$

with

$$Q_n = -\frac{(-i)^n}{n!}\int_0^\infty F^*(\kappa, t)\kappa^n e^{-2\kappa f}\,d\kappa + \\ + \frac{(-i)^{n+1}g^{\frac{1}{2}}}{n!}\int_0^t F^*(\kappa, \tau)\,d\tau \int_0^\infty L\kappa^{n+\frac{1}{2}}e^{-2\kappa f}\,d\kappa. \tag{13}$$

The boundary condition on the circle gives

$$P_1 = ca^2 + a^2 Q_1^*; \qquad P_n = a^{2n} Q_n^*. \tag{14}$$

Hence, for the quantities P_n we have the infinite set of equations

$$P_1(t) = ca^2 - ia^2 \int_0^\infty F(\kappa, t)\kappa e^{-2\kappa f}\,d\kappa - g^{\frac{1}{2}}a^2 \int_0^t F(\kappa,\tau)\,d\tau \int_0^\infty L^*\kappa^{\frac{3}{2}}e^{-2\kappa f}\,d\kappa,$$

$$P_n(t) = -\frac{i^n a^{2n}}{n!}\int_0^\infty F(\kappa,t)\kappa^n e^{-2\kappa f}\,d\kappa +$$
$$+\frac{i^{n+1}g^{\frac{1}{2}}a^{2n}}{n!}\int_0^t F(\kappa,\tau)\,d\tau \int_0^\infty L^*\kappa^{n+\frac{1}{2}}e^{-2\kappa f}\,d\kappa. \tag{15}$$

4. We shall only attempt an approximate solution of these equations as far as the second order, that is, up to $n = 2$. It may be noted that the condition at the free surface is satisfied exactly, but the condition on the circular boundary is only satisfied approximately to the order indicated. For the forces (X, Y) on the cylinder we use the general expression suitable for axes moving with the cylinder (2), which is in this case

$$X - iY = \tfrac{1}{2}\rho i \int \left(\frac{dw}{dz}\right)^2 dz + \pi\rho a^2\dot{c} - \rho i\frac{\partial}{\partial t}\int w^*\,dz^*. \tag{16}$$

We shall find it convenient to take the corresponding resistance in two parts; thus, to the present order,

$$R_1 = \mathrm{Re}\left\{-\tfrac{1}{2}\rho i\int\left(\frac{dw}{dz}\right)^2 dz\right\} = -\frac{4\pi\rho}{a^4}\mathrm{Re}\{P_1 P_2^*\} \tag{17}$$

$$R_2 = -\pi\rho a^2\dot{c} + \mathrm{Re}\left\{\rho i\frac{\partial}{\partial t}\int w^*\,dz^*\right\} = -\pi\rho a^2\dot{c} + 2\pi\rho\,\mathrm{Re}\left\{\frac{\partial}{\partial t}P_1^*\right\}. \tag{18}$$

Further, from (15), P_1 and P_2 are given by,

$$P_1 = ca^2 - \frac{a^2}{4f^2}P_1 - \frac{ia^2}{8f^3}P_2 - g^{\frac{1}{2}}a^2\int_0^t \{-iP_1(\tau) - \kappa P_2(\tau)\}\,d\tau \int_0^\infty L^*\kappa^{\frac{3}{2}}e^{-2\kappa f}\,d\kappa \tag{19}$$

$$P_2 = -\frac{ia^4}{8f^3}P_1 - \frac{3a^4}{16f^4}P_2 - \tfrac{1}{2}ig^{\frac{1}{2}}a^4\int_0^t \{-iP_1(\tau) - \kappa P_2(\tau)\}\,d\tau \int_0^\infty L^*\kappa^{\frac{5}{2}}e^{-2\kappa f}\,d\kappa. \tag{20}$$

If we neglect gravity, we have approximately

$$P_1 = ca^2(1 - a^2/4f^2); \qquad P_2 = -ica^6/8f^3. \tag{21}$$

From (17) and (18), R_1 is zero and

$$R_2 = \pi\rho a^2\dot{c}(1 - a^2/2f^2), \tag{22}$$

the coefficient of $\pi\rho a^2\dot{c}$ being, to this order, the effective inertia coefficient for a free surface, neglecting gravity. The next step is to use these first approximations for P_1 and P_2 in the integrals in (19) and (20) and so obtain the next approximation.

5. Before proceeding, we may confirm this process by applying the method to the case of uniform velocity c, for which the results have previously been obtained by direct consideration of steady motion. We require the limiting values of the integrals in (19) and (20) for t becoming infinite, the quantities P being constant. Putting in the appropriate form for L from (6), we have, for instance, the integral

$$\int_0^t d\tau \int_0^\infty \{e^{i(\kappa c-g^{\frac12}\kappa^{\frac12})(t-\tau)}-e^{i(\kappa c+g^{\frac12}\kappa^{\frac12})(t-\tau)}\}\kappa^{\frac32}e^{-2\kappa f}\,d\kappa. \tag{23}$$

Integrating with respect to τ and taking the limiting value of the integral in κ for $t \to \infty$, it is readily found that (23) has the limiting value

$$2g^{\frac12}\kappa_0\,c^{-2}[\pi e^{-\alpha}+i\{\alpha^{-1}-e^{-\alpha}\mathrm{Ei}(\alpha)\}], \tag{24}$$

where $\kappa_0 = g/c^2$, $\alpha = 2\kappa_0 f$, and Ei is the exponential integral. The similar integral with $\kappa^{\frac52}$ in place of $\kappa^{\frac32}$ converges to

$$2g^{\frac12}\kappa_0^2\,c^{-2}[\pi e^{-\alpha}+i\{\alpha^{-2}+\alpha^{-1}-e^{-\alpha}\mathrm{Ei}(\alpha)\}]. \tag{25}$$

The integral with a factor $\kappa^{\frac72}$ is not required at this stage; being factored by P_2, it clearly does not enter into the second-order approximation.

Hence, to this order, (19) and (20) give, for uniform velocity,

$$P_1 = ca^2-\frac{a^2}{4f^2}P_1+2i\kappa_0^2\,ca^4[\pi e^{-\alpha}+i\{\alpha^{-1}-e^{-\alpha}\mathrm{Ei}(\alpha)\}],$$

$$P_2 = -i\frac{a^4}{8f^3}P_1-\kappa_0^3\,a^4P_1[\pi e^{-\alpha}+i\{\alpha^{-2}+\alpha^{-1}-e^{-\alpha}\mathrm{Ei}(\alpha)\}]. \tag{26}$$

The resistance R_2 is zero in this case; and from (17) and (26) we obtain

$$R_1 = 4\pi^2\rho c^2\kappa_0^3\,a^4e^{-\alpha}[1-2\kappa_0^2\,a^2\{\alpha^{-2}+2\alpha^{-1}-2e^{-\alpha}\mathrm{Ei}(\alpha)\}]. \tag{27}$$

This result agrees, to the second order, with the more general expressions obtained previously for the wave resistance at uniform velocity (**3**).

6. Returning to the general expressions (19) and (20), we shall examine, in particular, motion with uniform acceleration γ, starting from rest; thus we have $c = \gamma t$, $s = \frac12\gamma t^2$. The first approximation to P_1 is $\gamma a^2t(1-a^2/4f^2)$, and it is sufficient for the next stage to put $P_1 = \gamma a^2\tau$ in the integrals in (19) and (20). Hence, to the required order, we have

$$P_1 = \gamma a^2t-\frac14\frac{a^2}{f^2}P_1+ig^{\frac12}a^4\gamma\int_0^t \tau\,d\tau\int_0^\infty L^*\kappa^{\frac32}e^{-2\kappa f}\,d\kappa,$$

$$P_2 = -\frac{ia^4}{8f^3}P_1-\tfrac12 g^{\frac12}a^6\gamma\int_0^t \tau\,d\tau\int_0^\infty L^*\kappa^{\frac52}e^{-2\kappa f}\,d\kappa, \tag{28}$$

where $$L^* = e^{i\{\frac12\kappa\gamma(t^2-\tau^2)-g^{\frac12}\kappa^{\frac12}(t-\tau)\}}-e^{i\{\frac12\kappa\gamma(t^2-\tau^2)+g^{\frac12}\kappa^{\frac12}(t-\tau)\}}. \tag{29}$$

We now reduce the integrals to a more convenient form. The integration with respect to τ can be expressed in terms of Fresnel integrals; after some reduction we obtain the result

$$\int_0^t L^*\tau\,d\tau = p^{-2}e^{i(pt-q)^2}\{qP(pt-q)+qP(q)-\tfrac{1}{2}ie^{-iq^2}\}-$$
$$-p^{-2}e^{i(pt+q)^2}\{qP(pt+q)-qP(q)-\tfrac{1}{2}ie^{-iq^2}\}, \qquad (30)$$

where
$$2p^2 = \kappa\gamma, \qquad q^2 = g/2\gamma;$$

and
$$P(u) = C(u)-iS(u) = \int_0^u e^{-iu^2}\,du.$$

For the integration with respect to κ, we change the variable from κ to v, given by $\kappa = 4gv^2/\gamma^2t^2$; and we obtain finally

$$\int_0^t \tau\,d\tau \int_0^\infty L^*\kappa^{\frac{3}{2}}e^{-2\kappa f}\,d\kappa = A_2+iB_2 = k^9g^{-\frac{5}{2}}t^{-3}\int_0^\infty (A+iB)v^2e^{-8\beta v^2}\,dv, \quad (31)$$

$$\int_0^t \tau\,d\tau \int_0^\infty L^*\kappa^{\frac{5}{2}}e^{-2\kappa f}\,d\kappa = A_4+iB_4 = k^{13}g^{-\frac{7}{2}}t^{-5}\int_0^\infty (A+iB)v^4e^{-8\beta v^2}\,dv, \quad (32)$$

with

$$k^2 = 2g/\gamma; \qquad \beta = gf/\gamma^2t^2; \qquad p_1 = k(v-\tfrac{1}{2}); \qquad p_2 = k(v+\tfrac{1}{2});$$

$$A = C(p_1)\cos p_1^2+S(p_1)\sin p_1^2+C(p_2)\cos p_2^2+S(p_2)\sin p_2^2+$$
$$+\{C(\tfrac{1}{2}k)-k^{-1}\sin\tfrac{1}{4}k^2\}(\cos p_1^2-\cos p_2^2)+$$
$$+\{S(\tfrac{1}{2}k)+k^{-1}\cos\tfrac{1}{4}k^2\}(\sin p_1^2-\sin p_2^2); \quad (33)$$

$$B = C(p_1)\sin p_1^2-S(p_1)\cos p_1^2+C(p_2)\sin p_2^2-S(p_2)\cos p_2^2-$$
$$-\{S(\tfrac{1}{2}k)+k^{-1}\cos\tfrac{1}{4}k^2\}(\cos p_1^2-\cos p_2^2)+$$
$$+\{C(\tfrac{1}{2}k)-k^{-1}\sin\tfrac{1}{4}k^2\}(\sin p_1^2-\sin p_2^2). \quad (34)$$

7. For the resistance, we consider first the part R_1. This could be obtained to the second approximation, but it was thought sufficient meantime to examine only the first approximation. The general effect of the second approximation is known in the case of uniform velocity; it consists in increasing the value somewhat at lower speeds and diminishing it slightly at higher speeds. From some rough calculations it appears that the effect in the present case would be similar; but for a general idea of the effect of acceleration upon R_1, which reduces to the wave resistance for uniform velocity, it is sufficient to take the first approximation. From (17) and (28), we have

$$R_1 = 2\pi\rho\gamma g^{\frac{1}{2}}a^4tA_4 = 128\pi g\rho a^2k\beta^2(a^2/f^2)\int_0^\infty Av^4e^{-8\beta v^2}\,dv, \qquad (35)$$

in the notation given in (33).

For the second part of the resistance we include second-order terms: from (18) and (28) we have

$$R_2 = -\pi\rho a^2\gamma + 2\pi\rho a^2\gamma(1-a^2/4f^2) - 2\pi\rho\gamma g^{\frac{1}{2}}a^4\partial B_2/\partial t. \tag{36}$$

From (31) and (33), this leads to

$$R_2 = \pi\rho a^2\gamma p,$$

with

$$p = 1-(\tfrac{1}{2}+32k\beta^2 b)(a^2/f^2),$$

$$b = \int_0^\infty (-3+16\beta v^2)v^2 B e^{-8\beta v^2}\,dv. \tag{37}$$

Numerical computations have been made for the integrals in (35) and (37). The quantities A and B depend only upon the acceleration, while the instantaneous value of the velocity enters through β. The integrals were calculated for two different accelerations, and for about a dozen values of β in each case—ranging from $\frac{1}{16}$ to 40. For small values of β it was necessary to go as far as $v = 4{\cdot}0$ or further, but subdivisions of $0{\cdot}1$ for v were usually sufficient. For large values of β the necessary range for v was less, but subdivisions of $0{\cdot}02$ had to be taken, especially for the larger values of k. For various reasons it was difficult to obtain any high degree of accuracy in the final results; but it is considered that the calculations are sufficient to show the general character of the effect of acceleration upon the resistance.

8. Some of the results are shown in the curves of resistance. These curves show the resistance for a particular value of the ratio of the radius of the cylinder to the depth of its centre, namely the value given by $a^2/f^2 = 0{\cdot}1$. We have chosen to graph the curves on a base of velocity c, or γt, the abscissae being $c/(gf)^{\frac{1}{2}}$. This was partly so as to bring into the diagram the wave resistance curve for uniform velocity; this curve is shown as R_0 in the diagram.

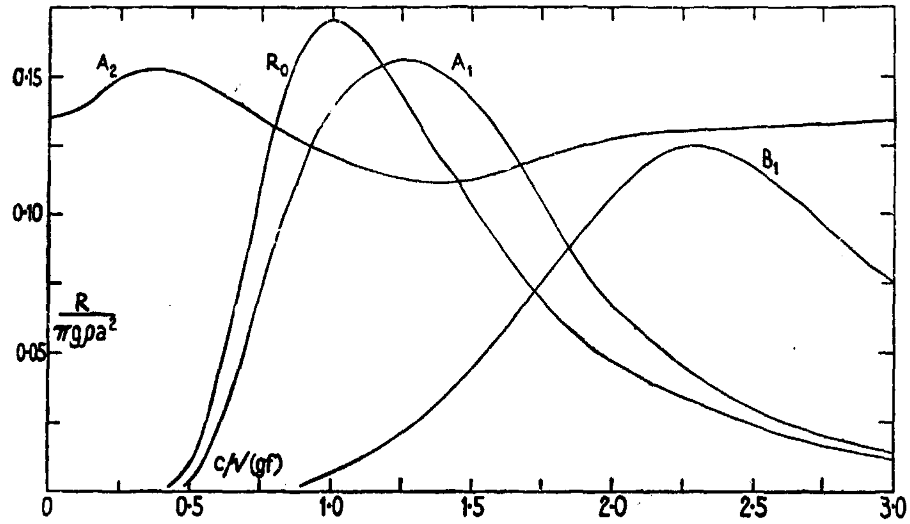
Taking the resistance R_1 first, the curve A_1 shows its value for $k^2 = 9\pi/2$, or for $\gamma/g = 0{\cdot}1418$; while the curve B_1 is for $k^2 = \pi/2$, or for $\gamma/g = 1{\cdot}276$. The effect of greater acceleration is shown in the lower maximum wave resistance and the higher velocity at which it occurs compared with the curve R_0 for uniform velocity. It should be noted that if we had graphed the curves on a time base, the abscissae for curve B_1 would be reduced to one-ninth compared with those for A_1.

We turn now to the resistance R_2, which is of greater interest. In general, the relative magnitudes of R_1 and R_2 depend upon the two ratios γ/g and a/f. In the diagram, the curve A_2 shows the resistance R_2 for the case $\gamma/g = 0{\cdot}1418$, and $a^2/f^2 = 0{\cdot}1$; the total resistance in that case is

given by A_1+A_2. It is seen, from (35) and (37), that the part of the total resistance which is simply proportional to the acceleration is

$$\pi\rho a^2\gamma(1-a^2/2f^2).$$

If we define the effective mass as the coefficient of γ in this term, then the inertia coefficient is the same as for a free surface neglecting gravity. We could, on the other hand, divide the total resistance by γ and so define



the effective mass at each instant. However, from the way in which R_1 and R_2 arise and from their variation, it seems convenient to refer to R_1 as the wave resistance and to regard R_2 as the product of the effective mass and the acceleration; with this convention, the inertia coefficient for motion with uniform acceleration from rest is given by the quantity p of (37). It can be seen from (37) that p converges to $1-a^2/2f^2$ for both $c\to 0$ and $c\to\infty$. In the particular case being considered, the inertia coefficient would be 0·95 for a free surface without gravity, and 1·05 for a rigid surface. Its variation with velocity can be seen from the curve A_2, which gives the resistance R_2. The coefficient p begins with the value 0·95, rises to a maximum of about 1·07 near $c/(gf)^{\frac{1}{2}}$ equal to 0·4, falls to a minimum of 0·78 near $c/(gf)^{\frac{1}{2}} = 1·4$, and then rises towards the value 0·95 with increasing velocity.

Similar calculations were made for the case $\gamma/g = 1·276$, for which the wave resistance is shown in the curve B_1. The curve for R_2 in this case is not shown in the diagram, because on the scale its magnitude would be nine times that of A_2. However, the curve, in its relation to B_1, is of the same type as in the case of the curves A_2 and A_1, but with less variation in the inertia coefficient; this coefficient begins at 0·95, rises to a maximum

of 0·975 near $c/(gf)^{\frac{1}{2}} = 1$, falls to a minimum of 0·91 near $c/(gf)^{\frac{1}{2}} = 2·5$, and then rises gradually towards 0·95. It is of special interest to notice that while the coefficient begins with the free surface value its rise towards the rigid surface value occurs before the wave resistance R_1 has become appreciable. A few calculations were also made for a very small acceleration, with $\gamma/g = 0·035$, to confirm the general trend of the variation; in this case, the coefficient p has risen to a value of 1·05 at about $c/(gf)^{\frac{1}{2}} = 0·2$.

Referring to (37), some of the approximate values found in these calculations are given for reference.

For $\gamma/g = 0·1418$, the quantity $32k\beta^2 b$ has the value $-0·52$, $-1·2$, $-1·1$, 0·92, 1·66, 0·4 for β equal to 40, 10, 4, 1, 0·5, 0·25 respectively. For $\gamma/g = 1·276$, the values of $32k\beta^2 b$ are $-0·04$, $-0·24$, 0·3, 0·4 for β equal to 10, 1, 0·25, 0·125 respectively.

The motion which has been examined in detail is uniform acceleration starting from rest. Similar calculations could be made for other cases of variable velocity, in particular for motion with uniform acceleration with a given initial velocity. In the latter case the results are not likely to be much different in general character; it appears that in any case the initial value of R_2 would be the inertia resistance for a free surface without gravity, and its subsequent variation would be similar to that shown by the present calculations.

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The forces on a submerged spheroid moving in a circular path

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Expressions are obtained for the tangential and radial forces on a sphere moving in a circular path at constant depth; similar calculations are made for a prolate spheroid, including in this case the couple acting on the spheroid. Numerical computations have been made, and curves are given to show the effect of curvature of the path upon the wave resistance.

1. The forces on a ship moving in a curved path are, no doubt, affected to some extent by the wave motion produced, but it is not easy to estimate the magnitude or nature of this influence. In the following paper an approach is made to some aspects of this problem by considering some cases of a submerged body moving in a circular path, namely, a sphere and a prolate spheroid. The motion of a sphere has been examined recently by Sretensky (1946), but the results given by him are incorrect. In the present work a different method is adopted; it is one which can be used for bodies of other forms, and also for non-uniform motion.

2. We may derive first expressions for the ideal case of a simple source moving in any manner at constant depth f below the free surface of the water. We take fixed axes with O in the free surface, Oz vertically upwards, and we use cylindrical coördinates (ϖ, θ, z). If at time τ the strength of the source is m and its horizontal distance from O is ϖ_0, the velocity potential due to an infinitesimal step in the motion is given, as in equation (27) of a previous paper (Havelock 1949), by

$$\phi = 2mg^{\frac{1}{2}}\delta\tau \int_0^\infty J_0(\kappa\varpi_0)\, e^{-\kappa(f-z)} \sin\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\, \kappa^{\frac{1}{2}}\, d\kappa. \tag{1}$$

We may regard the effect due to a point source, varying in strength and moving in any manner, as made up of the superposition of small steps of this nature. In particular, for the present problem, we suppose the source of constant strength and to be moving in a circle of radius h; further, we take the motion to start at $t = 0$ and the angular velocity to have a constant value Ω. Hence we obtain the velocity potential at any time t as

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} + 2mg^{\frac{1}{2}} \int_0^t d\tau \int_0^\infty J_0(\kappa\varpi_0)\, e^{-\kappa(f-z)} \sin\{g^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t-\tau)\}\, \kappa^{\frac{1}{2}}\, d\kappa, \tag{2}$$

where

$$\left.\begin{aligned} r_1^2 &= \varpi^2 + h^2 - 2\varpi h \cos(\theta - \Omega t) + (z+f)^2, \\ r_2^2 &= \varpi^2 + h^2 - 2\varpi h \cos(\theta - \Omega t) + (z-f)^2, \\ \varpi_0^2 &= \varpi^2 + h^2 - 2\varpi h \cos(\theta - \Omega\tau). \end{aligned}\right\} \tag{3}$$

For the relative steady state which is ultimately established we require the limiting form of (2) as $t \to \infty$. We substitute in (2)

$$J_0(\kappa\varpi_0) = J_0(\kappa\varpi)\, J_0(\kappa h) + 2\sum_1^\infty J_n(\kappa\varpi)\, J_n(\kappa h) \cos n(\theta - \Omega\tau). \tag{4}$$

We then integrate with respect to τ term by term and obtain the limiting form of the resulting integrals in κ as $t\to\infty$. This process readily gives the result

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} + 4m \sum_1^\infty P \int_0^\infty \frac{\kappa\, e^{-\kappa(f-z)}}{\kappa - n^2\Omega^2/g} J_n(\kappa\varpi) J_n(\kappa h) \cos n(\theta - \Omega t)\, d\kappa$$
$$- \frac{4\pi m\Omega^2}{g} \sum_1^\infty n^2 J_n(n^2\Omega^2\varpi/g) J_n(n^2\Omega^2 h/g) \exp[-n^2\Omega^2(f-z)/g] \sin n(\theta - \Omega t), \quad (5)$$

where P denotes the principal value of the integral. It may be verified directly that this solution satisfies the conditions for the quasi-steady state.

3. If a sphere, of radius a, is moving uniformly in a circle we may, as a first approximation, take it as equivalent to a doublet of moment M equal to $\frac{1}{2}a^3h\Omega$. It is easily seen that the velocity potential for this doublet can be derived from (5) by taking $\partial\phi/\partial t$ and replacing $mh\Omega$ by M. Thus we obtain

$$\phi = \frac{M\varpi \sin(\theta - \Omega t)}{r_1^3} - \frac{M\varpi \sin(\theta - \Omega t)}{r_2^3}$$
$$+ \frac{4M}{h} \sum_1^\infty P \int_0^\infty \frac{\kappa\, e^{-\kappa(f-z)}}{\kappa - n^2\Omega^2/g} nJ_n(\kappa\varpi) J_n(\kappa h) \sin n(\theta - \Omega t)\, d\kappa$$
$$+ \frac{4\pi M\Omega^2}{gh} \sum_1^\infty n^3 J_n(n^2\Omega^2\varpi/g) J_n(n^2\Omega^2 h/g) \exp[-n^2\Omega^2(f-z)/g] \cos n(\theta - \Omega t). \quad (6)$$

It may be noted that the second term in (6) is equivalent to

$$\frac{2M\varpi \sin(\theta - \Omega t)}{r_2^3} + \frac{4M\Omega^2}{gh} \sum_1^\infty P \int_0^\infty \frac{e^{-\kappa(f-z)}}{\kappa - n^2\Omega^2/g} n^3 J_n(\kappa\varpi) J_n(\kappa h) \sin n(\theta - \Omega t)\, d\kappa. \quad (7)$$

We may deduce the wave resistance from the energy propagated outwards through a cylindrical surface, namely,

$$-\int_{-\infty}^0 dz \int_0^{2\pi} \rho \frac{\partial\phi}{\partial t} \frac{\partial\phi}{\partial\varpi} \varpi\, d\theta. \quad (8)$$

Taking the cylinder of large radius, we require the first terms in the expansion of (6), which are seen to be of order $\varpi^{-\frac{1}{2}}$. One such term comes from the integral in (7). Referring to (4), since we are concerned with large values of ϖ, we may replace $J_n(\kappa\varpi)$ in the expansion by $H_n^{(1)}(\kappa\varpi)$ and take the real part. Thus we have to evaluate the real part of

$$P \int_0^\infty \frac{e^{-\kappa(f-z)}}{\kappa - \kappa_n} H_n^{(1)}(\kappa\varpi) J_n(\kappa h)\, d\kappa \quad (\varpi > h;\ z < 0), \quad (9)$$

where we have put $\kappa_n = n^2\Omega^2/g$. Regarding κ as a complex variable, we may change the path of integration to the positive half of the imaginary axis; taking account of the indentation at $\kappa = \kappa_n$, we obtain for (9)

$$\frac{2}{\pi} \int_0^\infty \frac{m \sin m(f-z) + \kappa_n \cos m(f-z)}{m^2 + \kappa_n^2} K_n(\varpi m) I_n(hm)\, dm$$
$$- \pi Y_n(\kappa_n\varpi) J_n(\kappa_n h) \exp[-\kappa_n(f-z)]. \quad (10)$$

Collecting the results from (6), (7) and (10), and using the asymptotic expansions for J_n and Y_n, we obtain, for ϖ large,

$$\phi \sim \frac{4M\Omega}{h} \left(\frac{2\pi}{g\varpi}\right)^{\frac{1}{2}} \sum_1^\infty n^2 J_n(\kappa_n h) \exp[-\kappa_n(f-z)] \cos\{n(\theta - \Omega t - \tfrac{1}{2}\pi) + \kappa_n\varpi - \tfrac{1}{4}\pi\}. \quad (11)$$

With this value of ϕ in (8) we obtain the rate of propagation of energy outwards, and this must be equal to $Rh\Omega$ with R the steady wave resistance. Finally, replacing M by $\frac{1}{2}a^3h$, we obtain for the wave resistance of the sphere

$$R = \frac{4\pi^2\rho a^6\Omega^4}{gh}\sum_1^\infty n^5 J_n^2(n^2\Omega^2 h/g)\exp[-2n^2\Omega^2 f/g]. \tag{12}$$

It is of interest to examine the limiting form of this expression as $h \to \infty$, $\Omega \to 0$, $h\Omega \to c$. The series then becomes an integral, and by using appropriate asymptotic expansions for Bessel functions of large order and large argument, it is found that (12) reduces to

$$R = 4\pi\rho a^6\kappa_0^4 c^2\int_0^{\frac{1}{2}\pi}\sec^5\beta\exp[-2\kappa_0 f\sec^2\beta]\,d\beta, \tag{13}$$

with $\kappa_0 = g/c^2$, and this is the wave resistance for a sphere in steady rectilinear motion with velocity c.

4. Returning to the general expression (6) we may evaluate the resultant fluid pressure on the sphere and so obtain both the radial force and the tangential force or wave resistance. The effective part of the pressure comes from $\rho\partial\phi/\partial t$, and we notice from (6) that the terms divide into two groups, (i) those symmetrical in the angle $\theta - \Omega t$, (ii) those anti-symmetrical in that angle. Obviously the resultant radial force on the sphere comes from the terms in group (i), while the tangential force is due to those in group (ii). It is necessary to note that, in using this method, the expression for the velocity potential must be carried to a further degree of approximation, because the boundary condition at the surface of the sphere must be satisfied to the same stage. Let (r, α, β) be spherical polar co-ordinates referred to the centre of the sphere so that

$$\left.\begin{aligned} \varpi\cos(\theta-\Omega t) &= h + r\sin\alpha\cos\beta,\\ \varpi\sin(\theta-\Omega t) &= r\sin\alpha\sin\beta,\\ z &= -f + r\cos\alpha. \end{aligned}\right\} \tag{14}$$

The first term in (6) is the doublet D giving the correct normal velocity at the surface of the sphere. The remaining terms in (6) may be expanded in the neighbourhood of the sphere in spherical harmonics so that we have (6) in the form

$$\phi = D + \sum_1^\infty (r/a)^n S_n(\alpha,\beta). \tag{15}$$

The required extension is then

$$\phi = D + \sum_1^\infty\left\{\left(\frac{r}{a}\right)^n + \frac{n}{n+1}\left(\frac{a}{r}\right)^{n+1}\right\}S_n(\alpha,\beta). \tag{16}$$

Taking the tangential resultant force, the effective terms in $\rho\partial\phi/\partial t$ from (6) are

$$(2\pi\rho a^3\Omega^2/g)\sum_1^\infty n^4 J_n(\kappa_n\varpi)J_n(\kappa_n h)\exp[-\kappa_n(f-z)]\sin n(\theta-\Omega t). \tag{17}$$

In this, we put

$$\begin{aligned} &J_n(\kappa_n\varpi)\exp[-\kappa_n(f-z)]\sin n(\theta-\Omega t)\\ &= \frac{(-i)^n}{2\pi}\exp[-\kappa_n(f-z)]\int_{-\pi}^{\pi}\exp[i\kappa_n\varpi\cos(\theta-\Omega t-u)]\sin nu\,du\\ &= \frac{(-i)^n}{2\pi}\exp[-2\kappa_n f]\int_{-\pi}^{\pi}\exp[i\kappa_n r(\sin\alpha\cos\beta\cos u + \sin\alpha\sin\beta\sin u - i\cos\alpha)\\ &\qquad - i\kappa_n h\cos u]\sin nu\,du. \end{aligned} \tag{18}$$

Expanding under the integral sign, we obtain the required expression in terms of spherical harmonics. Obtaining the resultant force involves multiplying the pressure by a surface harmonic of the first order and integrating over the sphere. Thus we only need the first-order term from (18), which is

$$\frac{(-i)^n}{2\pi} i\kappa_n a \exp[-2\kappa_n f] \int_{-\pi}^{\pi} (\sin\alpha\cos\beta\cos u + \sin\alpha\sin\beta\sin u - i\cos\alpha) \exp[-i\kappa_n h\cos u]\sin nu\,du. \quad (19)$$

In accordance with (16), this must be multiplied by $\frac{3}{2}$ to get the correct operative value of the pressure. We insert these results in (17), multiply by $\sin\alpha\sin\beta$ and integrate over the surface of the sphere. It is easily verified that this process gives the same expression (12) for the tangential resistance.

5. For the resultant radial force outwards, we carry out the same process on the pressure derived from the first three terms of (6), noting that in accordance with (16), the first-order surface harmonic from the second and third terms of (6) must be multiplied by $\frac{3}{2}$. We then multiply by $\sin\alpha\cos\beta$ and integrate over the surface of the sphere. The details of the calculation need not be given; after some reduction, we find for the resultant radial force outwards the expression

$$\tfrac{2}{3}\pi\rho a^3 h\Omega^2\left(1-\frac{3}{16}\frac{a^3}{f^3}\right) + 4\pi\rho a^6\Omega^2 \sum_1^\infty P\int_0^\infty \frac{\kappa^2 e^{-2\kappa f}}{\kappa - n^2\Omega^2/g} n^2 J_n(\kappa h) J_n'(\kappa h)\,d\kappa. \quad (20)$$

This expression does not lend itself readily to numerical computation. We notice, however, that the first term in (20) represents an effective mass $\frac{2}{3}\pi\rho a^3(1-3a^3/16f^3)$, which is the first approximation for a sphere under a free surface, neglecting gravity. On the other hand, when the angular velocity is small the last term in (20) approximates to $\frac{1}{4}\pi\rho a^6\Omega^2/f^3$, since $\Sigma n^2 J_n(\kappa h) J_n'(\kappa h) = \frac{1}{4}\kappa h$. Thus for small velocity, the effective mass approximates to $\frac{2}{3}\pi\rho a^3(1+3a^3/16f^3)$, as for a sphere under a rigid surface.

It is of some interest to make calculations from (12), so as to obtain some idea of the nature and magnitude of the effect of curvature of the path upon the wave resistance. Curves showing the results are given in figure 1. The abscissae are values of $h\Omega/\sqrt{(gf)}$, so as to include rectilinear motion for comparison; the ordinates are values of $R/M'g(a/f)^3$, where M' is the mass displaced by the sphere. Curve A is for steady rectilinear motion, that is, for the limiting case $h/f\to\infty$, and calculated from (13). Curve B is for $h=f$. Even in this case the mean curve approximates to A, but it is of interest to note the hump and hollows due to wave interference when the sphere is making complete circles. For curve C we have taken $h=4f$; it shows how with increasing radius of the circular path these interference effects disappear and the wave resistance approximates quite closely to that for straight-line motion at the same linear speed.

6. We consider now a prolate spheroid with its axis at a constant depth f below the surface, its centre C describing a horizontal circle of radius h with constant angular velocity Ω, the axis of the spheroid remaining at right angles to the rotating radius through C. We use the same fixed axes as before, with cylindrical co-ordinates $O(\varpi,\theta,z)$; and, when required, we use rotating axes $C(x,y,z)$ with Cx along the axis of the spheroid in the direction of motion and Cz vertically upwards.

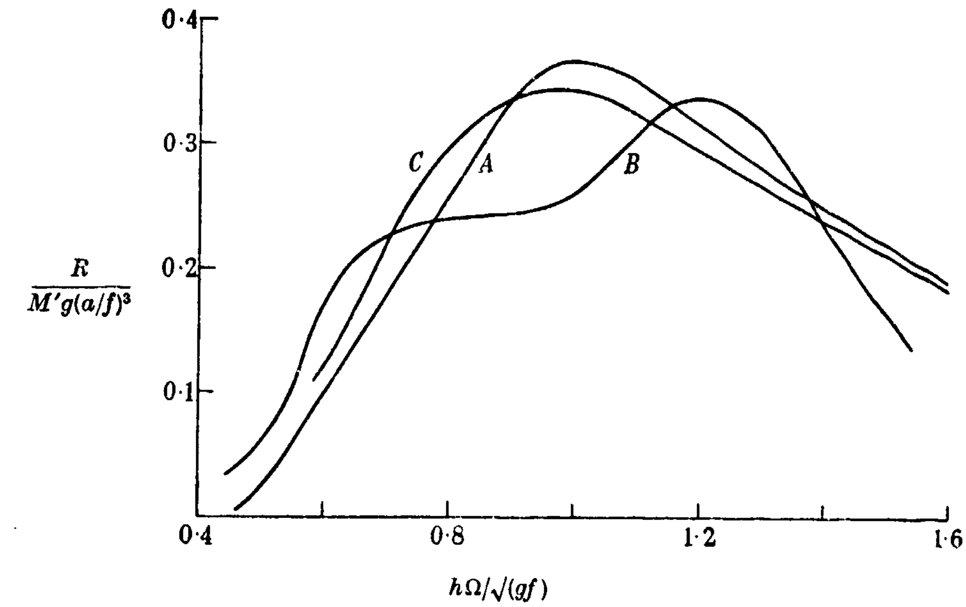

FIGURE 1

The motion of the spheroid is made up of a linear velocity $h\Omega$ parallel to Cx and a rotation Ω about Cz. In terms of spheroidal co-ordinates given by

$$x = ae\mu\zeta, \quad y = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega, \quad z = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega, \tag{21}$$

the known solution for this motion in an infinite liquid is (Lamb 1932)

$$\phi = 2Aaeh\Omega A P_1(\mu)\,Q_1(\zeta) - \tfrac{2}{3}Ba^2e^2\Omega P_2^1(\mu)\,Q_2^1(\zeta)\cos\omega, \tag{22}$$

with

$$\left.\begin{aligned} A^{-1} &= 2e/(1-e^2) - \log\{(1+e)/(1-e)\}, \\ B^{-1} &= \{3(2-e^2)/e^2\}\log\{(1+e)/(1-e)\} - 2(6-7e^2)/e(1-e^2). \end{aligned}\right\} \tag{23}$$

It is well known that the linear motion can be expressed in terms of a certain source distribution along the axis of the spheroid, and it can easily be shown that the angular motion can be ascribed to a doublet distribution along the axis. In fact, (22) is equivalent to

$$\phi = Ah\Omega\int_{-ae}^{ae}\frac{k\,dk}{\{(x-k)^2+y^2+z^2\}^{\frac{3}{2}}} + B\Omega y\int_{-ae}^{ae}\frac{k(a^2e^2-k^2)\,dk}{\{(x-k)^2+y^2+z^2\}^{\frac{3}{2}}}. \tag{24}$$

We may now obtain the required solution by integration of the expression for a source given in (6). For the first term in (24) we have to replace a typical factor $J_n(\kappa h)\cos n(\theta-\Omega t)$ in (6) by $J_n\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\cos n(\theta-\Omega t-\alpha)$, where $\tan\alpha = k/h$; and, taking account of the integration in k, this may be replaced by

$$J_n\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin n\alpha\,\sin n(\theta-\Omega t).$$

Further, so far as the co-ordinates x, y, z are concerned, the second term in (24) may be derived by taking $\partial/\partial y$ of the first term; and when the expressions are put in terms of the fixed co-ordinates this is equivalent to operating by $\partial/\partial h$. Also we have

$$\frac{\partial}{\partial h}[J_n\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin n\alpha] = \tfrac{1}{2}\kappa[J_{n-1}\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin(n-1)\alpha - J_{n+1}\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin(n+1)\alpha]. \tag{25}$$

Carrying out these operations we obtain for the rotating spheroid at depth f, its centre describing a circle of radius h,

$$\phi = \int_{-ae}^{ae} \{Ah\Omega kF + B\Omega k(a^2e^2 - k^2)\,G\}\,dk, \tag{26}$$

with

$$F = \frac{1}{r_1} - \frac{1}{r_2} + \sum_1^\infty P\int_0^\infty \frac{\kappa\,e^{-\kappa(f-z)}}{\kappa-\kappa_n} J_n(\kappa\varpi)\,J_n\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin n\alpha\,\sin n(\theta-\Omega t)\,d\kappa$$
$$+ (4\pi\Omega^2/g)\sum_1^\infty n^2 J_n(\kappa_n\varpi)\,J_n\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\exp[-\kappa_n(f-z)]\sin n\alpha\,\cos n(\theta-\Omega t), \tag{27}$$

$$G = \frac{h-\varpi\cos(\theta-\Omega t)}{r_1^3} - \frac{h-\varpi\cos(\theta-\Omega t)}{r_2^3}$$
$$+\sum_1^\infty P\int_0^\infty \frac{\kappa^2\,e^{-\kappa(f-z)}}{\kappa-\kappa_n} J_n(\kappa\varpi)\,[J_{n+1}\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin(n+1)\alpha$$
$$-J_{n-1}\{\kappa(h^2+k^2)^{\frac{1}{2}}\}\sin(n-1)\alpha]\sin n(\theta-\Omega t)\,d\kappa$$
$$+\frac{2\pi\Omega^4}{g^2}\sum_1^\infty n^4 J_n(\kappa_n\varpi)\,[J_{n+1}\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin(n+1)\alpha$$
$$-J_{n-1}\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin(n-1)\alpha]\exp[-\kappa_n(f-z)]\cos n(\theta-\Omega t). \tag{28}$$

In this $\kappa_n = n^2\Omega^2/g$, $\tan\alpha = k/h$, and

$$\left.\begin{aligned} r_1^2 &= \{\varpi\sin(\theta-\Omega t)-k\}^2 + \{h-\varpi\cos(\theta-\Omega t)\}^2 + (z+f)^2,\\ r_2^2 &= \{\varpi\sin(\theta-\Omega t)-k\}^2 + \{h-\varpi\cos(\theta-\Omega t)\}^2 + (z-f)^2.\end{aligned}\right\} \tag{29}$$

By comparison with § 3, we see that for ϖ large we have

$$\phi \sim 2h\Omega^2\left(\frac{2\pi}{g\varpi}\right)^{\frac{1}{2}}\sum_1^\infty\{2nAL_n + n^3B(\Omega^2/gh)\,M_n\}$$
$$\exp[-\kappa_n(f-z)]\cos\{n(\theta-\Omega t-\tfrac{1}{2}\pi)+\kappa_n\varpi-\tfrac{1}{4}\pi\}, \tag{30}$$

with

$$\left.\begin{aligned} L_n &= \int_{-ae}^{ae} kJ_n\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin n\alpha\,dk,\\ M_n &= \int_{-ae}^{ae} k(a^2e^2-k^2)\,[J_{n+1}\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin(n+1)\alpha\\ &\qquad -J_{n-1}\{\kappa_n(h^2+k^2)^{\frac{1}{2}}\}\sin(n-1)\alpha]\,dk.\end{aligned}\right\} \tag{31}$$

Using this in (8) we obtain the rate of propagation of energy outwards; if R is the tangential resistance and G the couple required to maintain the uniform motion, this leads to

$$Rh\Omega + G\Omega = \frac{4\pi^2\rho h^2\Omega^5}{g}\sum_1^\infty n^3\left(2AL_n + B\frac{\Omega^2}{gh}n^2M_n\right)^2\exp[-2n^2\Omega^2f/g]. \tag{32}$$

7. We may obtain the resistance, the couple and the radial force by calculating the resultant fluid pressure on the spheroid. For the wave resistance the only part of the pressure which gives a resultant is the term $\rho\,\partial\phi/\partial t$, and we have

$$R = \iint pl\,dS = \rho a^2(1-e^2)\int_{-1}^{1}\int_0^{2\pi}\frac{\partial\phi}{\partial t}\,\mu\,d\mu\,d\omega, \tag{33}$$

in terms of the co-ordinates (21), with $\zeta = \zeta_0 = 1/e$ on the spheroid. Taking from (26) the terms which contribute to the resultant tangential force, we have

$$\frac{\partial \phi}{\partial t} = \int_{-ae}^{ae} \{A h \Omega k F_1 + B \Omega k (a^2 e^2 - k^2) G_1\} dk, \tag{34}$$

with

$$F_1 = \frac{4\pi\Omega^3}{g} \sum_1^{\infty} n^3 J_n(\kappa_n \varpi) J_n\{\kappa_n (h^2 + k^2)^{\frac{1}{2}}\} \exp[-\kappa_n (f-z)] \sin n\alpha \sin n(\theta - \Omega t), \tag{35}$$

and a similar expression for G_1, derived from (28). If (34) is expanded in the form

$$\frac{\partial \phi}{\partial t} = \sum_{r=1}^{\infty} \sum_{s=0}^{r} (A_r^s \cos s\omega + B_r^s \sin s\omega) P_r^s(\mu) P_r^s(\zeta), \tag{36}$$

we must add a similar expression with $Q_r^s(\zeta)$ in place of $P_r^s(\zeta)$ so as to maintain the boundary condition at the surface of the spheroid; for this part of ϕ this is $\partial\phi/\partial\zeta = 0$ for $\zeta = \zeta_0$. Hence on the spheroid we have

$$\frac{\partial \phi}{\partial t} = \sum_{r=1}^{\infty} \sum_{s=0}^{r} C_r^s (A_r^s \cos s\omega + B_r^s \sin s\omega) P_r^s(\zeta_0) P_r^s(\mu),$$

with
$$C_r^s = 1 - P_r^{s\prime}(\zeta_0) Q_r^s(\zeta_0) / P_r^s(\zeta_0) Q_r^{s\prime}(\zeta_0). \tag{37}$$

We now expand (34) in the form (36), noting that for the value of R we only require the term in $P_1^0(\mu)$. For this purpose we have

$$J_n(\kappa_n \varpi) \exp[\kappa_n(f-z)] \sin n(\theta - \Omega t)$$
$$= \frac{(-i)^n}{2\pi} \exp[-2\kappa_n f + \kappa_n z] \int_{-\pi}^{\pi} \exp[i\kappa_n(x \sin u - y \cos u) + i\kappa_n h \cos u] \sin nu\, du, \tag{38}$$

with the origin now at the centre of the spheroid. Substituting from (21), we multiply by $\mu\, d\mu\, d\omega$ and integrate over the surface of the spheroid. It can be shown that

$$\int_{-1}^{1} \mu\, d\mu \int_0^{2\pi} \exp[i\kappa_n\{a\mu \sin u - b(1-\mu^2)^{\frac{1}{2}} (\cos u \cos\omega + i \sin\omega)\}]\, d\omega$$
$$= 4\pi i (\pi/2\kappa_n a e^3)^{\frac{1}{2}} \sin^{-\frac{1}{2}} u J_{\frac{3}{2}}(\kappa_n a e \sin u). \tag{39}$$

Using (39) in (38), we have, so far as this typical term is concerned, the integral

$$4\pi i (\pi/2\kappa_n a e^3)^{\frac{1}{2}} \int_{-\pi}^{\pi} \exp[i\kappa_n h \cos u] J_{\frac{3}{2}}(\kappa_n a e \sin u) \sin^{-\frac{1}{2}} u \sin nu\, du. \tag{40}$$

It is of interest to find that (40) can be put in the form $(4\pi^3/a^2e^3)^{\frac{1}{2}} i^n L_n$, in the notation of (31). Collecting these results and including the factor C_1^0 from (37), we obtain for the wave resistance

$$R = \frac{8\pi^2 \rho h \Omega^4}{g} A \sum_1^{\infty} n^3 L_n \left(2AL_n + B\frac{\Omega^2}{gh} n^2 M_n\right) \exp[-2n^2\Omega^2 f/g]. \tag{41}$$

We could obtain the couple G by similar calculations; or, using (32), we have

$$G = \frac{4\pi^2 \rho h \Omega^6}{g^2} B \sum_1^{\infty} n^5 M_n \left(2AL_n + B\frac{\Omega^2}{gh} n^2 M_n\right) \exp[-2n^2\Omega^2 f/g]. \tag{42}$$

The radial force can be obtained by the same method from the remaining terms in the velocity potential; but the expressions are lengthy and not suitable for numerical computation.

If we take limiting values as $h \to \infty$, $\Omega \to 0$, $h\Omega \to c$, the couple G becomes zero and R reduces to the wave resistance given previously (Havelock 1931) for the linear motion of a spheroid. On the other hand, if we take $h = 0$, we find that $L_n = 0$, $M_{2n+1} = 0$, and we obtain the couple for pure rotation as

$$G = \frac{512\pi^2\rho a^6 e^6}{g}\,\Omega^4 B^2 \sum_1^\infty n^5 P_{2n}^2 \exp\left[-8n^2\Omega^2 f/g\right],$$

with
$$P_{2n} = \int_0^1 (1-u^2)\,J_{2n}(4n^2\Omega^2 aeu/g)\,du. \tag{43}$$

8. For numerical computation the integrals for L, M can be expanded in various forms; for instance, one which proved useful can be derived from the expansion

$$J_n\{p(1+u^2)^{\frac{1}{2}}\}\sin(n\tan^{-1}u) = (J_{n-1}+J_{n+1})(\tfrac{1}{2}pu) - (J_{n-3}+3J_{n-1}+3J_{n+1}+J_{n+3})\frac{(\frac{1}{2}pu)^3}{3!}+\dots, \tag{44}$$

the Bessel functions having the argument p. For some values of the parameters it was found more convenient to evaluate the integrals by direct quadrature.

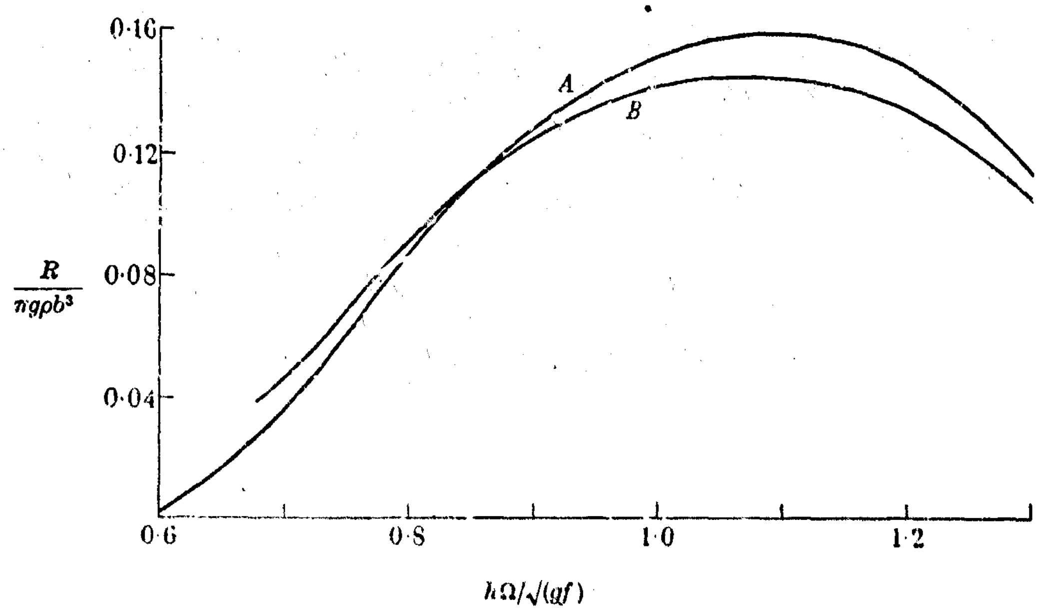

FIGURE 2

As a particular case we take a spheroid for which $2a = 5b$, so that $e = 0{\cdot}9165$; and for the depth we take $f = 2b$. This was one of the cases for which calculations were made previously for rectilinear motion. To bring out the effect of curvature we take for the radius of the path $h = 5b$. The results for the wave resistance are shown in figure 2. The ordinates are values of $R/\pi g\rho b^3$, and the abscissae are $h\Omega/\sqrt{(gf)}$. The curve A is for linear motion and is taken from the paper already quoted (Havelock

1931). Curve B shows the effect of circular motion in this particular case; the difference between the two curves is quite small even in this rather extreme case. A curve is not given for the couple G, as the quantities M are rather difficult to evaluate with sufficient accuracy for this purpose; however, approximate computations were made and the maximum value appears to be at about $h\Omega/\sqrt{(gf)} = 1{\cdot}25$ with a value of $G/\pi g\rho b^4$ of about 0·026. It might be expected that the couple would be small for a solid of revolution in this particular case; it would probably be larger for a flat ellipsoid, for which similar calculations could be made by the methods used in the present work, the appropriate source and doublet distributions being then over the plane area enclosed by the elliptic focal conic.

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Wave Resistance Theory and Its Application to Ship Problems[1]

By T. H. Havelock, Visitor[2]

It is now just over fifty years since the first mathematical analysis was made of the wave resistance of a ship form, and during the latter half of that period there has been a considerable output of work, both theoretical and experimental. It is impossible to give any adequate survey of this work here, and fortunately it is unnecessary to make the attempt; there are excellent summaries which have been published from time to time, and in particular I would refer, for a comprehensive account with references, to Wigley's recent paper, "The Present Position of the Calculation of Wave Resistance" (L'Association Technique Maritime et Aeronautique, Paris, 1949).

In the following notes I deal first with a solid body which is completely submerged; a short descriptive account of one method of developing the mathematical theory is followed by some recent results on motion in a curved path and on accelerated motion. The second section deals with floating bodies, or surface ships. Reference is made to the need for improving the approximate theory for models of fine form and extending its range of application; and a short account is given of some attempts, dealing in particular with (1) models of fuller form, (2) models of non-mathematical form and methods of approximate calculation, (3) the inclusion of the effects of viscosity and the possible interaction between frictional resistance and wave resistance.

Submerged Bodies. Consider a solid body wholly submerged in water and moving in a horizontal line with given velocity. Assuming the water to be frictionless, the fluid motion is specified by a velocity potential ϕ satisfying given boundary conditions: (1) the normal fluid velocity on the solid is equal to the normal velocity of the solid at each point, (2) the pressure is constant at the free surface of the water, (3) for deep water the velocity diminishes to zero with increasing depth. We may also impose a condition for the motion far in advance of the solid, such as, for instance, to insure that in the usual phrase the solid is advancing into still water. In general, this problem has only been attacked by some method of continued approximation. We may suppose that the wave motion at the surface is a relatively small effect, and we take ϕ_0 for the velocity potential as if the solid were moving in an infinite liquid, and satisfying condition (1). We then add a correcting potential ϕ_1 so that $\phi_0 + \phi_1$ satisfies condition (2) at the free surface; and then a potential ϕ_2 to maintain condition (1), and so on. Thus we may picture the solution ϕ as an infinite series $\phi_0 + \phi_1 + \phi_2 + \ldots$. We may assume this process to be convergent; but the expression of it in any particular mathematical form would involve consideration of convergence and of the uniqueness of the solution so obtained. It has only been possible to carry out this process in any detail for solids of simple form, such as a circular cylinder, sphere, or spheroid. In fact, for most cases it has not been carried further than the first three terms; while for bodies of ship-shape form nearly all the results have meantime been built up on the first two terms—denoted here by $\phi_0 + \phi_1$. Assume now that we know the first function ϕ_0, giving the solution if we neglect the wave motion completely, and consider the determination of the next function ϕ_1. There are various methods available; the one I wish to outline may not be the best from a mathematical point of view, but it has some advantages for descriptive purposes. The method is one which was used long ago by Kelvin for the waves produced by a pressure disturbance traveling over the surface of the water. Consider for a moment the classical problem of the traveling pressure point. Instead of treating this directly as a continuous process, we may regard the motion as the limit of

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a succession of small steps, at each step an impulse being applied to the surface of the water. Each impulse starts a series of ring-waves traveling out in all directions; and to get the total effect at any time we have simply to sum up the effects due to all the previous elementary steps, the well-known wave pattern emerging from the mutual interference of these elementary ring-waves. The process can be expressed mathematically to give the complete solution of this problem.

Returning to the submerged solid, we regard the continuous motion as the limit of elementary steps and examine what happens at any given step. We picture the solid as suddenly started from rest with a given velocity and then stopped after a short interval of time. For this impulsive motion ϕ_0 is the potential as if the solid were started from rest in an infinite liquid. But the form of the surface condition for this step is that there shall be no impulse at the free surface and we must add the appropriate function ϕ_1. This may be written down directly as a reflected potential, but we may picture it in this way. Suppose the water continued above the free surface and place in it the image of the given solid. When the solid is moved through its elementary step, we move the image suddenly through an equal small step in the opposite direction. The potential for these two motions in an infinite liquid gives the required approximation $\phi_0 + \phi_1$. We may notice, in passing, that gravity does not come into play during this impulsive motion. We now calculate the vertical velocity of the free surface, and the result of the step from rest to rest is that the free surface is left with a known elevation. The subsequent motion due to this elevation can be worked out, the elevation spreading out in all directions in the form of free gravity waves. Finally, for any continuous motion of the solid we sum up the total effect of all the previous elementary steps in the motion. The process can be set out in mathematical form, and so we obtain the first approximations for the assigned motion; it may be remarked that further approximations are possible by generalizing this process. An interesting point is that this formulation of the problem automatically leads to the so-called practical solution with the solid advancing into still water, and with the main wave pattern to the rear. This result is connected with the fact that for water waves the group velocity is less than the wave velocity; if the contrary had been the case, we should have arrived at a steady state with the solid pushing the wave pattern in advance instead of leaving it to the rear. It will be seen also from this description that this impulse method can be applied equally well to nonuniform motion or to motion of any kind in a curved path.

Although not necessary, it is convenient often to introduce the idea of sources and sinks. The potential ϕ_0 due to the motion of the solid as if in an infinite liquid can be regarded as due to a distribution of sources and sinks, or other singularities, on or within the boundary of the solid and an elementary step in the motion corresponds to establishing this distribution for a short interval of time. Consider in Fig. 1, a point source of strength m established at time τ at the point $(0, 0, -f)$ in the liquid, where we have taken the origin O in the free surface with OZ vertically upwards. During the short interval of time $\delta\tau$ we have the velocity potential

$$\phi = \frac{m}{r_1} - \frac{m}{r_2}$$

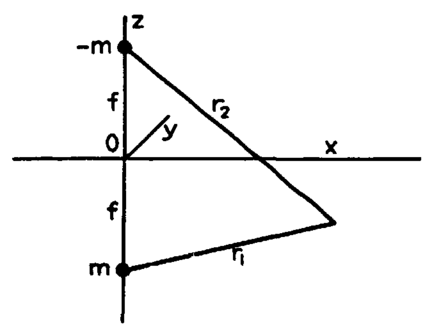


FIG. 1

with

$$r_1^2 = x^2 + y^2 + (z + f)^2; \quad r_2^2 = x^2 + y^2 + (z - f)^2$$

The initial elevation left by the elementary step is

$$\zeta_0 = \frac{2mf\,\delta\tau}{(x^2 + y^2 + f^2)^{3/2}} = \frac{m\delta\tau}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} e^{-kf} \times$$
$$\times \cos[k(x\cos\theta + y\sin\theta)]k\,dk$$

and the motion at any subsequent time t due to this elevation is given by

$$\phi = \frac{mg^{1/2}\delta\tau}{\pi}\int_{-\pi}^{\pi} d\theta \int_0^{\infty} e^{-k(f-z)} \times$$
$$\times \cos[k(x\cos\theta + y\sin\theta)]\sin[g^{1/2}k^{1/2}(t-\tau)]k^{1/2}\,dk$$

In particular, suppose the source starts from rest at time $t = 0$, is of constant magnitude, and moves with uniform velocity c in a horizontal line parallel to OX. The velocity potential at time t is given by

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} + \frac{mg^{1/2}}{\pi}\int_0^t d\tau \int_{-\pi}^{\pi} d\theta \int_0^{\infty} e^{-k(f-z)} \times$$
$$\cos[k\{(x + ct - c\tau)\cos\theta + y\sin\theta\}] \times$$
$$\sin[g^{1/2}k^{1/2}(t-\tau)]k^{1/2}\,dk$$

The limiting form as $t \to \infty$ gives the steady state which is ultimately established for uniform motion in a straight line; namely,

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} - \frac{4gm}{c^2} \int_0^{\pi/2} e^{-k_0(f-z)\sec^2\theta} \sin(k_0 \sec\theta) \times$$
$$\times \cos(k_0 y \sin\theta \sec^2\theta) \sec^2\theta \, d\theta - \frac{4gm}{\pi c^2}$$
$$P \int_0^{\pi/2} \sec^2\theta d\theta \int_0^{\infty} \frac{e^{-k(f-z)} \cos(kx\cos\theta)\cos(ky\sin\theta)}{k - k_0 \sec^2\theta} dk$$

where $k_0 = g/c^2$, and the origin O is now a moving origin vertically over the source.

Calculating the surface elevation from this expression, it is found that the wave pattern at a great distance to the rear approximates to the form

$$\zeta \to \frac{4k_0 m}{c} \int_{-\pi/2}^{\pi/2} e^{-k_0 f \sec^2\theta}$$
$$\cos\left[k_0 \sec^2\theta \, (x\cos\theta + y\sin\theta)\right] \sec^3\theta \, d\theta$$

From these results for a single source we can derive expressions for other singularities, or for any distribution of sources and sinks. Knowing the wave pattern at a great distance to the rear, we can, from energy considerations, write down the corresponding wave resistance of the solid body which is represented by the given distribution. It may be remarked that the forces and moments on the submerged body can be calculated as the resultant of the fluid pressures on its surface, but in that case the approximation must be carried to the next stage, that is, to the stage $\phi_0 + \phi_1 + \phi_2$ in the notation used here; this is necessary in order to satisfy the condition at the surface of the solid to the required degree of approximation and it is a point which has sometimes been overlooked.

We leave this brief description of fundamental theory with the remark that nearly all the work on such problems has been limited to uniform motion in a straight line. More recently, Sretensky has given some formulae for accelerated motion; and Brard has examined the motion of a source in a straight line, the strength of the source being subject to periodic variation, with a view to applying the results to the interesting problem of the pitching of a ship under way.

Using the integration method outlined in the foregoing, I have worked out the case of a sphere moving with uniform velocity in a circular path at constant depth below the surface. If a is the radius of the sphere, h the radius of the circular path, f the depth of the center of the sphere, and c the linear velocity in the path, the wave resistance is given by

$$R = \frac{4\pi^3 \rho a^6 c^4}{g h^6} \sum_{n=1}^{\infty} J_n^2\left(\frac{n^2 c^2}{gh}\right) e^{-(2n^2c^2f/gh^2)}$$

J_n denoting the Bessel function.

If we make h tend to infinity, keeping c constant, this reduces to the known result for a sphere in linear motion with uniform velocity c, namely

$$R_0 = 4\pi\rho a^6 k_0^4 c^2 \int_0^{\pi/2} e^{-2k_0 f \sec^2\beta} \sec^5\beta \, d\beta$$

These expressions can be evaluated numerically, and Fig. 2 shows some results.

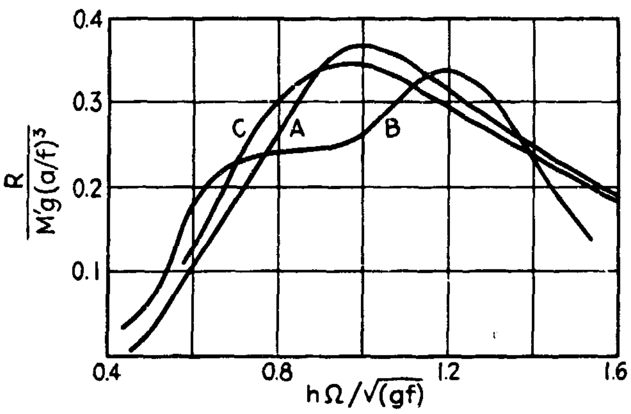


FIG. 2

Curve A is the resistance-velocity curve for linear motion with constant velocity. Curve B is for circular motion with $h = f$; we notice here the humps and hollows due to the motion of the sphere in the waves produced by previous revolutions in the path. For curve C, $h = 4f$ and we see how quickly these effects diminish with increasing radius of the path and the resistance approximates quite closely to that for linear motion at the same speed. A more interesting case is that of a prolate spheroid whose center is describing a circular path. The motion of the spheroid involves both translation in the direction of the axis at each instant and rotation about a vertical axis; the analysis is rather complicated but expressions were obtained for the wave resistance, the radial force outwards and the couple on the spheroid.

Fig. 3 shows calculations of the wave resistance for a spheroid whose length is 2½ times the maximum breadth, the radius of the path being equal to the length of the spheroid. Curve A is for linear motion and curve B for motion in this circular path; even in this extreme case the wave resistance is not much affected by the curvature of the path. These problems are, no doubt, chiefly of academic interest in themselves; but the development of such work may have a bearing on questions of great practical interest in the theory of steering, stability and so forth.

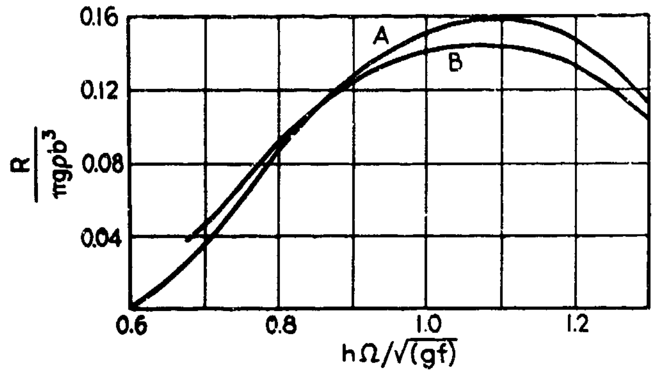


FIG. 3

Another matter of interest is the question of accelerated motion and effective mass or effective moment of inertia in such cases. There has been some discussion, for instance, about the suitable condition to take at the free surface of the water for an approximate estimate of effective inertia in ship problems; we may, on the one hand, neglect the wave motion completely and take the surface to be rigid, or, on the other hand, we may neglect gravity and treat it as a free surface. Very little work has been done in this field and it seemed worth while to attempt a more detailed examination of some simple case which could be carried far enough to allow of numerical calculation. I have worked out the problem to a certain stage for a circular cylinder moving with constant acceleration and starting from rest; some of the results are shown in Fig. 4.

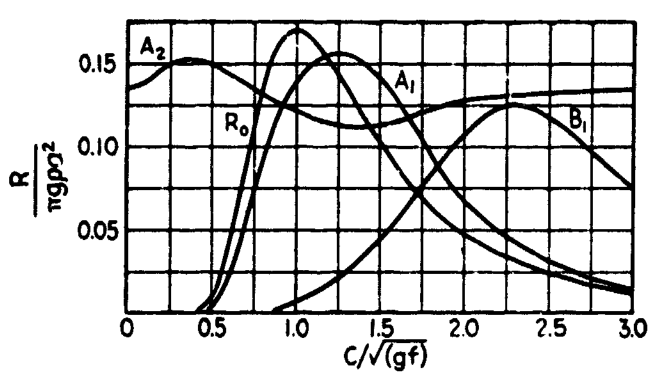


FIG. 4

The abscissae give the velocity acquired from rest with the given acceleration, and the total resistance at each speed has been divided into two parts. For the first case, the acceleration is $g/7$ and the two parts of the resistance are shown by the curves A_1 and A_2. From the way in which the two parts emerge from the calculations, it is convenient to call A_1 the wave resistance, and for comparison the curve R_0 shows the wave resistance for uniform motion at each speed. At uniform speed the ordinates of A_2 would, of course, be zero; in accelerated motion it is appropriate to call A_2 the inertia part of the resistance, although, as the curve shows, it depends upon the velocity as well as upon the acceleration. If we had used the approximations of treating the surface of the water (1) as rigid, or (2) as free but neglecting gravity, the part A_1 would be zero and A_2 a straight line of constant ordinate; for (1) the ordinate would be 0.15 on the diagram, and for (2) it would be 0.135. It is interesting to note how, in fact, A_2 varies between these extremes as the velocity increases. The curve B_1 is the wave resistance part for a greater acceleration; namely, $1.276g$. The corresponding curve B_2 is similar in character to A_2 but is not shown on the diagram as its ordinates would be nine times those of A_2.

These results are obviously not of much value for direct application; but they may serve to show the need for further work in a region which has been somewhat neglected, in which there are problems which could be studied both theoretically and experimentally with a view to practical applications.

Floating Bodies of Ship Form. If the solid is only partially immersed in the water we have a much more difficult problem, even when we assume the water to be frictionless. In the usual theory of wave motion we neglect the square of the fluid velocity. Further, except in special circumstances, the first two or three terms of an approximation similar to that for a submerged solid may be inadequate.

Then there are also complications arising from the intersection of the solid and the water, with the different conditions over the two surfaces; and, in general, any solution which has been obtained involves a mathematical infinity in the vertical component of fluid velocity at the bow and stern. Meantime most of the work on ship forms has been limited to cases of small ratio of beam to length where these difficulties may be neglected in the first place, and further approximations made later to improve the theory. However, a more direct approach is much to be desired, so as to give an adequate theory of wave resistance for a floating solid. In particular, a detailed study of simple forms would be valuable, for instance, a vertical circular cylinder, or a sphere or spheroid half immersed in water. One may apply to such problems a remark made by Kelvin in regard to the motion of a wholly submerged circular cylinder, which was solved some years later by Lamb; after suggesting the problem he left it with the remark, "it is a mathemati-

cal problem which presents interesting difficulties, worthy of serious work for anyone who may care to undertake it."

It may be added that such work would give a better idea of what has been neglected in the present approximate theory, and might lead to a fresh approach to the problem of the ship with more usual values of the ratio of beam to length.

The approximate solution for a slender ship form was given more than fifty years ago by Michell in a classical paper, which unfortunately was overlooked and forgotten for many years. Michell's approach was different from that outlined in the previous section. He considered a semi-infinite uniform stream of water with a free upper surface and bounded by a vertical plane parallel to the stream; and he solved the problem of the motion due to a given distribution of normal velocity over this vertical plane. A ship of narrow beam placed in the stream was pictured as producing a normal velocity outwards on both sides of amount given approximately by the product of the stream velocity and the horizontal gradient of the level lines of the form; finally, this was treated as a given distribution of horizontal velocity outwards on the two sides of the longitudinal vertical section of the ship. Such a discontinuity of normal velocity is equivalent, of course, to a corresponding distribution of sources and sinks over this vertical plane; and so we arrive at Michell's results as a particular case of the source distributions we have considered in the previous section. In particular, we may quote for reference the well-known resistance integrals. With one-half of the submerged form given by $y = f(x, z)$ we have

$$R = \frac{4k_0 g\rho}{\pi} \int_0^{\pi/2} (I^2 + J^2) \sec^3 \theta \, d\theta; \quad k_0 = \frac{g}{c^2}$$

with

$$I + iJ = \iint \left[\frac{\partial f}{\partial x}\right] e^{-k_0 z \sec^2 \theta + i k_0 x \sec \theta} \, dx \, dz$$

taken over the longitudinal vertical section of the ship.

Although, as might be expected, this formula does not enable us to predict with certainty the resistance of a given model at a given speed, it proved to be near enough to the general run of the resistance-velocity curve to give much interesting qualitative information: in particular, in the changes produced by small variations in the form of the model and the general explanation of such changes.

Fig. 5 shows the resistance curve A for the simple parabolic model given by $y = b(1 - x^2/l^2)(1 - z^2/d^2)$, for the case with the draft one-sixteenth of the length, showing the humps and hollows which are so much exaggerated at low speeds compared with experimental curves. It is interesting to recall that Kelvin ended his lecture on "Ship Waves" (1887), in which he first described the ship-wave pattern, by making "with some diffidence" a practical suggestion. It was to the effect that since wave disturbance is so much a surface effect, it might be an advantage to put as much displacement as possible below the waterline, assuming no doubt that one would not then increase other resistance by a greater amount. It is, of course, well known that the form of the lower level lines has comparatively little influence on the wave resistance as compared with the form of the level lines near the LWL; and one can see this confirmed by work on pressure distribution round the hull, such as that of Eggert. This point may be illustrated quite simply in Fig. 5 by inverting the parabolic model and putting the keel in the surface; thus the equation of the form is now

$$y = b(1 - x^2/l^2)(2z/d - z^2/d^2)$$

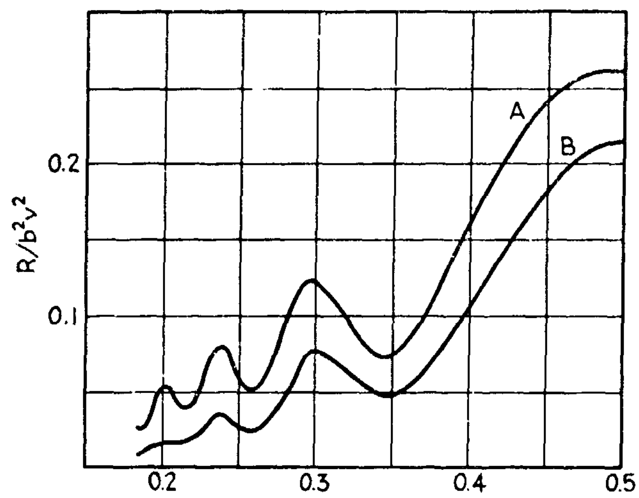


Fig. 5

Curve B shows the result. The operative factor is the ratio of draft to wave length at each speed. As one would expect, the wave resistance in the second curve is negligible at low speeds, but ultimately would rise to equality with curve A; the difference is rather striking even when the wave length is several times the draft.

I wish to refer now to some attempts which have been made to improve the theory and to extend its range of application. It may be remarked that the Michell resistance integrals can be applied to a much greater range of forms than was at

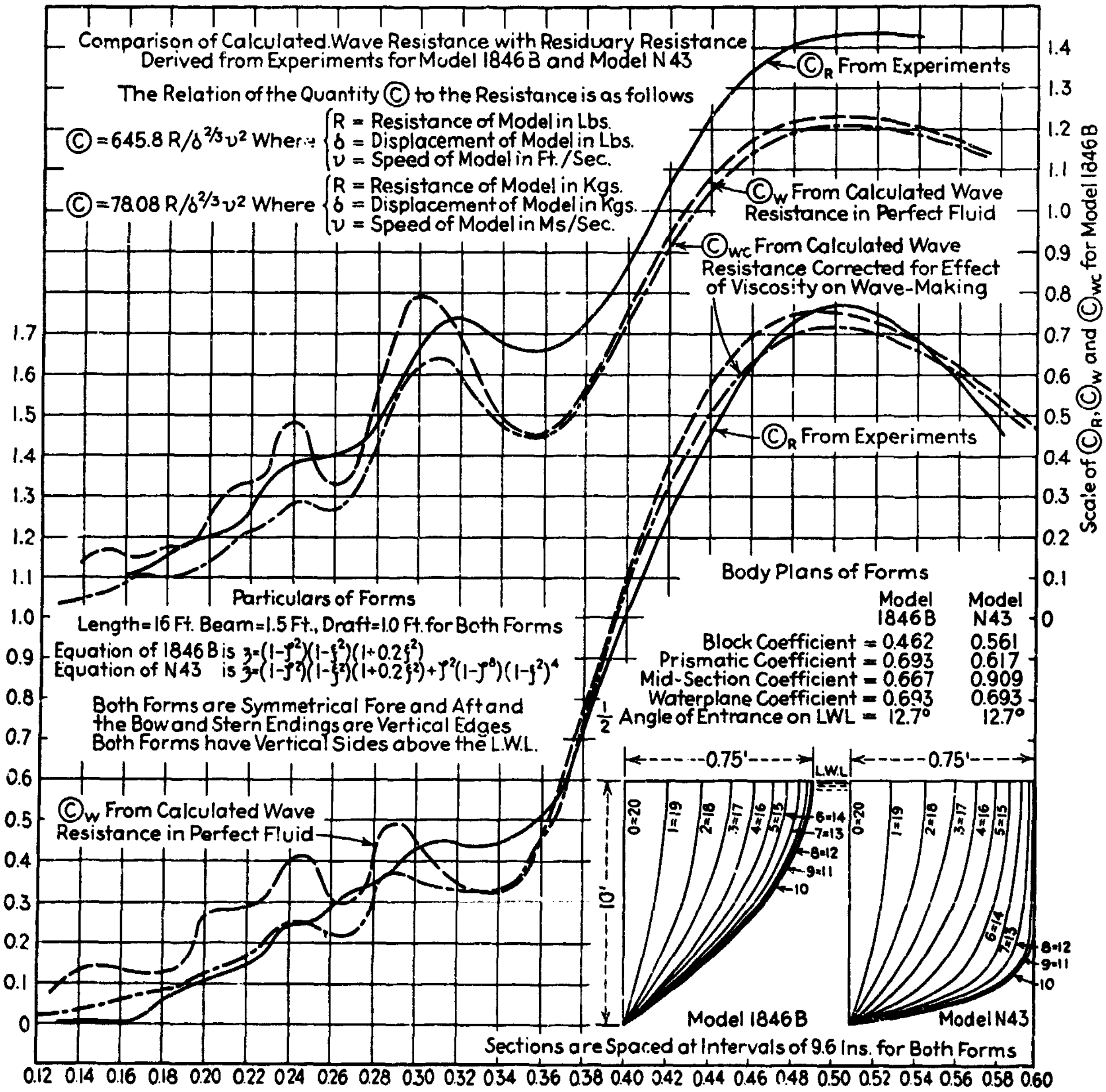


Fig. 6

one time thought permissible. Recently Wigley and Lunde have worked with forms of fuller mid-section and Fig. 6 shows some of their results.

The original fine form was altered by adding a bulge which widened the form amidships as indicated, and the resistance curves for the two models are shown. The comparison is made in a more striking manner in Fig. 7, which shows the difference between the two models, with calculated and observed values.

The models were tested in different tanks (Teddington and Trondheim) and the lack of agreement at very high speed is probably a depth effect due to the difference in depth of the two tanks. As the authors remark, the presence of a full mid-section, and therefore of a rather flat bottom, does not cause more discrepancies between calculation and fact than occur with finer mid-sections.

It is desirable to be able to calculate results for non-mathematical forms or for ordinary ship models. In essence the object is to replace the continuous distribution which represents the ship by a finite number of elements; these elements must be such that their super-position gives an approximation to the form of the model, and the elements must be of a simple character so that the necessary functions for each element can

be calculated and tabulated in sufficient detail. The element proposed by Guilloton is a semi-infinite wedge; or, if we prefer, we may think of it as a certain semi-infinite source distribution. Guilloton has tabulated many of the necessary functions and has had noteworthy success in calculating wave profiles and so forth; and the application of the method to a survey of stream lines around a fine hull promises results of great value, especially if it can be carried out for models for which experimental results are available. Another method, proposed for approximate calculation at high speeds, is to replace the continuous source distribution over the longitudinal vertical plane by a finite number of sources and sinks of suitable magnitudes and positions; it is ob-

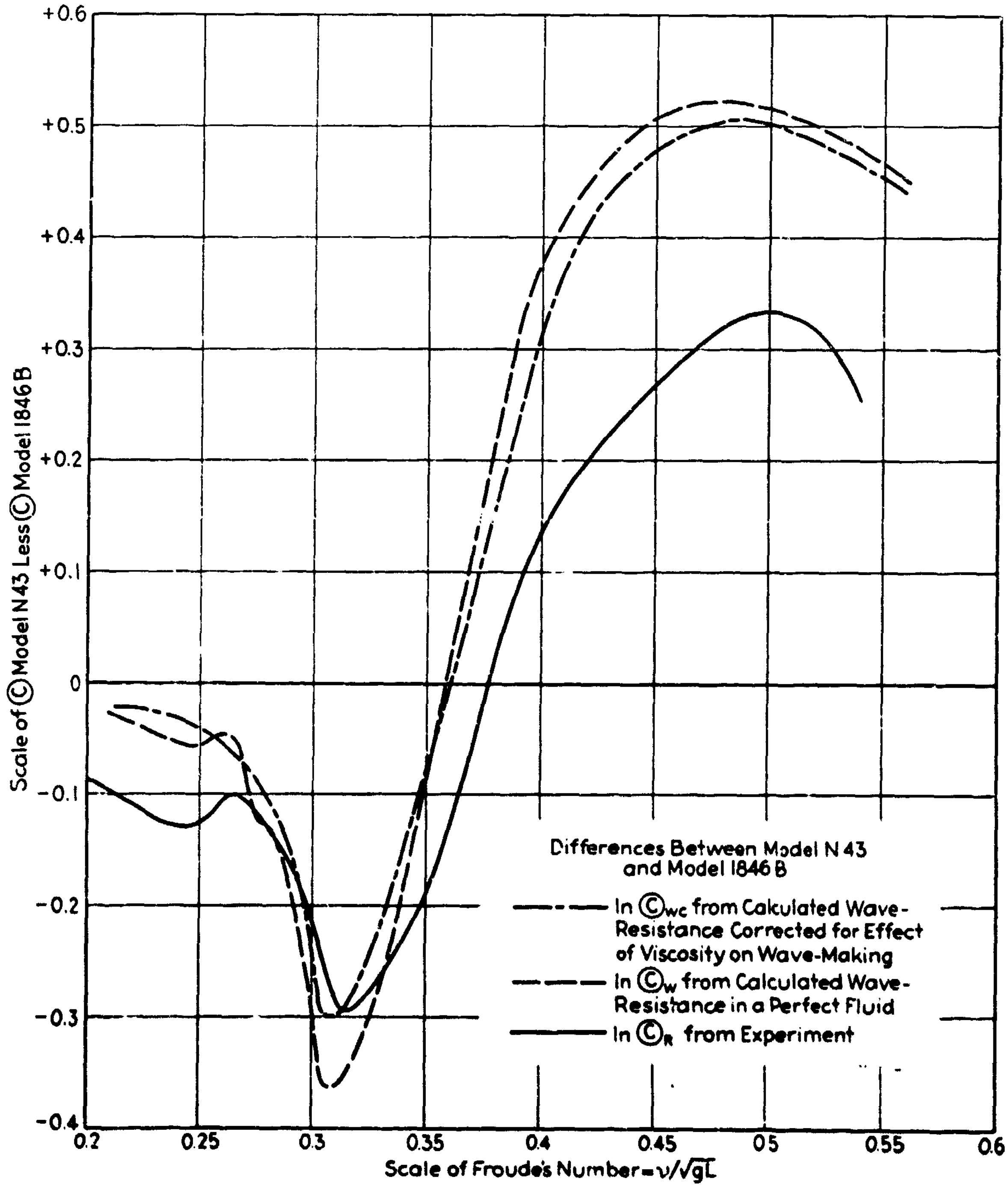


Fig. 7

vious that this method would not be worth while for low speeds, as the number of elements would be too large and other methods of calculation, such as that used by Weinblum, would be less laborious. However, one possible extension is of some interest; we may subdivide the ship into compartments also by longitudinal vertical planes, so that the sources are not just located in one plane but are distributed in space. This represents an attempt to extend the theory to models of fuller form than can be represented adequately by a plane distribution; although the method is rather crude, it might give some better idea of the effect of finite beam. I reproduce some diagrams to show the sort of results which have been obtained by these methods.

Fig. 8 shows calculated and observed wave profile for a certain model. The calculations were made both by the wedge method and by the source method, and there is not much difference in the first approximation; it should be added that Guilloton has considered various second order corrections by his method, and his corrected curve in this diagram shows extremely good agreement with the observed profile.

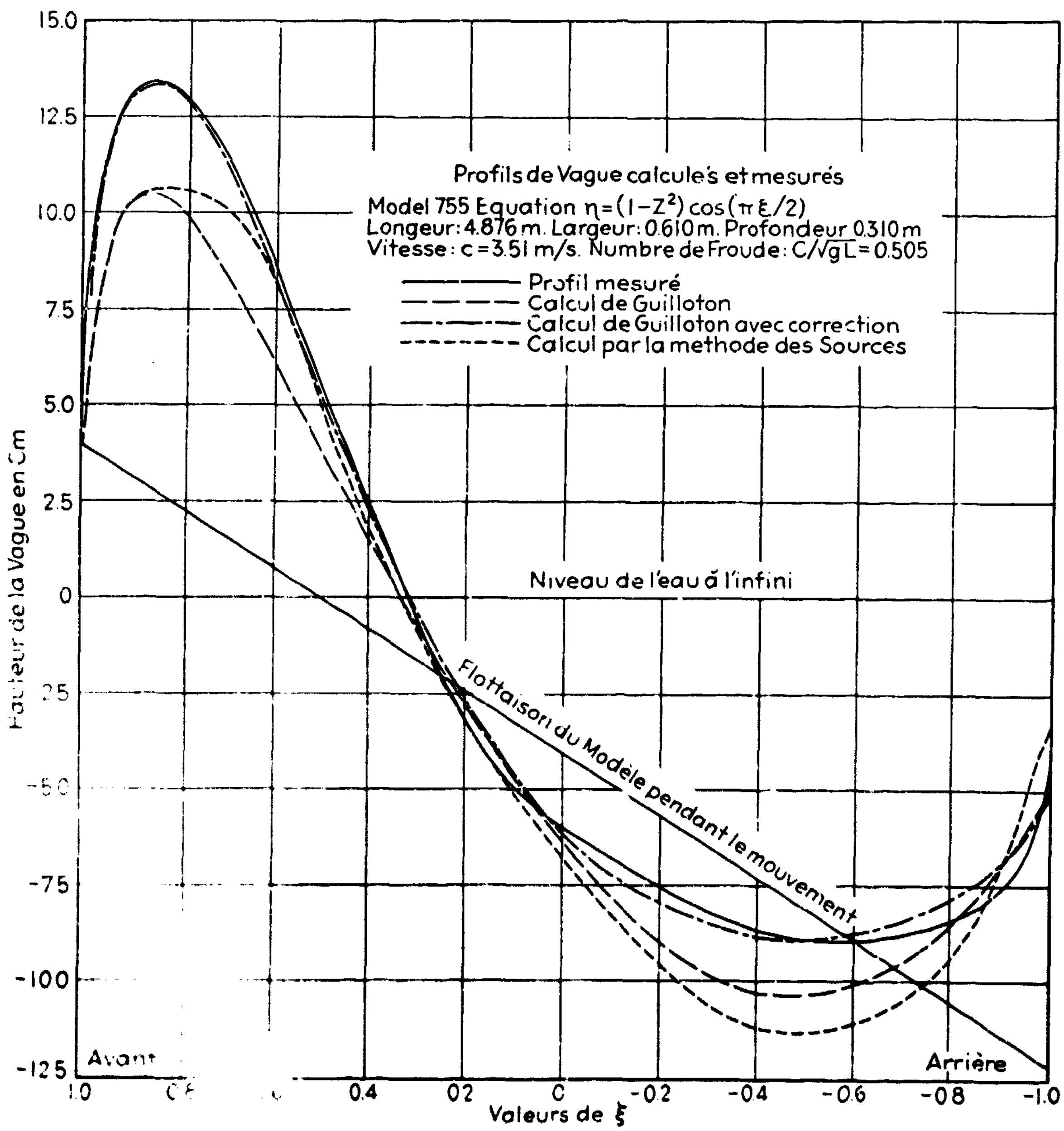


Fig. 8

Fig. 9 is from Guilloton's work on stream lines.

Fig. 10 shows resistance curves for two models, the calculations being made by the source method. The forms were not experimental models but were actual ships, of high-speed form and not symmetrical fore and aft. The models were divided into ten compartments and the strengths and positions of the sources determined directly from the plans of the model, the chief point of the work being to show that the calculations can be carried out in such cases.

Finally, I reproduce in Fig. 11 a diagram from Lunde's recent paper in which he examined the effect of placing sources and sinks off the longitudinal vertical section. Here the model was of destroyer type, but it is unnecessary to enter into details of the comparison except to note that some improvement was obtained by the space distribution of the sources.

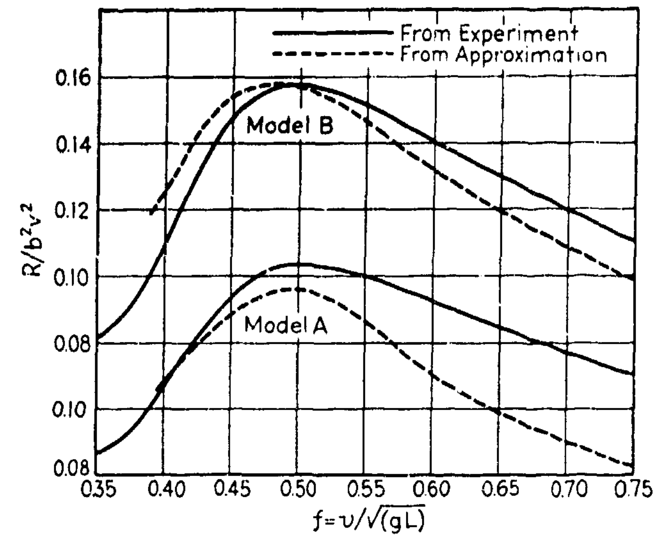
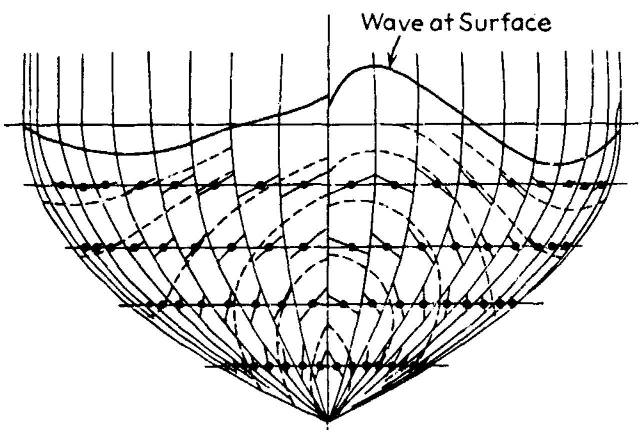


FIG. 10



Model No.755 (Mr. Wigley) for $\frac{v}{\sqrt{gL}} = 0.274$

Tangents $\frac{\partial e}{\partial x}$ to the Streamlines

In Dotted Lines, Approximate Traces of some Streamlines

FIG. 9

In some cases, and not only in those cases which have been reproduced here, one may suspect that the agreement with experimental results is too good; or perhaps one should say rather that the agreement may be deceptive when pushed too far in view of certain considerations which have been neglected. There are, for instance, the effects of trim and sinkage at higher speeds, of which it is possible to make a rough estimate; but, specially, there is the question of the effects of viscosity.

We talk of comparing calculated wave resistance with experiment, but there is no such thing as an experimentally measured wave resistance; for that we must wait for the day when someone invents a frictionless liquid. The only experimental result is the measured total resistance. We may adopt the usual procedure, which has been so well justified for most practical purposes, of considering the frictional resistance and the wave resistance separately, and we use some standard method, Froude's coefficients or some more recent formulation, for determining the frictional resistance. Then we begin to realize, when we require greater accuracy, that the ship is not a plank and that we should make some allowance for the effect of form upon frictional resistance; and, as the importance of boundary layer theory becomes recognized in ship problems, we find how necessary it is to know more of the conditions in the boundary layer, the extent of laminar flow, turbulent flow, separation and so forth, a matter which may be described as a burning question at the moment. It may well be the case that some of the differences between calculated wave resistance and so-called experimental values may prove to be due to error in estimating the frictional resistance. No doubt as we push on to greater accuracy, we may find it inadequate to treat the two parts of the resistance as independent; the problem is one, and the two must have mutual interaction, the important point being whether it is of appreciable magnitude. On the one hand, it is obvious that viscosity effects have a very considerable direct influence upon the wave-making; on the other hand, it seems possible that the wave motion may have an appreciable effect upon conditions in the boundary layer in special circumstances, as, for instance, the positions of crests and troughs in relation to the lines of the model.

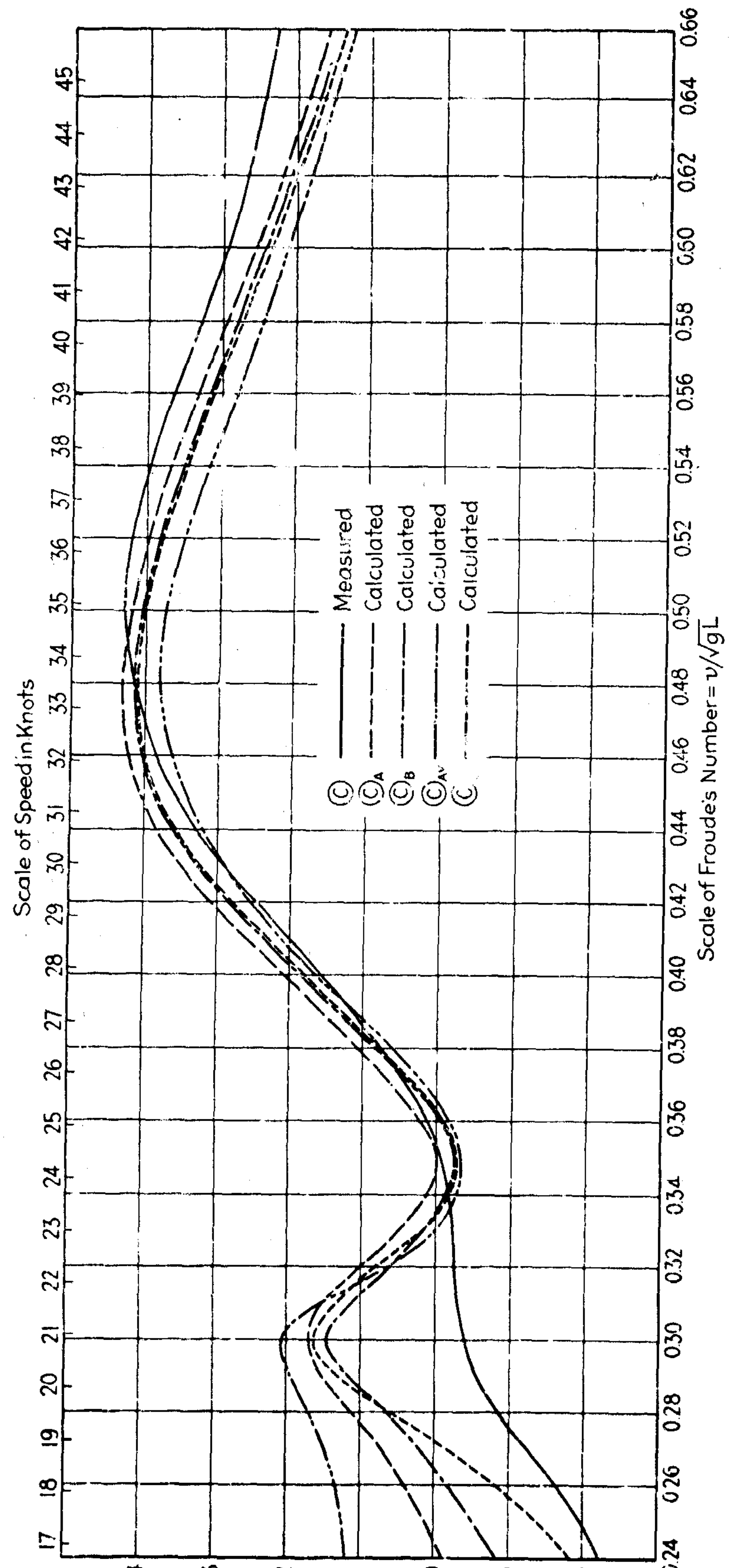

Scale of Speed in Knots
17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45
Scale of Froude's Number = v/√gL
0.24 0.26 0.28 0.30 0.32 0.34 0.36 0.38 0.40 0.42 0.44 0.46 0.48 0.50 0.52 0.54 0.56 0.58 0.60 0.62 0.64 0.66
Scale of ©
1.4 1.3 1.2 1.1 1.0 0.9 0.8
© Measured
©A Calculated
©B Calculated
©AV Calculated
© Calculated

Fig. 11

In regard to the effect of viscosity upon wave-making, some attempts have been made to allow for this, but no adequate theory has yet been provided. It is well known that at low speeds we do not observe the oscillations in the resistance curve indicated by theory for a frictionless liquid and due to interference between bow and stern waves; in fact, the wave resistance is due very largely to the bow and entrance only, the effect of viscosity being to reduce the wave-making properties of the stern. We may begin then simply by introducing an empirical reduction factor into the calculations, and for simplicity this factor was taken as constant and operative over the whole of the rear half of the model. This idea was improved by Wigley and made more useful from a practical point of view; comparing calculated and observed results for a large number of models, Wigley deduced a simple expression for such a reduction factor and for its dependence upon velocity. When we remember other considerations which have not been taken into account, it must be admitted that this viscosity correction probably includes other effects than those due to viscosity alone; nevertheless it serves a very useful purpose. The difference made by this correction can be seen in the curves of Figs. 6 and 7. The latter diagram illustrates a promising field of application of the theory as it stands at present; although it is not possible to give with sufficient certainty absolute values of the resistance, yet it is within reach to forecast differences made in the resistance curves for two models of a series with small variations in form. However, for a satisfactory account of viscous effects it will be necessary to link up wave theory and boundary layer theory. Starting with a much simplified conception, consider a ship of streamline form with its boundary layer over the surface and becoming of any appreciable thickness only near the stern. The displacement thickness of the layer gives some measure of the amount by which the stream lines of the flow are displaced outwards; suppose then that we take the effective form of the ship for wave-making as the actual form increased by the displacement thickness of the boundary layer. Some calculations were made on these lines recently; but, needless to say, it was not possible to deal with actual boundary-layer structure. What was done was to make small modifications of the lines near the stern such as might reasonably be ascribed to boundary layer effect, the main point being that these modifications were confined to quite a small region near the stern. The purpose of the calculations was to illustrate the possible effect of such boundary-layer modifications of the form and to see if they were sufficient to eliminate the excessive resistance oscillations at low speed given by theory for a frictionless liquid, while at the same time not materially affecting values at high speeds.

Figs. 12 and 13 show some of the results, with the modified forms and the corresponding resistance curves. They agree fairly well with the anticipated effect, except that the hollow at a Froude number of about 0.34 still remains too pronounced; but the latter is a persistent disagreement between calculated and observed results for which some other explanation must be found.

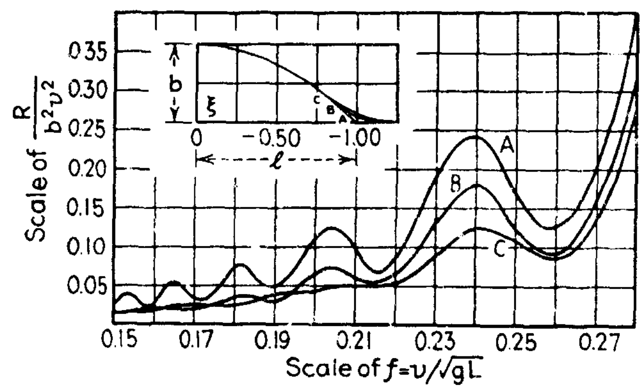


FIG. 12

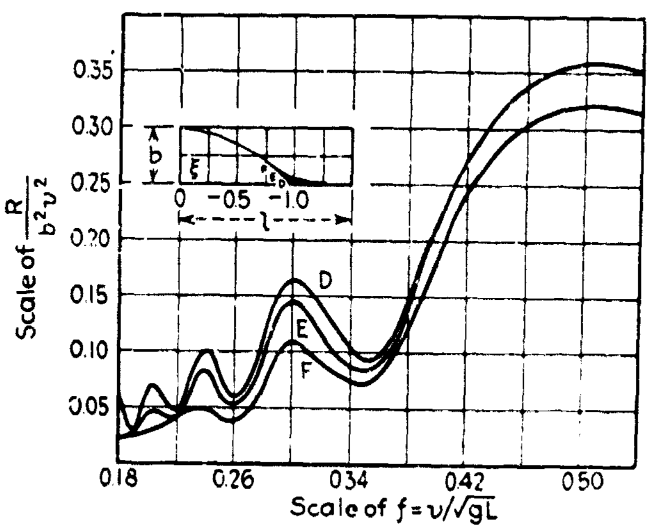


FIG. 13

This inadequate survey of wave-resistance theory and its applications may be concluded by indicating briefly some directions in which further work would be specially useful. Even with the theory as it stands at present, much could be done to extend its range of application: for instance, by a systematic study of methods of approximation and by the computation of necessary tables of functions, so that numerical calculations could be carried out more readily. But the two main problems, broadly speaking, are those of the ship of finite beam and of the effects of viscosity. It may well be that in both cases it may only prove possible to advance by successive stages of approximation to a solution: but the former problem, leaving viscosity out of account, is essentially a mathematical one for which a new approach is much to be desired. On the other hand, our knowledge of boundary-layer conditions is insufficient and the latter problem is pre-eminently one for combined theoretical and experimental investigation. Indeed the whole subject calls for a close association between mathematical and experimental work, especially if we keep in view its practical application to ship problems.

Note: The illustrations are from the following sources:

Figs. 2 and 3: T. H. Havelock, *Proceedings* of the Royal Society, (A), Volume 201, page 297 (1950).

Fig. 4: T. H. Havelock, Quarterly Journal of Mechanics and Applied Mathematics, Volume 2, page 419 (1949).

Figs. 6 and 7: W. C. S. Wigley and J. K. Lunde, *Transactions* of the Institution of Naval Architects, Volume 90, page 92 (1948).

Fig. 8: W. C. S. Wigley, *Bulletin*, L'Association Technique Maritime et Aéronautique, Volume, 48, page 533 (1949).

Fig. 9: R. S. Guilloton, *Transactions* of the Institution of Naval Architects, Volume 90, page 48 (1949).

Fig. 10: T. H. Havelock, *Transactions* of the North East Coast Institution of Engineers and Shipbuilders, Volume 60, page 47 (1943).

Fig. 11: J. K. Lunde, *Transactions* of the Institution of Naval Architects, Volume 91, page 182 (1949).

Figs. 12 and 13: T. H. Havelock, *Transactions* of the Institution of Naval Architects, Volume 90, page 259 (1948).

THE MOMENT ON A SUBMERGED SOLID OF REVOLUTION MOVING HORIZONTALLY

By T. H. HAVELOCK (*King's College, Newcastle upon Tyne*)

[Received 20 February 1951]

SUMMARY

The moment, due to surface waves, on a submerged solid of revolution moving axially at constant depth below the surface of the water is examined in detail.

1. A SUBMERGED solid of revolution moves axially with uniform velocity and with its axis at a constant depth below the surface of the water. If the solid is such that the motion in an infinite liquid can be represented by a known source-sink distribution along the axis, the horizontal and vertical forces on the solid due to the wave motion can readily be obtained to the usual approximation; however, for the moment about a transverse horizontal axis it is necessary to obtain the velocity potential to a higher degree of approximation, a point which was noticed in an early paper on the circular cylinder (1) but which has sometimes been overlooked. In the present note we consider a prolate spheroid, for which this extension can be carried out; the form of the additional term in the moment in this case suggests an approximation applicable to other elongated solids of revolution, such as a Rankine ovoid, generated by an axial source distribution.

2. We suppose the spheroid to be held at rest in a uniform stream of velocity c in the negative direction of Ox, the axis being at a depth f below the free surface of the water. We take O at the centre of the spheroid, Ox along the axis, Oy transversely, and Oz vertically upwards. Using the known axial distribution for motion in an infinite liquid, the velocity potential is given by

$$\phi = cx + Ac\int_{-ae}^{ae} \frac{k\,dk}{\{y^2+z^2+(x-k)^2\}^{\frac{1}{2}}} -$$
$$-\frac{Ac}{2\pi}\int_{-ae}^{ae} k\,dk \int_{-\pi}^{\pi} d\theta \int_0^\infty \frac{\kappa+\kappa_0\sec^2\theta}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\, e^{-\kappa(2f-z)+i\kappa\varpi}\,d\kappa, \qquad (1)$$

where

$$A^{-1} = 2e/(1-e^2)-\log\{(1+e)/(1-e)\}, \qquad \varpi = (x-k)\cos\theta+y\sin\theta,$$

$\kappa_0 = g/c^2$, and the limit is taken as $\mu \to 0$.

[Quart. Journ. Mech. and Applied Math., Vol. V, Pt. 2 (1952)]
5092.18

The first two terms in (1) satisfy the condition at the surface of the solid. The third term, which we shall denote by ϕ_3, is the first approximation to the wave motion; its form is determined so as to ensure that the three terms together satisfy the condition at the free surface of the water (**2**).

The second term in (1), which is the velocity potential for the axial motion of a prolate spheroid, is usually given (**3**) as $2AcaeP_1(\mu)Q_1(\zeta)$ in terms of coordinates specified by $x = ae\mu\zeta$, $y = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega$, $z = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega$; it can readily be verified that the two forms are equivalent. This equivalence is a particular case of a general relation which does not seem to have been stated explicitly, and the opportunity is taken of recording it here in view of its use in problems dealing with the motion of a spheroid. The relation expresses prolate spheroidal harmonics in terms of axial distributions of poles or multi-poles. Using the appropriate form of the known general expansion of reciprocal distance (**4**), it follows at once that

$$P_n(\mu)Q_n(\zeta) = \frac{1}{2}\int_{-ae}^{ae} \frac{P_n(k/ae)\,dk}{\{y^2+z^2+(x-k)^2\}^{\frac{1}{2}}}.$$

For the general case, forming the corresponding expansion for the potential of a multi-pole, it can be shown that

$$P_n^s(\mu)Q_n^s(\zeta)e^{is\omega} = \frac{1}{2}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^s \int_{-ae}^{ae} \frac{(a^2e^2-k^2)^{\frac{1}{2}s}P_n^s(k/ae)\,dk}{\{y^2+z^2+(x-k)^2\}^{\frac{1}{2}}}.$$

We use the theorem that the forces on the solid can be obtained as the resultant of forces on the internal sources, the force on a typical source m being $-4\pi\rho m\mathbf{q}$, where $\mathbf{q}$ is the fluid velocity at the point other than that due to the source itself; in fact, we may omit the part of the velocity due to all the other internal sources and sinks. Thus for the horizontal force, or wave resistance R, we have

$$R = -4\pi\rho\int_{-ae}^{ae} Acx(\partial\phi_3/\partial x)\,dx, \tag{2}$$

taken along the axis $y = z = 0$.

Taking ϕ_3 from (1) and omitting terms which, on account of the integrations in x and k, give no contribution to the final result, this reduces to

$$R = -16\rho\kappa_0^2c^2A^2\int_{-ae}^{ae} x\,dx\int_{-ae}^{ae} k\,dk\int_0^{\frac{1}{2}\pi}\sec^3\theta\,d\theta\int_0^\infty \frac{e^{-2\kappa f+i\kappa(x-k)\cos\theta}}{\kappa-\kappa_0\sec^2\theta+i\mu\sec\theta}\,d\kappa, \tag{3}$$

where the imaginary part is to be taken.

The integration in κ may be transformed in the usual manner by treating κ as a complex variable and integrating round a suitable quadrant according as $x-k > $ or < 0. Finally we obtain

$$R = 32\pi^2 g\rho a^3 e^3 A^2 \int_0^{\frac{1}{2}\pi} \sec^2\theta \, J^2_{\frac{3}{2}}(\kappa_0 ae \sec\theta) e^{-2\kappa_0 f \sec^2\theta} \, d\theta, \tag{4}$$

which is the known expression for the wave resistance.

The vertical force, Y, apart from that due to buoyancy, can be obtained similarly from

$$Y = 4\pi\rho \int_{-ae}^{ae} Acx(\partial\phi_3/\partial z) \, dx. \tag{5}$$

This involves the real part of the contour integrals in κ referred to above, and leads to double integrals; the expression for Y can easily be written down, but it is not very suitable for numerical computation.

3. The moment of the forces about Oy requires more consideration, and we shall take it in two parts.

We calculate first the moment on the initial source distribution arising from the vertical component of the velocity derived from the term ϕ_3 in (1); we denote this part by G_1. Thus

$$G_1 = 4\pi\rho \int_{-ae}^{ae} Acx^2(\partial\phi_3/\partial z) \, dx, \tag{6}$$

taken along the axis.

But we have to proceed to a further approximation to the velocity potential, because the uniform stream produces on this second approximation a contribution to the moment of the same order as G_1; we denote this second part by G_2. Let ϕ_4 be the term to be added to (1) for the next approximation. This term represents some distribution of sources and sinks within the spheroid; if M is the total moment of this distribution resolved parallel to Oz, then we have

$$G_2 = -4\pi\rho c M, \tag{7}$$

and the total moment on the solid to this stage is $G_1 + G_2$.

4. From (6) and (1), we have

$$G_1 = -4\rho A^2 c^2 \int_{-ae}^{ae} x^2 \, dx \int_{-ae}^{ae} k \, dk \int_{-\pi}^{\pi} d\theta \int_0^\infty \frac{\kappa(\kappa+\kappa_0 \sec^2\theta)}{\kappa - \kappa_0 \sec^2\theta + i\mu \sec\theta} \times$$
$$\times e^{-2\kappa f + i\kappa(x-k)\cos\theta} \, d\kappa. \tag{8}$$

Treating the integration in κ as before, and carrying out the integrations in x and k, this leads, after some reduction, to

$$G_1 = 64\pi\rho a^2e^2A^2(c^2/\kappa_0)\int_0^{\frac{1}{2}\pi}\sec\theta\left(\frac{\sin p}{p}-\cos p\right)\times$$
$$\times\left(p\sin p+2\cos p-2\frac{\sin p}{p}\right)e^{-2\kappa_0 f\sec^2\theta}\,d\theta, \quad (9)$$

with $p = \kappa_0 ae\sec\theta$.

We now determine the next approximation to ϕ so as to satisfy the condition at the surface of the spheroid, namely that the normal component of velocity from $\phi_3+\phi_4$ must be zero over the spheroid.

We use coordinates μ, ζ, ω given by

$$x = ae\mu\zeta, \quad y = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega, \quad z = ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega, \quad (10)$$

the spheroid being given by $\zeta = \zeta_0 = 1/e$.

If, in the neighbourhood of the spheroid, ϕ_3 is expressed in the form

$$\phi_3 = \sum_{r=1}^{\infty}\sum_{s=0}^{r}(A_r^s\cos s\omega+B_r^s\sin s\omega)P_r^s(\mu)P_r^s(\zeta), \quad (11)$$

then the required next approximation is given by

$$\phi_4 = -\sum_{r=1}^{\infty}\sum_{s=0}^{r}\frac{P_r^{s\prime}(\zeta_0)}{Q_r^{s\prime}(\zeta_0)}(A_r^s\cos s\omega+B_r^s\sin s\omega)P_r^s(\mu)Q_r^s(\zeta). \quad (12)$$

By considering the behaviour of the terms in ϕ_4 as $\zeta\to\infty$, we see that the only one which contributes to the moment M referred to in (7) is the term in $P_1^1(\mu)Q_1^1(\zeta)\cos\omega$; this latter quantity approximates to $-2a^2e^2z/3r^3$ as $\zeta\to\infty$. Alternatively, we may get the same result from the expression of this term as a line distribution of doublets parallel to Oz along the axis of the spheroid between the two foci. Hence, putting in the value of the factor $P_1^{1\prime}(\zeta_0)/Q_1^{1\prime}(\zeta_0)$, we have

$$M = \tfrac{2}{3}a^2e^2BA_1^1, \quad (13)$$

with $$B^{-1} = \tfrac{1}{2}\log\{(1+e)/(1-e)\}+e(2e^2-1)/(1-e^2).$$

To determine A_1^1 we take from the expression for ϕ_3 in (1) the term in the integrand involving the coordinates, namely

$$\exp\kappa(z+ix\cos\theta+iy\sin\theta),$$

and expand the value of this on the spheroid in the form

$$\sum_{r=0}^{\infty}\sum_{s=1}^{r}(C_r^s\cos s\omega+D_r^s\sin s\omega)P_r^s(\mu). \quad (14)$$

The coefficient C_1^1 is then given by

$$\frac{4\pi}{3} C_1^1 = \int_{-1}^{1} (1-\mu^2)^{\frac{1}{2}}\, d\mu \int_0^{2\pi} \exp\{\kappa b(1-\mu^2)^{\frac{1}{2}}(\cos\omega + i\sin\omega\sin\theta) + $$
$$+ i\kappa a\mu\cos\theta\}\cos\omega\, d\omega. \quad (15)$$

The integrations can be reduced to known forms, and we obtain

$$C_1^1 = 3(\pi/2\kappa a^3 e^3)^{\frac{1}{2}} b \sec^{\frac{1}{2}}\theta\, J_{\frac{3}{2}}(\kappa a e\cos\theta). \quad (16)$$

Hence, from (11), the corresponding term in the integral for A_1^1 is

$$3(\pi/2\kappa a e)^{\frac{1}{2}} \sec^{\frac{1}{2}}\theta\, J_{\frac{3}{2}}(\kappa a e\cos\theta). \quad (17)$$

Using (7) and (17) in (1) we obtain the expression for G_2, which may be written in the form

$$G_2 = 16\rho a e c^2 A B \int_{-ae}^{ae} k\, dk \int_0^{\frac{1}{2}\pi} \sec^2\theta\, d\theta \int_0^{\infty} D\left(\frac{\sin q}{q} - \cos q\right) e^{-2\kappa f - i\kappa k\cos\theta}\, d\kappa, \quad (18)$$

where the real part is to be taken, and

$$q = \kappa a e\cos\theta, \qquad D = (\kappa + \kappa_0\sec^2\theta)/\kappa(\kappa - \kappa_0\sec^2\theta + i\mu\sec\theta).$$

After carrying out the integrations in κ and k this leads to

$$G_2 = -64\pi\rho a^2 e^2 A B(c^2/\kappa_0) \int_0^{\frac{1}{2}\pi} \left(\frac{\sin p}{p} - \cos p\right)^2 e^{-2\kappa_0 f\sec^2\theta} \sec\theta\, d\theta, \quad (19)$$

with $p = \kappa_0 a e\sec\theta$.

For computation it is convenient to express these results in terms of the so-called spherical Bessel functions, of which tables are available. If we write

$$S_{n+\frac{1}{2}}(p) = (\pi/2p)^{\frac{1}{2}} J_{n+\frac{1}{2}}(p),$$

$$I_1 = \int_0^{\frac{1}{2}\pi} S_{\frac{1}{2}}(p) S_{\frac{3}{2}}(p) e^{-2\kappa_0 f\sec^2\theta} \sec^4\theta\, d\theta,$$

$$I_2 = \int_0^{\frac{1}{2}\pi} S_{\frac{3}{2}}^2(p) e^{-2\kappa_0 f\sec^2\theta} \sec^3\theta\, d\theta,$$

we have

$$R = 64\pi g\rho\kappa_0 a^4 e^4 A^2 I_2,$$
$$G_1 = 64\pi g\rho a^4 e^4 A^2(\kappa_0 a e I_1 - 2I_2),$$
$$G_2 = -64\pi g\rho a^4 e^4 A B I_2. \quad (20)$$

5. These results may be checked, to some extent, by taking the limiting case of a sphere. In the first place we may calculate directly the case for the sphere by the same method. For a sphere of radius a, we obtain

$$G_1 = 4\pi\rho c^2 a^6 \kappa_0^3 \int_0^{\frac{1}{2}\pi} \sec^5\theta\, e^{-2\kappa_0 f\sec^2\theta}\, d\theta. \quad (21)$$

For G_2 we expand the corresponding term ϕ_3 in spherical harmonics, and we find that G_2 reduces to the same expression (21) with a negative sign; thus the total moment is zero, as should be the case. Turning to the expressions in (9) and (19) for the prolate spheroid, we find their limiting values for $e \to 0$ reduce to the correct values for the sphere.

6. Returning to the spheroid, we notice that G_1 may be positive or negative according to the speed; on the other hand, G_2 is essentially negative. Further, from (20) we have

$$G_2 = -BR/\kappa_0 A. \tag{22}$$

If k_1 and k_2 are the inertia coefficients for axial motion and transverse motion respectively, we have $2e^3A = (1-e^2)(1+k_1)$ and a similar relation between B and k_2. Hence (22) may be put in the convenient form

$$G_2 = -(1+k_2)R/\kappa_0(1+k_1). \tag{23}$$

The ratio $(1+k_2)/(1+k_1)$ is unity for a sphere, and approximates to two for a spheroid of large length-beam ratio. When $c \to \infty$, or $\kappa_0 \to 0$, the integrals in the expressions for R, G_1, and G_2 all reduce to the integral given in (21), which can be expressed in terms of Bessel functions; hence we may find the limiting values of these quantities as the speed increases indefinitely. It appears that as $c \to \infty$, R becomes zero of order c^{-2}; on the other hand G_1 and G_2 approach finite limiting values, with

$$G_1 \to \tfrac{8}{9}\pi g\rho a^6e^6A^2/f^2, \qquad G_2 \to -\tfrac{8}{9}\pi g\rho a^6e^6AB/f^2. \tag{24}$$

Thus the moment G approaches the limiting value

$$G = G_1+G_2 \to -\tfrac{2}{9}\pi g\rho a^2b^4(1+k_1)(k_2-k_1)/f^2, \tag{25}$$

and this is negative for a prolate spheroid.

Some numerical values have been calculated from (20) for a spheroid of a length-beam ratio of 10. The moment at low speeds may be positive or negative and is small numerically; after a Froude number, $c/\sqrt{(2ga)}$, of more than about 0·4 the moment remains negative and increases rapidly towards its limiting value.

It may seem unexpected, as compared with surface ships, to find the moment remaining negative at high speeds. The model of a surface ship is usually allowed to trim and at high speeds it takes up a position with bow up and stern down, corresponding to a positive moment; the attitude of the model is then roughly parallel to the mean line of the water surface in its vicinity. But the submerged spheroid we have been considering has its axis maintained horizontal; so we may describe it roughly as being in a stream whose effective direction in the vicinity of the spheroid is inclined to the axis and this provides a moment tending to increase this angle, that is, a negative moment. For a numerical case take a spheroid with $a = 10b$

and immersed to a depth $f = 2{\cdot}5b$; we calculate the part G_2 of the moment for a Froude number $0{\cdot}5$. For a spheroid in a uniform stream at a small angle δ to the axis the moment tending to increase this angle is $\frac{4}{3}\pi\rho ab^2c^2(k_2-k_1)\delta$. Comparing these two moments in this particular case, we find that G_2 would be accounted for in this way by an angle δ of about $0{\cdot}03$, which seems not unreasonable. However, this comparison cannot be pressed far: it is only intended to indicate a possible physical explanation of the negative moment at high speeds.

7. Consider now any solid of revolution which, so far as axial motion in an infinite liquid is concerned, can be specified by a known axial source distribution. The part G_1 of the moment can be obtained at once by the method used in the previous sections; but it is not possible, in general, to calculate the part G_2. Turning to the connexion between G_2 and R for the spheroid given in (23), it is proposed now to use this as a suitable approximation for any solid of revolution, and in particular for one of large ratio of length to beam. The inertia coefficient k_1 can be calculated; if M is the total moment of the given axial source distribution and V is the volume of the solid, we have $4\pi M = (1+k_1)V$. It is not possible, in general, to calculate k_2. However, for a long slender solid, k_1 is small; on the other hand, k_2 approximates to the value unity which it has for the transverse motion of a circular cylinder. Thus, in such a case, it is sufficient for a fairly close approximation to take

$$G_2 = -2R/\kappa_0. \tag{26}$$

where R is the wave resistance of the submerged solid. The simplest case is that of the solid specified by a single source and sink. If m is the strength of the source or sink, $2h$ the distance apart, $2l$ the axial length of the solid, and $2b$ the maximum beam, we have

$$4mlh = c(l^2-h^2)^2; \qquad 4mh = cb^2(h^2+b^2)^{\frac{1}{2}}. \tag{27}$$

Taking the axis at depth f, the velocity potential can be written down to the same approximation as for the spheroid in (1). The process of determining R and the part G_1 of the moment is the same as before, and the details need not be given. Using (27) to express m in terms of the dimensions, we obtain

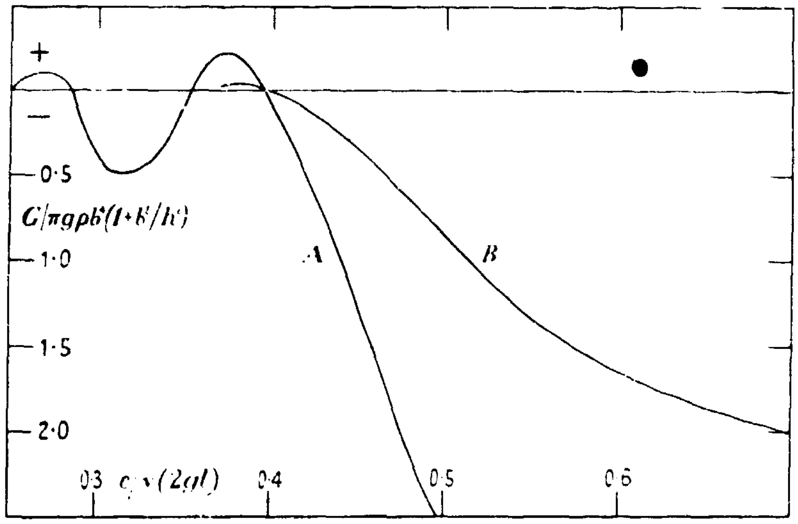
$$R = 2\pi g\rho\kappa_0 b^4(1+b^2/h^2)\int_0^{\frac{1}{2}\pi}\{1-\cos(2\kappa_0 h\sec\theta)\}e^{-2\kappa_0 f\sec^2\theta}\sec^3\theta\,d\theta. \tag{28}$$

$$G_1 = 2\pi g\rho\kappa_0 hb^4(1+b^2/h^2)\int_0^{\frac{1}{2}\pi}\sin(2\kappa_0 h\sec\theta)e^{-2\kappa_0 f\sec^2\theta}\sec^4\theta\,d\theta. \tag{29}$$

For G_2 we should work out the next approximation for the velocity potential, as in the case of the spheroid; but this does not seem feasible for the given solid. Meantime, as already indicated, we shall take (26) as giving a sufficient approximation and thus we assume

$$G_2 = -4\pi g\rho b^4(1+b^2/h^2)\int_0^{\frac{1}{2}\pi}\{1-\cos(2\kappa_0 h\sec\theta)\}e^{-2\kappa_0 f\sec^2\theta}\sec^3\theta\,d\theta. \quad (30)$$

Computation of the total moment G can be made from the integrals in



(29) and (30) either by direct quadrature or by asymptotic expansions suitable for large values of the parameter $2\kappa_0 h$. To show the nature of the results calculations have been made for an ovoid with $h = 10b$, giving a length-beam ratio of about 10·5. Two depths of immersion were taken, and the results are shown in the figure with values of $G/\pi g\rho b^4(1+b^2/h^2)$ graphed on a base of Froude number $c/\sqrt{(2gl)}$. Curve A is for $f = 2{\cdot}5b$, and curve B for $f = 5b$. Curve A shows the typical oscillations at low speeds due to interference between bow and stern waves; these would no doubt be damped by viscous effects in an actual liquid. For curve B at greater depth these oscillations are too small to be shown on the scale of the diagram.

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ISSUED FOR WRITTEN DISCUSSION

Members are invited to submit written contributions to this paper, which should reach the Secretary not later than 15th September, 1952: they should be typewritten.
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SHIP VIBRATIONS: THE VIRTUAL INERTIA OF A SPHEROID IN SHALLOW WATER

By PROFESSOR SIR THOMAS H. HAVELOCK, M.A., D.Sc., F.R.S. (Honorary Member and Associate Member of Council)

Summary

It is known that certain motions of the surface of a spheroid expressed by spheroidal harmonics are similar to flexural 2- and 3-node vibrations, and can be used to obtain virtual inertia coefficients for motion in an infinite liquid. These calculations are now extended so as to include the effect of a plane boundary, and are given in a general form which includes translation and rotation as well as the flexural vibrations.

Consideration is given, in particular, to the vertical and horizontal vibrations of a floating spheroid, half-immersed, in water of given depth. Graphs are obtained for the variation of the relative increase of inertia coefficient with the depth of water. These show how the variation depends upon the type of vibration, and a result of special interest is the striking difference between horizontal and vertical vibrations; the relative increase is less for the horizontal vibrations, and decreases much more rapidly with increasing depth of water.

PART I

1. In this part we give a general account of the work, leaving details of the analysis to Part II.

In calculating the frequencies of the natural flexural vibrations of a ship, allowance has to be made for the added inertia due to the surrounding water. This is usually carried out by a two-dimensional strip method which consists in obtaining a suitable expression for an elementary transverse section and integrating longitudinally; an empirical factor is then added to allow for the fact that the motion of the water is three-dimensional. The only direct three-dimensional calculations which have been made are for a prolate spheroid deeply immersed, or in an infinite liquid. It was shown by Lewis,[1] and about the same time independently by Lockwood Taylor,[2] that certain motions of the surface of the spheroid expressible by spheroidal harmonics are approximately the same as for the 2-node and 3-node flexural vibrations, and so can be used to give an estimate of the increase of kinetic energy due to the surrounding water.

Recently the influence of depth of water upon the added inertia has become of interest. Here, again, the calculations have been made by the two-dimensional method extended to allow for finite depth of water: reference may be made, in particular, to work by Prohaska.[3]

In the present paper no attempt is made to examine afresh the general theory of the natural vibrations of a solid which is partially, or wholly, immersed in water, although a more complete theory is much to be desired; nor is any attempt made to deal explicitly with solids of ship form. Although the analysis may have wider applications, the main object of the paper is to carry out three-dimensional calculations for a prolate spheroid so as to include the effect of finite depth of water, and, in particular, to examine the vertical and horizontal vibrations of a spheroid floating in water of finite depth.

2. After a brief summary of the analysis for a spheroid in an infinite liquid (§ 6), we proceed to the case of finite depth of water. We consider a prolate spheroid, major axis $2a$ and transverse axis $2b$, wholly immersed in water with its axis horizontal and at a height f above the bed; in the first place we suppose the water deep enough so that we can ignore any effect of the upper free surface. The surface of the spheroid is given a prescribed motion and we calculate the kinetic energy of the resulting fluid motion. Naturally, an exact solution is not obtained, and the degree of approximation may be indicated by reference to known simple cases. If a circular cylinder is moved transversely to its length, either parallel to the boundary or at right angles to it, the approximate relative increase in the virtual inertia coefficient is $b^2/2f^2$. For a three-dimensional case, the only known result appears to be the similar approximation given by Stokes for a sphere; if the motion is parallel to the boundary the relative increase is $3b^3/16f^3$, while for motion at right angles to the boundary it is $3b^3/8f^3$. We obtain the corresponding approximation for a prolate spheroid. The analysis is given in general form for motion of the spheroid specified by a harmonic of order n, for motion both parallel to the boundary and at right angles to it; particular cases of the solution include translation and rotation of the spheroid and also 2- and 3-node vibrations.

3. We turn next to the more interesting problem of a floating spheroid, which we suppose to be half immersed in the water. For a complete theory we should include the surface waves produced by the vibrations, but we neglect these meantime; having in view application to ship vibrations we adopt what seems to be the appropriate simplification, the so-called free surface condition neglecting gravity. A modification of the previous section gives expressions for the relative increase in inertia coefficient for the various types of motion and in § 11 we consider the vertical vibrations of the floating

spheroid. Numerical computations have been made for a spheroid whose length $2a$ is just over 10 times the beam $2b$, and the results are shown in Fig. 1; the computations were troublesome, and a high degree of accuracy was not attempted.

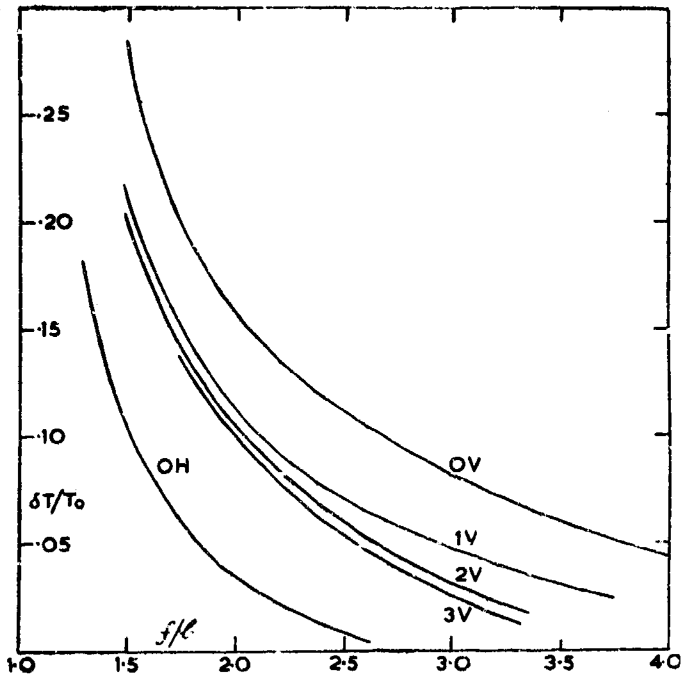

FIG. 1.—RELATIVE INCREASE OF VIRTUAL INERTIA COEFFICIENT ($\delta T/T_0$) FOR RATIO OF DEPTH OF WATER TO DRAFT (f/b), VERTICAL VIBRATIONS (V), HORIZONTAL VIBRATIONS (H).

The ordinates are the relative increase in inertia coefficient, that is the ratio of the increase to the value in deep water; the abscissae are the values of f/b, or the ratio of depth of water to the maximum draught. The curves in question are those marked V. Those marked 0 V and 1 V are for translation and rotation respectively; but we may regard the set of curves as representing vertical vibrations specified by the number of nodes, 0, 1, 2, 3, respectively. From this point of view, it is of interest to note the varying influence of depth according to the type of vibration; it is clear, for instance, that using values derived from pure translation would give misleading results for 2- or 3-node vibrations. The curves V in Fig. 1 were obtained from the general results given in equation (35). These expressions have simple approximate forms when the spheroid is very long; the values are 0·658, 0·470, 0·439, 0·429 times b^2/f^2 for $n = 1, 2, 3, 4$ respectively. In the present case, for which the length-beam ratio is 10, the curves approximate fairly closely to these values for small depth of water. As regards actual measurements, there are no experimental results which are strictly comparable. Prohaska[3] has given a formula $2\,C_B\,d^2/f^2$, where C_B is the block coefficient and d is the mean draught. As the form indicates, this is based on two-dimensional theory, with the coefficient chosen to agree as well as possible with results from actual ship forms. The prolate spheroid is not a normal ship form, nevertheless it is of interest to note that this formula gives $0{\cdot}466\,b^2/f^2$, which may be compared with the approximate values given above.

4. The remaining sections of the work are devoted to the similar horizontal vibrations of the floating spheroid, dealing first with deep water. It is generally known that if inertia coefficients for horizontal motions are calculated using the free surface condition, the values are much less than if the rigid surface condition had been used. If a circular cylinder, half immersed, is oscillating horizontally at right angles to its axis the inertia coefficient is $4/\pi^2$ compared with the usual value unity. For a log of square cross-section, Wendel[4] has calculated that the value for horizontal motion is about 0·337 of the value for vertical motion; for a general account of virtual inertia coefficients reference may be made here to a recent paper by Weinblum.[5] Calculations for three-dimensional motion do not seem to have been published, though no doubt the general nature of the results is known. We give in § 12, general expressions for a prolate spheroid, half immersed, from which the inertia coefficients could be found for the various types of horizontal motion we have been considering; these include translation, rotation, and 2-node and 3-node vibrations. Approximate calculations have been made for the particular case of a length-beam ratio of 10, and these indicate that the values are of the order of 0·4 of the values for a deeply submerged spheroid.

5. The last section deals with the same problem for water of finite depth. Here the mathematical difficulties are such as to preclude a general form of solution for the various types of vibration. However, taking the simplest type $n = 1$, an approximation is obtained in (53) for the relative increase in inertia coefficient due to the finite depth of water; it is considered that this approximation is sufficient to show the essential character of the effect of depth of water. Taking the same particular case of a length/beam ratio of 10, numerical computations have been made from this expression and the results are shown in the curve labelled OH in Fig. 1. The two curves to be compared are the curves OV and OH; they are both for the same type of vibration, the former being vertical and the latter horizontal. The point of special interest is the remarkable difference between vertical and horizontal vibrations as regards the influence of shallow water. This difference is expressed simply if we take the approximate values for a long spheroid; in that case, it is easily shown that the expression (53) for horizontal motion is of the order of $(b/f)^4$, while we have already seen that for vertical motion the approximation is of order $(b/f)^2$. This may be confirmed by working out a simple two-dimensional case, a circular cylinder half-immersed. In this case the conditions of the problem may be satisfied to any required degree of accuracy in the ratio b/f; it may be sufficient to state the results here. If the motion is vertical, the inertia coefficient in deep water is unity; the relative increase in shallow water is given by

$$0{\cdot}8225(b/f)^2 + 0{\cdot}3382(b/f)^4 + 0{\cdot}1391(b/f)^6 + \ldots$$

2

If the motion is horizontal the inertia coefficient in deep water is $4/\pi^2$, and the relative increase in shallow water is given by $0{\cdot}6314(b/f)^4 - 0{\cdot}2190(b/f)^6 + \ldots$ If we graph these two expressions we obtain curves of the same character as the curves OV and OH in Fig. 1.

As regards observed results for actual ship vibrations, it has been stated that there is no measurable change of frequency of horizontal vibrations in shallow water, in striking contrast to the observations on vertical vibrations. If that should prove to be the case, it would confirm the assumptions underlying the present analysis; however, it would be of value to have a direct examination of the problem under conditions which would allow both of theoretical calculation and of precise experimental determination.

PART II

Infinite Liquid

6. Consider a prolate spheroid, of semi-axes a, b and eccentricity e, in an infinite liquid. We take axes with O at the centre, Ox along the axis of the spheroid, Oy transversely, and Oz vertically downwards. We shall use non-dimensional space-co-ordinates, giving the ratio of any distance to the length $a\,e$. We have then spheroidal co-ordinates (μ, ζ, ω) with

$$x = \mu\zeta;\quad y = (1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega;$$
$$z = (1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega \quad . \quad (1)$$

The spheroid is given in these co-ordinates by $\zeta = \zeta_0 = 1/e$. Consider the fluid motion given by the velocity potential

$$\phi = C\,P_n^1(\mu)Q_n^1(\zeta)\sin\omega\cos\sigma t \quad . \quad . \quad (2)$$

This motion would be produced by a distribution of normal velocity over the surface of the spheroid given by

$$-\partial\phi/\partial\nu = -(C/ae)[(1-e^2)/(1-e^2\mu^2)]^{\frac{1}{2}}$$
$$P_n^1(\mu)\dot{Q}_n^1(\zeta_0)\sin\omega\cos\sigma t \quad . \quad (3)$$

where the dot denotes differentiation, a notation we shall use throughout. We make the usual approximation for vibrations of the spheroid of small amplitude, assuming this to be equivalent to a distribution of normal velocity given over the spheroid in its mean position.

It is well known that, with suitable values of the constant C, for $n = 1$ or $n = 2$ the fluid motion given by (2) can be produced by motion of the spheroid as a rigid body; if $n = 1$, this motion is translation parallel to Oz, while if $n = 2$ it is rotation round Oy (e.g. Lamb, *Hydrodynamics*, p. 141). For higher orders of harmonics, deformation of the spheroid is necessary The present application is, chiefly, to the transverse flexural vibrations of a spheroid of large ratio of length to beam. We may then regard the deformation as a simple shear of transverse sections of the spheroid. It can be shown that the normal velocity (3) is produced by such a transverse motion with the velocity distribution along the axis proportional to $P_n(x/a)$. For instance, with $n = 3$ the nodes of the vibration are given by $x/a = \pm\frac{1}{5}\sqrt{5}$, while for $n = 4$ we have a 3-node vibration with nodes at $x/a = 0, \pm\sqrt{3/7}$. It is possible to improve this approximation to the natural vibrations, as pointed out by Lewis[1] and by Lockwood Taylor,[2] by taking combinations of spheroidal harmonics or by other refinements. But the additional complication is not worth while for the present purpose; we are concerned not so much with the absolute value of the inertia coefficient as with its relative increase in shallow water.

From (2) and (3) we obtain the kinetic energy of the fluid by integrating over the surface of the spheroid; and we have

$$T = -\tfrac{1}{2}\rho\int\phi\,(\partial\phi/\partial\nu)\,dS \quad . \quad . \quad . \quad (4)$$
$$= -\pi\rho a(1-e^2)/e\,.\,[n(n+1)/(2n+1)]$$
$$C^2 Q_n^1(\zeta_0)\,\dot{Q}_n^1(\zeta_0)\cos^2\sigma t \quad . \quad (5)$$

The kinetic energy of the spheroid can be obtained from the corresponding velocity distribution in the solid, and hence the virtual inertia coefficient; but these results are already known.

Semi-Infinite Liquid, with Rigid Boundary

7. Let the axis of the spheroid be parallel to a plane rigid boundary given by $z = f/a\,e$. If T_0 is the kinetic energy of the fluid for a given type of motion when the spheroid is in an infinite liquid, and δT is the increase in kinetic energy due to the boundary, we are concerned with the ratio $\delta T/T_0$, which is, of course, the relative increase in the corresponding virtual inertia coefficient. If we imagine this quantity expressed in powers of the ratio b/f, the approximation at which we aim is the leading term in such an expression. This can be obtained in the following way. Let ϕ_0 give the motion in an infinite liquid with the given normal velocity over the spheroid. Let ϕ_1 be the image of this in the boundary, giving zero normal velocity over the boundary; and let ϕ_2 be the image of ϕ_1 in the spheroid, so that the normal velocity over the spheroid is unaltered. Then, using $\phi_0 + \phi_1 + \phi_2$ in the usual surface integral for the kinetic energy, we obtain this approximation.

It is convenient to give here some formulae which are used throughout the analysis.

We require the expansion of the inverse distance between two points whose spheroidal co-ordinates are $(\mu\,\zeta\,\omega)$ and $(\mu_1\,\zeta_1\,\omega_1)$; this is (Hobson[6])

$$r^{-1} = \sum_0^\infty (2n+1)\,P_n(\mu_1)\,Q_n(\zeta_1)\,P_n(\mu)\,P_n(\zeta)$$
$$+ 2\sum_1^\infty (2n+1)\sum_{s=1}^{n}(-1)^s\left[\frac{(n-s)!}{(n+s)!}\right]^2$$
$$P_n^s(\mu_1)\,Q_n^s(\zeta_1)\,P_n^s(\mu)\,P_n^s(\zeta)\cos s(\omega_1-\omega) \quad (6)$$

an expansion which is valid for $\zeta_1 > \zeta$.

We also need the relation

$$P_n^s(\zeta)\,\dot{Q}_n^s(\zeta) - \dot{P}_n^s(\zeta)\,Q_n^s(\zeta)$$
$$= (-1)^{s+1}\left[(n+s)!/(n-s)!\right]/(\zeta^2-1) \quad . \quad (7)$$

Another relation, given by the present writer[7] is largely the basis of the following calculations; it is, in the present notation,

$$P_n^s(\mu)\, Q_n^s(\zeta)\, e^{is\omega} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^s \int_{-1}^{1} \frac{(1-h^2)^{\frac{1}{2}s}\, P_n^s(h)\, dh}{[(x-h)^2 + y^2 + z^2]^{\frac{1}{2}}} \quad . \quad (8)$$

This result expresses the spheroidal harmonic as a line distribution of multi-poles along the axis of the spheroid between the two foci.

Transverse Motion at Right Angles to the Boundary

8. We begin with the form for an infinite liquid,

$$\phi_0 = P_n^1(\mu)\, Q_n^1(\zeta) \sin\omega \quad . \quad . \quad . \quad (9)$$

For convenience, we omit the time factor and the constant C. Take parallel axes O′(x', y', z') with the origin O′ at the point $x = 0$, $y = 0$, $z = 2f/a\,e$; and let μ', ζ', ω' be spheroidal co-ordinates referred to these new axes. It is easily seen that to maintain zero normal velocity over the plane $z = f/a\,e$, we must take

$$\phi_1 = -P_n^1(\mu')\, Q_n^1(\zeta') \sin\omega' \quad . \quad . \quad (10)$$

To obtain ϕ_2 we expand (10) in the neighbourhood of the spheroid $\zeta = \zeta_0$ in the form

$$-\sum_{n=0}^{\infty} \sum_{s=0}^{n} (A_n^s \cos s\,\omega + B_n^s \sin s\,\omega)\, P_n^s(\mu)\, P_n^s(\zeta) \quad (11)$$

Then ϕ_2 is given by

$$\phi_2 = \Sigma\,\Sigma\, (A_n^s \cos s\,\omega + B_n^s \sin s\,\omega)\, P_n^s(\mu)\, Q_n^s(\zeta)\, \dot{P}_n^s(\zeta_0)/\dot{Q}_n^s(\zeta_0) \quad . \quad . \quad . \quad (12)$$

General expressions for the coefficients could be obtained; but it is easily seen that in order to calculate the kinetic energy we only need the coefficient B_n^1, noting that the normal velocity on the spheroid is unaltered and also using the orthogonal properties of the functions. From (8) we have

$$P_n^1(\mu')\, Q_n^1(\zeta') \sin\omega' = \frac{1}{2}\frac{\partial}{\partial z'}\int_{-1}^{1} \frac{(1-k^2)^{\frac{1}{2}}\, P_n^1(k)\, dk}{[(x'-k)^2 + y'^2 + z'^2]^{\frac{1}{2}}}$$

$$= -\frac{1}{2}\frac{\partial}{\partial q}\int_{-1}^{1} \frac{(1-k^2)^{\frac{1}{2}}\, P_n^1(k)\, dk}{[(x-k)^2 + y^2 + (z-q)^2]^{\frac{1}{2}}} \quad . \quad . \quad . \quad (13)$$

with $q = 2f/a\,e$. If the point ($x\;y\;z$) is ($\mu\;\zeta\;\omega$) and the point ($k\;o\;q$) is ($\mu_1\;\zeta_1\;\omega_1$) in the same spheroidal co-ordinates, we have in (6) the expansion of the inverse distance between these two points, with

$$k = \mu_1\,\zeta_1;\; 0 = (1-\mu_1^2)^{\frac{1}{2}}\,(\zeta_1^2 - 1)^{\frac{1}{2}} \cos\omega_1;$$

$$q = (1-\mu_1^2)^{\frac{1}{2}}\,(\zeta_1^2 - 1)^{\frac{1}{2}} \sin\omega_1$$

Further, the expansion is valid in the neighbourhood of the spheroid, since $\zeta_1 > \zeta_0$.

We substitute this expansion for the denominator in (13), and B_n^1 is the coefficient of the term $P_n^1(\mu)\, P_n^1(\zeta) \sin\omega$.

Hence, from (6) and (13) we have

$$B_n^1 = \frac{2n+1}{n^2(n+1)^2}\frac{\partial}{\partial q}\int_{-1}^{1} (1-k^2)^{\frac{1}{2}}\, P_n^1(k)\, P_n^1(\mu_1)\, Q_n^1(\zeta_1) \sin\omega_1\, dk \quad (14)$$

We may put this into a symmetrical form by noting that $P_n^1(\mu_1)\, Q_n^1(\zeta_1) \sin\omega_1$ is the value of this spheroidal harmonic at the point ($k\;o\;q$); hence, from (8)

$$P_n^1(\mu_1)\, Q_n^1(\zeta_1) \sin\omega_1 = \frac{1}{2}\frac{\partial}{\partial q}\int_{-1}^{1} \frac{(1-h^2)^{\frac{1}{2}}\, P_n^1(h)\, dh}{[(k-h)^2 + q^2]^{\frac{1}{2}}} \quad (15)$$

This gives

$$B_n^1 = \frac{2n+1}{n^2(n+1)^2}\frac{\partial^2}{dq^2}\int_{-1}^{1}\int_{-1}^{1} \frac{(1-h^2)^{\frac{1}{2}}\,(1-k^2)^{\frac{1}{2}}\, P_n^1(h)\, P_n^1(k)}{[(k-h)^2 + q^2]^{\frac{1}{2}}}\, dh\, dk \quad (16)$$

To calculate the kinetic energy, we see that $\partial\phi/\partial\zeta$ on the spheroid is $P_n^1(\mu)\, \dot{Q}_n^1(\zeta_0) \sin\omega$; and from (9), (11), (12) the corresponding term in the value of ϕ on the spheroid is

$$[Q_n^1(\zeta_0) - B_n^1\, P_n^1(\zeta_0) + B_n^1\, \dot{P}_n^1(\zeta_0)\, Q_n^1(\zeta_0)]/\dot{Q}_n^1(\zeta_0)\,.\, P_n^1(\mu) \sin\omega$$

or using (7)

$$[1 - n(n+1)\, B_n^1/(\zeta_0^2 - 1)\, Q_n^1(\zeta_0)\, \dot{Q}_n^1(\zeta_0)]\; Q_n^1(\zeta_0)\, P_n^1(\mu) \sin\omega \quad . \quad (17)$$

It is obvious that the kinetic energy is increased by the factor within square brackets in (17); hence from (16) and (17), the relative increase in kinetic energy, or in the inertia coefficient is given by

$$\delta T/T_0 = -(2n+1)\, D_n/2n(n+1)(\zeta_0^2 - 1)\, Q_n^1(\zeta_0)\, \dot{Q}_n^1(\zeta_0) \quad . \quad (18)$$

with

$$D_n = \frac{\partial^2}{\partial q^2}\int_{-1}^{1}\int_{-1}^{1} \frac{(1-h^2)^{\frac{1}{2}}\,(1-k^2)^{\frac{1}{2}}\, P_n^1(h)\, P_n^1(k)}{[(k-h)^2 + q^2]^{\frac{1}{2}}}\, dh\, dk \quad . \quad . \quad . \quad (19)$$

Transverse Motion Parallel to the Boundary

9. When the vibrations are parallel to the boundary, we begin with

$$\phi_0 = P_n^1(\mu)\, Q_n^1(\zeta) \cos\omega \quad . \quad . \quad . \quad (20)$$

In this case, we have

$$\phi_1 = P_n^1(\mu')\,Q_n^1(\zeta')\cos\omega'$$

$$= -\tfrac{1}{2}\lim_{p\to 0}\frac{\partial}{\partial p}\int_{-1}^{1}\frac{(1-k^2)^{\frac12}\,P_n^1(k)\,dk}{[(x-k)^2+(y-p)^2+(z-q)^2]^{\frac12}} \qquad (21)$$

Using the expansion (6) in (21), with the point $(k\,p\,q)$ being $(\mu_1\,\zeta_1\,\omega_1)$ in spheroidal co-ordinates, we obtain the coefficient of $P_n^1(\mu)\,P_n^1(\zeta)\cos\omega$ as

$$A_n^1 = \frac{2n+1}{n^2(n+1)^2}\frac{\partial}{\partial p}$$

$$\int_{-1}^{1}(1-k^2)^{\frac12}\,P_n^1(k)\,P_n^1(\mu_1)\,Q_n^1(\zeta_1)\cos\omega_1\,dk \qquad (22)$$

Further, we have

$$P_n^1(\mu_1)\,Q_n^1(\zeta_1)\cos\omega_1 = \frac{1}{2}\frac{\partial}{\partial p}\int_{-1}^{1}\frac{(1-h^2)\,P_n^1(h)\,dh}{[(k-h)^2+p^2+q^2]^{\frac12}} \qquad (23)$$

Taking the limit of $\partial^2/\partial p^2$ as $p\to 0$, this leads to a fractional increase of kinetic energy, or of the inertia coefficient, given by

$$\delta T/T_0 =$$

$$-(2n+1)\,E_n/2n(n+1)(\zeta_0^2-1)\,Q_n^1(\zeta_0)\,\dot{Q}_n^1(\zeta_0) \qquad (24)$$

with

$$E_n = \int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)^{\frac12}(1-k^2)^{\frac12}\,P_n^1(h)\,P_n^1(k)}{[(k-h)^2+q^2]^{3/2}}\,dh\,dk \qquad (25)$$

These integrals can, of course, be evaluated explicitly in closed expressions for any given value of n. However, the expressions become very lengthy for the higher orders and we shall not reproduce them here in general. For numerical computation it proved somewhat better to express the double integrals in terms of subsidiary single integrals. Also one can obtain, either from the double integrals or from the explicit expressions, approximate forms suitable for q large or q small.

10. We give now the results for some special cases.

$n = 1$. For motion at right angles to the boundary, (18) and (19) give

$$\delta T/T_0 = -\tfrac{3}{4}\,D/(\zeta_0^2-1)\,Q_1^1(\zeta_0)\,\dot{Q}_1^1(\zeta_0) \quad . \quad (26)$$

with

$$D = \frac{\partial^2}{\partial q^2}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)(1-k^2)}{[(k-h)^2+q^2]^{\frac12}}\,dh\,dk$$

$$= \tfrac{8}{3}\log\{[2+(4+q^2)^{\frac12}]/q\}$$

$$+\tfrac{1}{45}(12\,q^{-2}-41-19\,q^2-2\,q^4)(4+q^2)^{\frac12}$$

$$+\tfrac{16}{3}\,q+\tfrac{4}{45}\,q^3 \quad . \quad (27)$$

For motion parallel to the boundary, (24) and (25) give

$$\delta T/T_0 = -\tfrac{3}{4}\,E/(\zeta_0^2-1)\,Q_1^1(\zeta_0)\,Q_1^1(\zeta_0) \quad . \quad (28)$$

with

$$E = \int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)(1-k^2)}{[(k-h)^2+q^2]^{3/2}}\,dh\,dk$$

$$= -\tfrac{8}{3}\{\log[2+(4+q^2)^{\frac12}]/q\}$$

$$+2(\tfrac{8}{15}\,q^{-2}+\tfrac{52}{45}+\tfrac{4}{45}\,q^2)(4+q^2)^{\frac12}$$

$$-\tfrac{8}{3}\,q-\tfrac{8}{45}\,q^3 \qquad (29)$$

It is readily verified that as $q\to\infty$, or $e\to 0$, we recover the known results for a sphere, namely $3\,b^3/8\,f^3$ and $3b^3/16f^3$ respectively. We may also find the limiting values for a long spheroid with b/a small, by making $e\to 1$, $a\to\infty$, while retaining $b^2 = a^2(1-e^2)$. The limiting values are the same for the two types of motion, as is the case for a circular cylinder; but the value of the ratio is $2b^2/5f^2$ instead of $b^2/2f^2$. Thus in this respect the circular cylinder is not the limit of a long spheroid. This value can also be obtained directly by the two-dimensional strip method. For this purpose we consider a circular section of the spheroid of radius y; take its contribution to the kinetic energy of the fluid motion as proportional to $y^2(1+y^2/2f^2)$, and integrate along the axis of the spheroid.

$n = 2$. Considering only motion in a vertical plane (18) and (19) give

$$\delta T/T_0 = -15\,A/4\,(\zeta_0^2-1)\,Q_2^1(\zeta_0)\,\dot{Q}_2^1(\zeta_0)$$

with

$$A = \frac{\partial^2}{\partial q^2}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)(1-k^2)\,h\,k}{[(k-h)^2+q^2]^{\frac12}}\,dh\,dk \qquad (30)$$

$n = 3$. For a 2-node vertical vibration

$$\delta T/T_0 = -21\,A/32\,(\zeta_0^2-1)\,Q_3^1(\zeta_0)\,\dot{Q}_3^1(\zeta_0)$$

with

$$A = \frac{\partial^2}{\partial q^2}\int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)(5h^2-1)(1-k^2)(5k^2-1)}{[(k-h)^2+q^2]^{\frac12}} \quad . \quad . \quad . \quad (31)$$

$n = 4$. For a 3-node vertical vibration,

$$\delta T/T_0 = -45\,A/32\,(\zeta_0^2-1)\,Q_4^1(\zeta_0)\,\dot{Q}_4^1(\zeta_0)$$

with

$$A = \frac{\partial^2}{\partial q^2}$$

$$\int_{-1}^{1}\int_{-1}^{1}\frac{(1-h^2)(7h^3-3h)(1-k^2)(7k^3-3k)}{[(k-h)^2+q^2]^{\frac12}}\,dh\,dk \quad . \quad . \quad . \quad (32)$$

For a long spheroid the limiting values of these expressions are [illegible], [illegible], [illegible] b^2/f^2 for $n = 2, 3, 4$ respectively. These can also be obtained by the two-dimensional strip method, taking into account the distribution of transverse velocity along the axis of the spheroid.

Vertical Oscillations of a Floating Spheroid

11. We suppose the spheroid to be floating with one-half immersed. As before, O is at the centre of the spheroid, and Oz vertically downwards; the free surface of the water is the plane $z = 0$, while the bed is the plane $z = f/a\,e$. We have now to consider the condition at the free surface. For simple harmonic motion of frequency σ we have the usual linearized condition $\phi + (g/\sigma^2)\,\partial\phi/\partial z = 0$. Using this condition we should have to take into account the wave motion of the free surface, but that is beyond the scope of the present work. There are two limiting simplifications which may be made according to the conditions of the particular problem; we may take $\phi = 0$, the free surface condition neglecting gravity, or we may take $\partial\phi/\partial z = 0$, the rigid surface condition. Taking into account the frequency of ship vibrations, the appropriate alternative seems to be $\phi = 0$; the measure of agreement between calculated and observed frequencies justifies this assumption as a working hypothesis.

The conditions for the velocity potential are now (i) the given normal velocity over the submerged half of the spheroid, (ii) $\partial\phi/\partial z = 0$ for $z = f/a\,e$, (iii) $\phi = 0$ for $z = 0$. This is the same as considering the complete spheroid in water contained between two parallel planes $z = \pm f/a\,e$, with the normal velocity given over the whole surface. For a vertical vibration we begin, as before, with

$$\phi_0 = P_n^1(\mu)\,Q_n^1(\zeta)\sin\omega \quad . \quad . \quad . \quad (33)$$

In order to satisfy conditions (ii) and (iii) we now have an infinite series of image systems alternatively positive and negative, associated with the points $z = \pm 2\,s f/a\,e$.

Hence we have

$$\phi_1 = -2\sum_{s=1}^{\infty}(-1)^{s-1}P_n^1(\mu_s)\,Q_n^1(\zeta_s)\sin\omega_s \quad (34)$$

We have obtained in the previous sections, the value of ϕ_2 for any one of these image systems, and also its contribution to $\delta\,T/T_0$. Hence for the vertical vibrations we have

$$\delta\,T/T_0 = -(2n+1)\sum_{1}^{\infty}(-1)^{s-1}D_{ns}/\,n(n+1)(\zeta_0^2-1)\,Q_n^1(\zeta_0)\,\dot{Q}_n^1(\zeta_0) \quad . \quad (35)$$

with D_{ns} given by (19) with $q = 2\,s f/a\,e$.

For instance, for $n = 1$, $\delta\,T/T_0$ is given by (26) with

$$D = 2\sum_{1}^{\infty}(-1)^{s-1}D_s \quad . \quad . \quad . \quad (36)$$

with D_s given by (27) with $q = 2\,s f/a\,e$.

The limiting form of this result for a long spheroid is $(2\,b^2/5\,f^2)\,\Sigma(-1)^{s-1}s^{-2}$, or $\pi^2 b^2/30\,f^2$.

For any given case, having computed and graphed the double integral involved as a function of the parameter q, it is a simple matter to obtain from the graph the summation with respect to s.

Horizontal Vibrations of Floating Spheroid

12. *Deep water.* If we retain the same condition at the free surface, the horizontal vibrations are a more difficult problem. With the given normal velocity on the immersed half of the spheroid, the conditions are now

$$\partial\phi_0/\partial\zeta = P_n^1(\mu)\cos\omega;\ \ \zeta = \zeta_0;\ \ 0 \leqslant \omega \leqslant \pi$$
$$\phi_0 = 0;\ \ \omega = 0 \quad . \quad . \quad . \quad . \quad (37)$$

This is equivalent to considering the complete spheroid in an infinite liquid with the conditions

$$\partial\phi_0/\partial\zeta = P_n^1(\mu)\cos\omega;\ \ 0 \leqslant \omega \leqslant \pi$$
$$\partial\phi_0/\partial\zeta = -P_n^1(\mu)\cos\omega;\ \ -\pi \leqslant \omega \leqslant 0$$
$$\phi_0 = 0;\ \ \omega = 0 \quad . \quad . \quad . \quad . \quad (38)$$

To satisfy these conditions, we express the value of $\partial\phi_0/\partial\zeta$ in a suitable infinite series of Legendre functions, of the form

$$\sum_m\sum_s A_m^s\,P_m^s(\mu)\sin s\,\omega \quad . \quad . \quad . \quad (39)$$

Forming the series by the usual methods, it is seen that s must be even, and the coefficients are given by

$$\frac{2\pi}{2m+1}\,\frac{(m+s)!}{(m-s)!}\,A_m^s = \frac{4\,s}{s^2-1}\int_{-1}^{1}P_n^1(\mu)\,P_m^s(\mu)\,d\mu \quad (40)$$

It follows that if n is even, the coefficients are only different from zero if m is odd; while if n is odd, m must be even. The velocity potential is then given by

$$\phi_0 = \sum_m\sum_s A_m^s/\dot{Q}_m^s(\zeta_0)\,.\,P_m^s(\mu)\,Q_m^s(\zeta)\sin s\,\omega \quad (41)$$

This form of solution gives the assigned normal velocity on the spheroid for all points other than those for which ω is actually zero, that is for points not actually on the water line. There is, in fact, a discontinuity in the normal velocity on crossing the water line; there will be a corresponding infinity in the tangential velocity at these points. However, as in similar problems involving what is effectively flow round a sharp edge, the usual surface integrals for the kinetic energy lead to a finite result.

From (38) and (41) we may obtain the kinetic energy, and if we introduce the suitable factor according to the motion of the solid, the corresponding inertia coefficient can be calculated.

13. *Water of Finite Depth.*—If the water is of finite depth we should have, as in § 11, an infinite series of image systems in subsidiary spheroidal co-ordinates, each system being an infinite series of terms. Further, expanding any one term so as to obtain the image in the spheroid would involve infinite series. Finally, in contrast to the conditions for vertical motion, from the form of the conditions all the terms in the series contribute to the kinetic energy. Thus, in general, the method becomes much too complicated.

However, when we form the expression for the kinetic energy for cases with which we are dealing, it appears that the first few terms of the series account for much

the greater part, and a useful approximation is obtained by taking only one or two terms. For instance, if we take the simplest case $n = 1$, representing a transverse vibration with no node, we have, omitting the time factor and any constant multiplier

$$\phi_0 = A_2^2/\dot{Q}_2^2(\zeta_0) \, . \, P_2^2(\mu)\, Q_2^2(\zeta) \sin 2\omega$$
$$+ A_4^2/\dot{Q}_4^2(\zeta_0) \, . \, P_4^2(\mu)\, Q_4^2(\zeta) \sin 2\omega + \ldots \quad (42)$$

the coefficients being given by (40) with $n = 1$. This gives $A_2^2 = 5/16$, $A_4^2 = 1/64, \ldots$

For the kinetic energy for the complete spheroid in an infinite liquid, we have the usual surface integral (4), in which $\partial\phi/\partial\zeta = \pm P_1^1(\mu)\cos\omega$ according as ω is positive or negative. We obtain

$$T = -\tfrac{1}{2}\pi\rho a(1-e^2)/e \, . \, [\tfrac{15}{16} Q_2^2(\zeta_0)/$$
$$\dot{Q}_2^2(\zeta_0) + \tfrac{5}{264} Q_4^2(\zeta_0)/\dot{Q}_4^2(\zeta_0) + \ldots] \quad (43)$$

For the case we have been using for numerical computation, $a e = 10 b$, the terms in the square brackets in (43) are $0{\cdot}9375 + 0{\cdot}0195 + \ldots$ Although the rest of the series converges rather slowly, much the greater part is given by the first term.

Consider now the same problem in water of depth f. To avoid prohibitive complications, we shall take only the first term. Although this leaves somewhat uncertain the degree of approximation, yet, as we are concerned more with relative increase than with absolute value, we may expect to get at least the main character of the variation with depth of water. Omitting unnecessary factors, we begin, in the notation of previous sections, with

$$\phi_0 = P_2^2(\mu)\, Q_2^2(\zeta) \sin 2\omega \quad (44)$$

and suppose the bed to be given by $z = f/a e$.

We now have an infinite series of image systems associated with the points $z = \pm 2 s f/a e$, and we have

$$\phi_1 = -2 \sum_{s=1}^{\infty} (-1)^{s-1} P_2^2(\mu_s)\, Q_2^2(\zeta_s) \sin 2\omega_s \quad (45)$$

To expand a typical system associated with $z = q$, we have, from (8),

$$P_2^2(\mu_s)\, Q_2^2(\zeta_s) \sin 2\omega_s$$
$$= \frac{\partial^2}{\partial y\, \partial z} \int_{-1}^{1} \frac{(1-k^2)\, P_2^2(k)\, dk}{[(x-k)^2 + y^2 + (z-q)^2]^{\frac{3}{2}}}$$
$$= \lim_{p \to 0} \frac{\partial^2}{\partial p\, \partial q} \int_{-1}^{1} \frac{(1-k^2)\, P_2^2(k)\, dk}{[(x-k)^2 + (y-p)^2 + (z-q)^2]^{\frac{3}{2}}} \quad (46)$$

with $k = \mu_1 \zeta_1$; $p = (1-\mu_1^2)^{\frac{1}{2}} (\zeta_1^2 - 1)^{\frac{1}{2}} \cos\omega_1$; $q = (1-\mu_1^2)^{\frac{1}{2}} (\zeta_1^2 - 1)^{\frac{1}{2}} \sin\omega_1$; we select the required term using the expansion (6), giving for the coefficient of $P_2^2(\mu)\, Q_2^2(\zeta) \sin 2\omega$ the expression

$$\frac{5}{288} \frac{\partial^2}{\partial p\, \partial q} \int_{-1}^{1} (1-k^2)\, P_2^2(k)\, P_2^2(\mu_1)\, Q_2^2(\zeta_1) \sin 2\omega_1 \, dk \quad (47)$$

Further we have

$$P_2^2(\mu_1)\, Q_2^2(\zeta_1) \sin 2\omega_1 = \frac{\partial^2}{\partial p\, \partial q} \int_{-1}^{1} \frac{(1-h^2)\, P_2^2(h)\, dh}{[(k-h)^2 + p^2 + q^2]^{\frac{3}{2}}} \quad (48)$$

Putting this into (46), and taking the limit as $p \to 0$, the contribution from a typical term may be written as $-\frac{5}{32} B_s$, with

$$B_s = \frac{\partial^2}{\partial q^2} \int_{-1}^{1} \int_{-1}^{1} \frac{(1-h^2)^2 (1-k^2)^2\, dh\, dk}{[(k-h)^2 + q^2]^{\frac{3}{2}}} \quad (49)$$

and the required term in the expansion is

$$\frac{5}{16} \sum_{s=1}^{\infty} (-1)^{s-1} B_s\, P_2^2(\mu)\, P_2^2(\zeta) \sin 2\omega \quad (50)$$

to which, in order to maintain the normal velocity on the spheroid, we add

$$-\frac{5}{16} \sum_{1}^{\infty} (-1)^{s-1} B_s\, \dot{P}_2^2(\zeta_0)/\dot{Q}_2^2(\zeta_0) \, . \, P_2^2(\mu)\, Q_2^2(\zeta) \sin 2\omega \quad (51)$$

Finally, after using (7), the value of ϕ on the spheroid, to this approximation, is

$$Q_2^2(\zeta_0)\, [1 - 15 \Sigma (-1)^{s-1} B_s /$$
$$2(\zeta_0^2 - 1)\, Q_2^2(\zeta_0)\, \dot{Q}_2^2(\zeta_0)]\, P_2^2(\mu) \sin 2\omega \quad (52)$$

Since the value of $\partial\phi/\partial\zeta$ on the spheroid is unaltered, the kinetic energy is increased by the factor in square brackets in (52). Hence the relative increase in kinetic energy, or in the inertia coefficient is given by

$$\delta T/T_0 = -15 \Sigma (-1)^{s-1} B_s / 2(\zeta_0^2 - 1)\, Q_2^2(\zeta_0)\, \dot{Q}_2^2(\zeta_0) \quad (53)$$

with B_s given by (49) in which $q = 2 s f/a e$.

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ISSUED FOR WRITTEN DISCUSSION

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THE FORCES ON A SUBMERGED BODY MOVING UNDER WAVES

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PART I. MOTION NORMAL TO THE WAVE CRESTS

Summary

A theoretical investigation of the forces and moments on a submerged spheroid moving close to the surface under waves. Expressions are obtained for the surging force, heaving force and pitching moment taking into account the speed of advance and also the disturbance of the wave train by the solid; graphs are given to show the variation of these quantities with the speed of advance and with the wavelength.

1. Introduction

The theory of the forces on a submerged spheroid moving through smooth water was examined in a previous paper (Ref. 1), and has been discussed in detail recently by Wigley in these TRANSACTIONS (Ref. 2). An interesting and important extension is the same problem when the spheroid is moving steadily either with or against a regular train of transverse waves. In considering the similar problem for a surface ship it is usual to assume the pressure on the ship to be that due to the undisturbed wave train, as, for example, in the so-called "Smith effect" or as in the classical theory of the motion of ships among waves as developed by Froude, Kriloff and others; broadly speaking, this is equivalent to neglecting the various virtual inertia coefficients of the ship. Moreover, the effect of the speed of advance of the ship is assumed to be simply an alteration of the frequency of encounter with the waves. A more adequate theory for surface ships presents great difficulties; however, for various reasons, it is possible to carry the theory further for a wholly submerged body under certain conditions, and the present paper deals with the motion of a submerged prolate spheroid. The mathematical analysis is given in Sections 2, 3, 4, and the notation and main results are summarized in Section 5. General remarks are made in Section 6, together with graphs for the force and the moment coefficients; a point of special interest is the effect of speed of advance and the difference between moving against the waves or with the waves.

2. Velocity Potential

A prolate spheroid, of major axis $2a$ and eccentricity e, is moving axially under water with velocity V parallel to the surface and there is a regular train of transverse waves, of wavelength $2\pi/\kappa$ and wave velocity c, moving in the opposite direction; the axis is at a depth d below the surface. It is convenient to reduce the spheroid to relative rest by superposing a uniform stream V in the opposite direction. We now take fixed axes with the origin O at the centre of the spheroid, Ox axially, Oy transversely, and Oz vertically upwards. We begin with the velocity potential for a spheroid in a uniform stream (as given for instance in Ref. 3),

$$\phi = Vx - aeVP_1(\mu)Q_1(\zeta)/\dot{Q}_1(\zeta_0) \quad . \quad (1)$$

when the dot denotes differentiation, and the spheroidal co-ordinates ζ, μ, ω are given by

$$\begin{aligned} x &= ae\mu\zeta; \\ y &= ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\sin\omega; \\ z &= ae(1-\mu^2)^{\frac{1}{2}}(\zeta^2-1)^{\frac{1}{2}}\cos\omega \quad . \quad . \quad (2) \end{aligned}$$

In these co-ordinates the spheroid is given by $\zeta = \zeta_0 = 1/e$. We add the velocity potential ϕ_1 giving the assigned train of waves moving in the negative direction of Ox on the surface of the stream; it is easily verified that

$$\phi_1 = -hce^{-\kappa d+\kappa z}\cos(\kappa x + \sigma t + \alpha) \quad . \quad (3)$$

with α an arbitrary phase angle, gives a wave train on the surface with elevation

$$\eta = h\sin(\kappa x + \sigma t + \alpha) \quad . \quad . \quad . \quad (4)$$

provided

$$\sigma = \kappa(V + c); \; c^2 = g/\kappa$$

We now take

$$\phi = Vx - aeVP_1(\mu)Q_1(\zeta)/\dot{Q}_1(\zeta_0) + \phi_1 + \phi_2 \; . \; (5)$$

with ϕ_1 given by (3). ϕ_2 represents the disturbance of the wave-train by the spheroid, and is to be determined so that the normal fluid velocity is zero over the spheroid: thus we must have

$$\partial(\phi_1 + \phi_2)/\partial\zeta = 0; \quad \zeta = \zeta_0 \quad . \quad . \quad (6)$$

We should also add a further term to (5) to represent the steady wave pattern produced by the spheroid itself, which would be determined as in Ref. 1 for motion through smooth water; to a first approximation the forces due to the transverse waves would be simply additive. Meantime we shall assume the conditions to be such that the former forces are small compared with the latter, an assumption which can be checked by calculation from the results given here in Section 5 and those given in Refs. 1 and 2.

To determine ϕ_2 we have to expand ϕ_1 in a suitable series of spheroidal harmonics so that condition (6) may be used. We shall take ϕ_1 in the form

$$\phi_1 = h\,c\,e^{-\kappa d + \kappa z + i(\kappa x + \sigma t + \alpha)} \quad . \quad . \quad (7)$$

where the real part is to be taken.

We first expand $\exp(\kappa z + i\kappa x)$ on the surface of the spheroid in spherical harmonics. On $\zeta = \zeta_0 = 1/e$, we have $x = a\mu$, $z = b(1-\mu^2)^{\frac{1}{2}}\cos\omega$, and we assume

$$\exp\left[\kappa b(1-\mu^2)^{\frac{1}{2}}\cos\omega + i\kappa a\mu\right] = \sum_n\sum_s A_n^s\,P_n^s(\mu)\cos s\omega \quad . \quad (8)$$

By the usual process for determining the coefficients we have

$$A_n^s = \frac{2n+1}{2\pi}\frac{(n-s)!}{(n+s)!}\int_0^{2\pi}\cos s\omega\,d\omega\int_{-1}^{1}e^{\kappa b(1-\mu^2)^{\frac{1}{2}}\cos\omega + i\kappa a\mu}\,P_n^s(\mu)\,d\mu \quad . \quad (9)$$

noting that A_n is given by (9) with $s = 0$, but with a factor $\frac{1}{2}$.

Taking the integration in ω first, we have

$$\int_0^{2\pi}\exp\left[\kappa b(1-\mu^2)^{\frac{1}{2}}\cos\omega\right]\cos s\omega\,d\omega = 2\pi\,I_s\left[\kappa b(1-\mu^2)^{\frac{1}{2}}\right] \quad . \quad (10)$$

where I_s is the Bessel function of that type.

It can also be shown that

$$\int_{-1}^{1}P_n^s(\mu)\,I_s\left[\kappa b(1-\mu^2)^{\frac{1}{2}}\right]e^{i\kappa a\mu}\,d\mu = (2\pi)^{\frac{1}{2}}(i)^{n-s}\,P_n^s(\zeta_0)\,J_{n+\frac{1}{2}}(\kappa a e)/(\kappa a e)^{\frac{1}{2}} \quad . \quad (11)$$

J denoting the ordinary Bessel function.

Hence we have

$$A_n^s = (2\pi)^{\frac{1}{2}}(i)^{n-s}(2n+1)\frac{(n-s)!}{(n+s)!}P_n^s(\zeta_0)\,J_{n+\frac{1}{2}}(\kappa a e)/(\kappa a e)^{\frac{1}{2}} \quad . \quad (12)$$

A_n being given by $s = 0$, with a factor $\frac{1}{2}$.

Hence the required expansion is

$$\exp(\kappa z + i\kappa x) = \sum_n\sum_s A_n^s\,P_n^s(\mu)\,P_n^s(\zeta)/P_n^s(\zeta_0)\cos s\omega$$
$$= \left(\frac{2\pi}{\kappa a e}\right)^{\frac{1}{2}}\sum_{n=0}^{\infty}\sum_{s=0}^{n}(i)^{n-s}(2n+1)\frac{(n-s)!}{(n+s)!}J_{n+\frac{1}{2}}(\kappa a e)\,P_n^s(\mu)\,P_n^s(\zeta)\cos s\omega \quad . \quad (13)$$

with a factor $\frac{1}{2}$ for the terms with $s = 0$.

To satisfy (6) we take for ϕ_2 a similar series with the typical term

$$A_n^s\,P_n^s(\mu)\,Q_n^s(\zeta)\left[\dot{P}_n^s(\zeta_0)/P_n^s(\zeta_0)\,\dot{Q}_n^s(\zeta_0)\right]\cos s\omega \quad (14)$$

Hence $\phi_1 + \phi_2$ is given by the real part of

$$-h\,c\,e^{-\kappa d + i(\sigma t + \alpha)}\sum_n\sum_s A_n^s\,P_n^s(\mu)\frac{\dot{Q}_n^s(\zeta_0)\,P_n^s(\zeta) - \dot{P}_n^s(\zeta_0)\,Q_n^s(\zeta)}{P_n^s(\zeta_0)\,\dot{Q}_n^s(\zeta_0)}\cos s\omega \quad . \quad (15)$$

with the coefficients A given by (12).

3. Pressure Equation

The variable part of the pressure, omitting the buoyancy term, is given by

$$p = \rho\,\partial\phi/\partial t - \tfrac{1}{2}\rho\,(q_\zeta^2 + q_\mu^2 + q_\omega^2) \quad . \quad (16)$$

with q_ζ, q_μ, q_ω for component velocities in the three corresponding directions. On the spheroid, the normal velocity q_ζ is zero. Also the tangential component q_ω comes only from $\phi_1 + \phi_2$ and is of the first order, and as usual in wave-theory we neglect q_ω^2. So we have only to consider the tangential component q_μ, which is given on the spheroid, using (5), by

$$q_\mu = -\left[(1-\mu^2)^{\frac{1}{2}}/a e(\zeta_0^2 - \mu^2)^{\frac{1}{2}}\right]\partial\phi/\partial\mu$$
$$= -\frac{(1-\mu^2)^{\frac{1}{2}}}{a e(\zeta_0^2 - \mu^2)^{\frac{1}{2}}}\left\{a\,V\left[1 - e\frac{Q_1(\zeta_0)}{\dot{Q}_1(\zeta_0)}\right] + \frac{\partial}{\partial\mu}(\phi_1 + \phi_2)\right\} \quad . \quad (17)$$

The square of the term in V in (17) corresponds to the pressure on a spheroid in steady axial motion in an infinite liquid, and the resultant forces and moments due to this term are zero. The square of the second term in (17) is of the second order and is neglected, and we are left with only the product term. Hence we need only consider, on the spheroid,

$$p = \rho\frac{\partial}{\partial t}(\phi_1 + \phi_2) - \frac{\rho\,V(1+k_1)}{a e^2}\frac{1-\mu^2}{\zeta_0^2-\mu^2}\frac{\partial}{\partial\mu}(\phi_1 + \phi_2) \quad . \quad (18)$$

where we have simplified the form by using the fact that the axial virtual inertia coefficient k_1, is given by

$$k_1 = -e\,Q_1(\zeta_0)/\dot{Q}_1(\zeta_0) \quad . \quad . \quad . \quad (19)$$

Turning to (15) for the value of $\phi_1 + \phi_2$ on the spheroid, we use the relation

$$P_n^s\dot{Q}_n^s - \dot{P}_n^s Q_n^s = (-1)^{s+1}(n+s)!/(n-s)!\,(\zeta_0^2 - 1) \quad (20)$$

Introducing the coefficients A from (12) we obtain, on the spheroid

$$\phi_1 + \phi_2 = (2\pi/\kappa a e)^{\frac{1}{2}}\,h\,c\,e^{-\kappa d}\,B\,e^{i(\sigma t + \alpha)}$$

with

$$B = \sum_{n=0}^{\infty}\sum_{s=0}^{n}(2n+1)(i)^{n-s}J_{n+\frac{1}{2}}(\kappa a e)\,P_n^s(\mu)\cos s\omega/(\zeta_0^2 - 1)\,\dot{Q}_n^s(\zeta_0) \quad . \quad (21)$$

with a factor $\frac{1}{2}$ for the terms in $s = 0$.

Thus the effective pressure for calculating the forces is given by

$$p = \left(\frac{2\pi}{\kappa a e}\right)^{\frac{1}{2}}\rho\,h\,c\,e^{-\kappa d}\left[i\sigma B - \frac{(1+k_1)V(1-\mu^2)}{a e^2(\zeta_0^2-\mu^2)}\frac{\partial B}{\partial\mu}\right]e^{i(\sigma t + \alpha)} \quad . \quad (22)$$

with B given by (21), and noting that eventually we take the real part of the expressions.

4. Resultant Forces

If p is the fluid pressure at any point of the spheroid; (l, m, n) the direction-cosines of the normal; (X, Z) the resultant forces in the directions Ox, Oz; M the resultant moment round Oy, we have

$$\mathrm{X} = -\iint p\, l\, d\mathrm{S};\quad \mathrm{Z} = -\iint p\, n\, d\mathrm{S};$$
$$\mathrm{M} = -\iint p\,(l\,z - n\,x)\, d\mathrm{S} \quad . \quad . \quad (23)$$

taken over the surface of the spheroid. In the spheroidal co-ordinates (2) we have

$$l = \mu\,(\zeta_0^2 - 1)^{\frac{1}{2}}/(\zeta_0^2 - \mu^2)^{\frac{1}{2}};$$
$$n = \zeta_0\,(1 - \mu^2)^{\frac{1}{2}} \cos\omega/(\zeta_0^2 - \mu^2)^{\frac{1}{2}};$$
$$d\mathrm{S} = a^2 e^2 (\zeta_0^2 - 1)^{\frac{1}{2}} (\zeta_0^2 - \mu^2)^{\frac{1}{2}}\, d\mu\, d\omega \quad . \quad (24)$$

Hence we have

$$\mathrm{X} = a^2 e^2 (\zeta_0^2 - 1) \int_0^{2\pi}\!\!\int_{-1}^{1} p\, \mathrm{P}_1(\mu)\, d\mu\, d\omega;$$
$$\mathrm{Z} = -a^2 e^2 \zeta_0 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{2\pi} \cos\omega \int_{-1}^{1} p\, \mathrm{P}_1^1(\mu)\, d\mu\, d\omega;$$
$$\mathrm{M} = \tfrac{1}{3} a^3 e^3 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{2\pi} \cos\omega \int_{-1}^{1} p\, \mathrm{P}_2^1(\mu)\, d\mu\, d\omega \quad . \quad . \quad . \quad . \quad . \quad (25)$$

For the horizontal force X, taking account of the integration in the angular co-ordinate ω, we see that we only need the terms in B independent of ω; and for the general term from the second part of (22) we require the value of the integral

$$\int_{-1}^{1} [(1 - \mu^2)/(\zeta_0^2 - \mu^2)]\, \mathrm{P}_1(\mu)\, \dot{\mathrm{P}}_n(\mu)\, d\mu \quad . \quad (26)$$

It is easily seen that (26) is zero if n is odd; and if n is even, and equal to $2m$, it can be shown that the value of (26) is

$$2(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_{2m}(\zeta_0) \quad . \quad . \quad . \quad (27)$$

Hence from (21) and (25) we have

$$\mathrm{X} = (2\pi)^{3/2} a^2 e^2 (\zeta_0^2 - 1)\, \rho\, h\, c\, (\kappa a e)^{-\frac{1}{2}} e^{-\kappa d}$$
$$[\sigma\, \mathrm{J}_{3/2}/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1(\zeta_0) - (1 + k_1)(\mathrm{V}/a e^2)$$
$$\sum_{m=1}^{\infty} (-1)^m (4m + 1)\, \mathrm{J}_{2m+\frac{1}{2}}]\, e^{i(\sigma t + \alpha)} \quad . \quad (28)$$

the argument of the Bessel functions being $\kappa a e$.

From the properties of these functions, the sum of the series in (28) is simply $-(\kappa a e)\, \mathrm{J}_{3/2}(\kappa a e)$.

Using properties of the spheroidal harmonics such as (20) it can be deduced from (19) that

$$1 + k_1 = -e/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1(\zeta_0) \quad . \quad . \quad (29)$$

Thus the quantity in square brackets in (28) reduces to

$$e^{-1}[-\sigma(1 + k_1) + \kappa \mathrm{V}(1 + k_1)]\, \mathrm{J}_{3/2}(\kappa a e) \quad . \quad (30)$$

Noting that $\sigma = \kappa(\mathrm{V} + c)$, this simplifies further; and we obtain, taking the real part,

$$\mathrm{X} = -(2\pi)^{3/2} g\, \rho\, b^2 h\, e^{-\kappa d} (1 + k_1)(\kappa a e^3)^{-\frac{1}{2}}$$
$$\mathrm{J}_{3/2}(\kappa a e) \cos(\sigma t + \alpha) \quad . \quad (31)$$

From the expression for the vertical force in (25), we have to evaluate for the general term the integral

$$\int_{-1}^{1} [(1 - \mu^2)/(\zeta_0^2 - \mu^2)]\, \mathrm{P}_1^1(\mu)\, \dot{\mathrm{P}}_n^1(\mu)\, d\mu \quad . \quad (32)$$

This integral is zero if n is odd. If n is even and equal to $2m$, it can be shown to have the value

$$(2/\zeta_0)(\zeta_0^2 - 1)^{3/2}\, \dot{\mathrm{Q}}_{2m}^1(\zeta_0) \quad . \quad . \quad . \quad (33)$$

From (21) and (25) we obtain

$$\mathrm{Z} = (2\pi)^{3/2} a\, b\, (\kappa a e)^{-\frac{1}{2}} \rho\, h\, c\, e^{-\kappa d}$$
$$[2 i \sigma\, \mathrm{J}_{3/2}/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1^1 + i \mathrm{V}(a e)^{-1}(1 + k_1)(\zeta_0^2 - 1)^{\frac{1}{2}}$$
$$\sum (-1)^m (4m + 1)\, \mathrm{J}_{2m+\frac{1}{2}}]\, e^{i(\sigma t + \alpha)} \quad . \quad . \quad . \quad (34)$$

the argument of the J functions being $\kappa a e$, and that of Q being ζ_0.

The series in (33) sums as in the previous case; further if k_2 is the virtual inertia coefficient for transverse motion of the spheroid, we have

$$k_2 = -\zeta_0\, \mathrm{Q}_1^1(\zeta_0)/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1^1(\zeta_0) \quad . \quad (35)$$

from which we can deduce

$$(\zeta_0^2 - 1)^{\frac{1}{2}}(1 + k_2) = 2/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1^1(\zeta_0) \; . \quad (36)$$

Thus the quantity in square brackets in (34) has $\mathrm{J}_{3/2}$ as a factor, and another factor is

$$(1 + k_2)(\mathrm{V} + c) - (1 + k_1)\mathrm{V}$$

Collecting these results we obtain finally

$$\mathrm{Z} = (2\pi)^{3/2} g\, \rho\, b^2 h\, e^{-\kappa d} (\kappa a e^3)^{-\frac{1}{2}}$$
$$[1 + k_2 + (k_2 - k_1)\mathrm{V}/c]\, \mathrm{J}_{3/2}(\kappa a e) \sin(\sigma t + \alpha) \; . \; (37)$$

Similarly, for the moment M from (21) and (25) we have to evaluate the integral

$$\int_{-1}^{1} [(1 - \mu^2)/(\zeta_0^2 - \mu^2)]\, \mathrm{P}_2^1(\mu)\, \dot{\mathrm{P}}_n^1(\mu)\, d\mu \quad . \quad (38)$$

In this case, (38) is zero when n is even. When n is odd and equal to $2m + 1$ it has the value

$$6(\zeta_0^2 - 1)^{3/2}\, \dot{\mathrm{Q}}_{2m+1}^1(\zeta_0) \quad . \quad . \quad . \quad (39)$$

for all values of m except $m = 0$; when $m = 0$, (38) has the special value

$$6(\zeta_0^2 - 1)^{3/2}\, \dot{\mathrm{Q}}_1^1(\zeta_0) - 8 \quad . \quad . \quad . \quad (40)$$

With these values in (21) and (25) we have

$$\mathrm{M} = (2\pi)^{3/2} a^3 e^3 (\kappa a e)^{-\frac{1}{2}} (\zeta_0^2 - 1)^{\frac{1}{2}} h\, c\, e^{-\kappa d}$$
$$\{2\sigma\, \mathrm{J}_{5/2}/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_2^1 + (1 + k_1)(\mathrm{V}/a e^2)$$
$$[-4\, \mathrm{J}_{3/2}/(\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_1^1 + (\zeta_0^2 - 1)^{\frac{1}{2}}$$
$$\sum_0^{\infty} (-1)^m (4m + 3)\, \mathrm{J}_{2m+3/2}]\}\, e^{i(\sigma t + \alpha)} \quad . \quad . \quad (41)$$

The series of Bessel functions has the sum $\kappa a e\, \mathrm{J}_{\frac{1}{2}}(\kappa a e)$. Also we substitute from (36) for $\dot{\mathrm{Q}}_1^1$ in terms of k_2. We may also introduce the virtual inertia coefficient for rotation; k' is defined as the relative increase in moment of inertia of the spheroid rotating like a solid of density ρ. It can be shown that

$$k' = -\mathrm{Q}_2^1(\zeta_0)/\zeta_0(\zeta_0^2 - 1)(2\zeta_0^2 - 1)\, \dot{\mathrm{Q}}_2^1(\zeta_0) \quad (42)$$

Using (20) and the expressions for P_2^1 and $\dot{\mathrm{P}}_2^1$ we deduce

$$2/(\zeta_0^2 - 1)\dot{\mathrm{Q}}_2^1 = \zeta_0(\zeta_0^2 - 1)^{\frac{1}{2}}[1 + (2\zeta_0^2 - 1)^2 k'] \quad (43)$$

We also replace $J_{3/2}$ in (41) in terms of $J_{1/2}$ and $J_{5/2}$ and after some reduction we obtain, after putting $\sigma = \kappa (V + c)$, the form

$$M = (2\pi)^{3/2} g \rho a b^2 h e^{-\kappa d} (\kappa a e)^{-\frac{1}{2}}$$

$$\left\{\left[1 + \left(\frac{\beta^2 + 1}{\beta^2 - 1}\right)^2 k'\right] J_{5/2} + \frac{V}{c}\left\{\left[1 + \left(\frac{\beta^2 + 1}{\beta^2 - 1}\right)^2 k'\right.\right.\right.$$

$$\left. - \tfrac{2}{3}(1 + k_1)(1 + k_2)\right] J_{5/2} + \tfrac{1}{3}(1 + k_1)$$

$$\left.\left.(1 - 2k_2) J_{1/2}\right\}\right\} \cos(\sigma t + \alpha) \quad . \quad . \quad . \quad . \quad (44)$$

where the argument of the J functions is $\kappa a e$, and β = length-beam ratio = a/b.

5. Summary of Notation and Results

We may express these results more conveniently in the following notation, in which we also define suitable force and moment coefficients.

- L = Length of spheroid = $2a$.
- e = Eccentricity = $(1 - b^2/a^2)^{\frac{1}{2}}$.
- β = Length-beam ratio = a/b.
- D = Displacement = $\frac{4}{3}\pi g \rho a b^2$.
- k_1, k_2, k' = Axial, transverse, rotational virtual inertia coefficients, as defined and evaluated, for instance, in Ref. 3.
- V = Speed of advance (positive against the waves).
- f = Froude number = $V/(gL)^{\frac{1}{2}}$.
- d = Depth of axis below the surface.
- h = Amplitude of waves = half wave height.
- λ = Wavelength = $2\pi/\kappa$.
- c = Wave velocity = $(g/\kappa)^{\frac{1}{2}}$.
- $2\pi/\sigma$ = Period of encounter.
- σ = $\kappa(V + c)$.
- θ = $2\pi(h/\lambda)e^{-2\pi d/\lambda}$ = Maximum effective wave slope at depth of axis.
- X, Z = Resultant forces in directions Ox, Oz.
- M = Resultant moment about Oy.
- C_x = Surging force coefficient = X (max)/D θ.
- C_z = Heaving force coefficient = Z (max)/D θ.
- C_{yy} = Pitching moment coefficient = M (max)/D L θ.

In this notation, the results obtained are

$$X = -D\theta C_x \cos(\sigma t + \alpha);\; Z = -D\theta C_z \sin(\sigma t + \alpha);$$

$$M = D L \theta C_{yy} \cos(\sigma t + \alpha) \quad . \quad . \quad (45)$$

$$C_x = \frac{3\sqrt{2}}{2\pi e^{3/2}}(1 + k_1)\left(\frac{\lambda}{L}\right)^{3/2} J_{3/2}\left(\frac{\pi e L}{\lambda}\right) \quad . \quad (46)$$

$$C_z = \frac{3\sqrt{2}}{2\pi e^{3/2}}\left[1 + k_2 + f\left(\frac{2\pi L}{\lambda}\right)^{\frac{1}{2}}(k_2 - k_1)\right]$$

$$\left(\frac{\lambda}{L}\right)^{3/2} J_{3/2}\left(\frac{\pi e L}{\lambda}\right) \quad . \quad (47)$$

$$C_{yy} = \frac{3\sqrt{2}}{4\pi e^{1/2}}\left(\frac{\lambda}{L}\right)^{3/2}\left\{\left[1 + \left(\frac{\beta^2 + 1}{\beta^2 - 1}\right)^2 k'\right] J_{5/2}\left(\frac{\pi e L}{\lambda}\right)\right.$$

$$+ f\left(\frac{2\pi L}{\lambda}\right)^{\frac{1}{2}}\left\{\left[1 + \left(\frac{\beta^2 + 1}{\beta^2 - 1}\right)^2 k'\right.\right.$$

$$\left. - \tfrac{2}{3}(1 + k_1)(1 + k_2)\right] J_{5/2}\left(\frac{\pi e L}{\lambda}\right)$$

$$\left.\left. + \tfrac{1}{3}(1 + k_1)(1 - 2k_2) J_{1/2}\left(\frac{\pi e L}{\lambda}\right)\right\}\right\} \quad . \quad (48)$$

6. General Discussion

The phase relations between the waves and the forces can be seen from a comparison of (4) with (45). It is of interest that these relations are unaltered by the improved theory: there is, for instance, a difference of phase of 90 deg. between the heaving force and the pitching moment. It is also confirmed that the period of the forces and moment is the period of encounter with the waves.

From (46) we have the unexpected result that the surging force coefficient is independent of the speed of advance. The coefficient is small and oscillating in value for small values of λ/L: for a long spheroid, the highest zero position is at about $\lambda/L = 0{\cdot}7$. The graph of C_x, except for a scale-factor, is the same as the curve labelled $f = 0$ in Fig. 1. For large values of λ/L, the surging force X approximates to

$$-(1 + k_1) D \theta \cos(\sigma t + \alpha)$$

assuming the wave slope to be kept constant.

Similarly from (47), the heaving force Z approximates to $-(1 + k_2) D\theta \sin(\sigma t + \alpha)$ for large values of λ/L. In general C_z varies with the speed of advance due to the difference between k_1 and k_2. We take for illustration the case of a long spheroid with e approximately unity and k_1, k_2 approximately 0, 1 respectively. Fig. 1 shows C_z on a base of λ/L for zero speed and for $f = 0{\cdot}5$ and $-0{\cdot}5$; we note that f positive is for motion against the waves and f negative for motion with the waves.

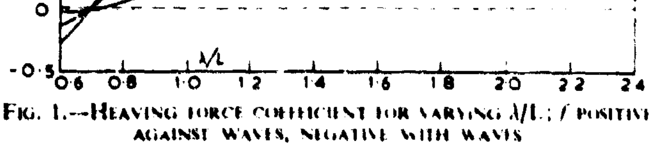

FIG. 1.—HEAVING FORCE COEFFICIENT FOR VARYING λ/L; f POSITIVE AGAINST WAVES, NEGATIVE WITH WAVES

There are several points of interest in the pitching moment coefficient (48). In passing, it may be remarked

that it reduces to zero for a sphere, as should be the case. The variation with the length-beam ratio is shown by the quantity in square brackets in (48); for instance, taking the corresponding values of the inertia coefficients this quantity reduces to

$$1 \cdot 823\, J_{5/2} + (2\,\pi\, L/\lambda)^{\frac{1}{2}} f (0 \cdot 485\, J_{5/2} - 0 \cdot 279\, J_{1/2});\ \beta = 5$$

$$1 \cdot 883\, J_{5/2} + (2\pi\, L/\lambda)^{\frac{1}{2}} f (0 \cdot 549\, J_{5/2} - 0 \cdot 313\, J_{1/2});\ \beta = 10$$

$$2\, J_{5/2} + (2\,\pi\, L/\lambda)^{\frac{1}{2}} f (0 \cdot 666\, J_{5/2} - 0 \cdot 333\, J_{1/2});\ \beta \to \infty$$

We choose the last case for numerical calculation; that is, we consider a long solid of revolution for which k_1, k_2, k' are approximately 0, 1, 1 respectively, and we obtain C_{yy} from (48) with these values. It is interesting that there are values of λ/L for which the pitching moment is independent of the speed of advance; for this case, these are the roots of the equation

$$2\, J_{5/2}\,(\pi\, L/\lambda) - J_{1/2}\,(\pi\, L/\lambda) = 0 \quad . \quad . \quad (49)$$

The two highest roots are approximately $\lambda/L = 0 \cdot 53$ and $\lambda/L = 1 \cdot 51$.

This point is brought out in Fig. 2, which shows C_{yy} on a base λ/L for zero speed and for $f = 0 \cdot 5$, $f = -0 \cdot 5$; the curves show how the effect of speed of advance on the pitching moment, with or against the waves, differs according to the value of λ/L. The same result is also shown in Fig. 3, which gives C_{yy} on a base f for given wavelengths. As this is a linear relation, the graphs are straight lines; those for $\lambda/L = 0 \cdot 53$, $1 \cdot 15$ are parallel to the base line.

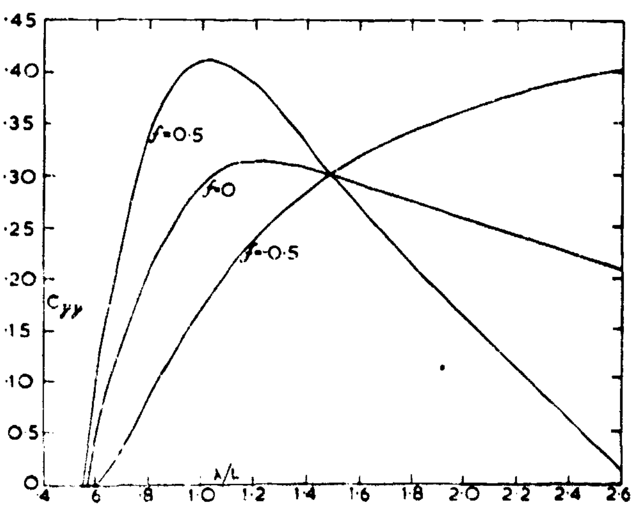

FIG. 2.—PITCHING MOMENT COEFFICIENT FOR VARYING λ/L; f POSITIVE AGAINST WAVES, NEGATIVE WITH WAVES

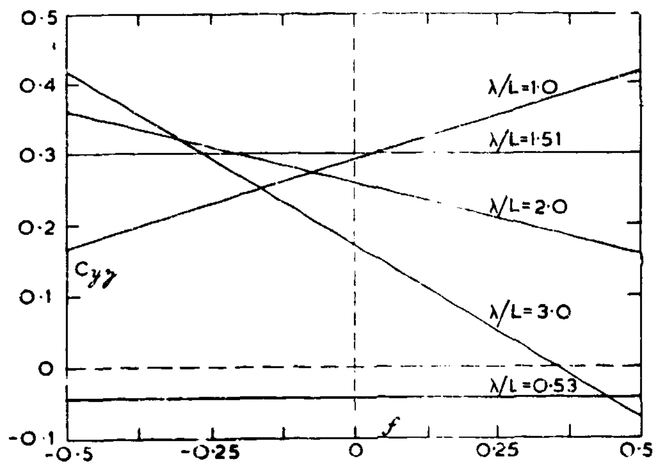

FIG. 3.—PITCHING MOMENT COEFFICIENT FOR GIVEN λ/L, VARYING SPEED OF ADVANCE

The results which have been given in this paper were obtained by direct calculation for a spheroid of any ratio of length to beam; nevertheless, in the form in which they are given in Section 5, they are probably a good approximation for any fairly long solid of revolution. The conclusions are not directly applicable to surface ships; however, they may possibly indicate the nature of the difference to be expected when speed of advance and other factors are taken into account.

The present analysis can be extended to include motion of the submerged solid obliquely to a train of waves, and it is hoped to examine the various forces and moments in subsequent work.

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PART II. MOTION OBLIQUE TO THE WAVE CRESTS

Summary

Expressions are obtained for the forces and moments on a submerged spheroid moving under waves at any angle of attack. Graphs are given to show how these quantities depend upon the wavelength and upon the speed and direction of advance of the body. It appears that, when account is taken of the speed, pitching and yawing moments are developed when the body is moving parallel to the wave crests.

1. The results given in Part I (which we shall refer to as I) can be extended to cover motion in any direction. As the analysis is carried out by the same steps as in I, we need only indicate the necessary modifications. The solid moves in the direction Ox, and the wave train moves at an angle γ to Ox, with γ between 0 deg. and 90 deg. the motion is against the waves, while if it is between 90 deg. and 180 deg. the motion is with following waves.

We have equation I(5) as before; and it is readily verified that we have now

$$\phi_1 = -hc\,e^{-\kappa d - \kappa z}\cos\left[\kappa\,(x\cos\gamma + y\sin\gamma) + \sigma t + \alpha\right] \quad . \quad . \quad . \quad . \quad (1)$$

giving a surface elevation

$$\eta = h \sin\left[\kappa\,(x \cos\gamma + y \sin\gamma) - \sigma t - \alpha\right] \quad . \quad (2)$$

provided $\quad \sigma = \kappa\,(\mathrm{V}\cos\gamma - c); \quad c^2 = g/\kappa \quad . \quad . \quad (3)$

To obtain ϕ_2 we have now to expand

$$\exp \kappa \left[z + i\,(x \cos\gamma + y \sin\gamma)\right]$$

in a series of spheroidal harmonics. It is found that instead of I(10) we require the two integrals

$$\int_0^{2\pi} e^{\kappa b (1-\mu^2)^{\frac{1}{2}} (\cos\omega \cos\gamma + i \sin\gamma \sin\omega)} \cos s\,\omega\, d\,\omega$$

$$= 2\pi\, \mathrm{C}_s\, \mathrm{I}_s\left[\kappa\, b\,(1-\mu^2)^{\frac{1}{2}} \cos\gamma\right] \quad . \quad (4)$$

and the same integral with $\sin s\,\omega$ for $\cos s\,\omega$, equal to

$$\mathrm{S}_s\, \mathrm{I}_s\left[\kappa\, b\,(1-\mu^2)^{\frac{1}{2}} \cos\gamma\right] \quad . \quad . \quad . \quad (5)$$

in which

$$\left.\begin{aligned} \mathrm{C}_s &= \tfrac{1}{2}\left[(1+\sin\gamma)^s + (1-\sin\gamma)^s\right]/\cos^s\gamma \\ \mathrm{S}_s &= \tfrac{1}{2}\left[(1+\sin\gamma)^s - (1-\sin\gamma)^s\right]/\cos^s\gamma \end{aligned}\right\} \quad . \quad (6)$$

Proceeding as in I Section 2, the only alteration required is to replace $\kappa\, a\, e$ by $\kappa\, a\, e \cos\gamma$ and $\mathrm{A}_n^s \cos s\,\omega$ by $\mathrm{A}_n^s\,(\mathrm{C}_s \cos s\,\omega + i\, \mathrm{S}_s \sin s\,\omega)$.

Thus, as in I Section 3, we have for the effective part of the pressure

$$p = \left(\frac{2\pi}{\kappa\, a\, e \cos\gamma}\right)^{\frac{1}{2}} \rho\, h\, c\, e^{-\kappa d}$$

$$\left[i\,\sigma\,\mathrm{B} - \frac{(1+k_1)\,\mathrm{V}\,(1-\mu^2)}{a\,e^2\,(\zeta_0^2-\mu^2)}\frac{\partial\,\mathrm{B}}{\partial\,\mu}\right] e^{i(\sigma t - \alpha)} \quad . \quad (7)$$

with

$$\mathrm{B} = \Sigma\,\Sigma\,(2n+1)\,(i)^{n+s}\,\frac{\mathrm{J}_{n+\frac{1}{2}}(\kappa\, a\, e\cos\gamma)}{(\zeta_0^2-1)\,\mathrm{Q}_n^{s\,\prime}(\zeta_0)}$$

$$\mathrm{P}_n^s(\mu)\,(\mathrm{C}_s \cos s\,\omega + i\,\mathrm{S}_s \sin s\,\omega) \quad . \quad (8)$$

with a factor $\frac{1}{2}$ for the terms with $s = 0$.

The forces X, Z and the pitching moment M are given by I(25); and, owing to the presence of terms in $\sin s\,\omega$, we have a swaying force Y and a yawing moment M′ given by

$$\mathrm{Y} = -a^2 e^2 \zeta_0 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{2\pi} \sin\omega \int_{-1}^{1} p\, \mathrm{P}_1^1(\mu)\, d\,\mu\, d\,\omega$$

$$\mathrm{M}' = \tfrac{1}{3}\, a^3 e^3 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{2\pi} \sin\omega \int_{-1}^{1} p\, \mathrm{P}_2^1(\mu)\, d\,\mu\, d\,\omega$$

$$. \quad . \quad . \quad . \quad (9)$$

The evaluation of the forces and moments follows as in I Section 4, noting that we must introduce the appropriate values of the coefficients given in (6) and also that $\sigma = \kappa\,(\mathrm{V}\cos\gamma - c)$.

2. We add to the notation specified in I Section 5

C_y swaying force coefficient $= \mathrm{Y}\,(\max)/\mathrm{D}\,\theta$

C_{zz} yawing moment coefficient $= \mathrm{M}'\,(\max)/\mathrm{D}\,\mathrm{L}\,\theta$

The components are given by

$$\left.\begin{aligned} \mathrm{X} &= \mathrm{D}\,\theta\,\mathrm{C}_x \cos(\sigma t - \alpha); \\ \mathrm{Y} &= \mathrm{D}\,\theta\,\mathrm{C}_y \cos(\sigma t - \alpha); \\ \mathrm{Z} &= \mathrm{D}\,\theta\,\mathrm{C}_z \sin(\sigma t - \alpha); \\ \mathrm{M} &= \mathrm{D}\,\mathrm{L}\,\theta\,\mathrm{C}_{yy} \cos(\sigma t - \alpha); \\ \mathrm{M}' &= \mathrm{D}\,\mathrm{L}\,\theta\,\mathrm{C}_{zz} \sin(\sigma t - \alpha) \end{aligned}\right. \quad . \quad . \quad . \quad (10)$$

These relations give the connection between the phases of the components and the phase of the surface waves; if in varying the angle of attack γ from 0 deg. to 180 deg. any coefficient C passes through zero and changes sign, the corresponding force or moment changes in phase by 180 deg.

The force and moment coefficients are given by

$$\mathrm{C}_x = \frac{3\sqrt{2}}{2\pi e^{3/2}}\left(\frac{1}{\cos\gamma}\right)^{\frac{1}{2}}\left(\frac{\lambda}{\mathrm{L}}\right)^{3/2}(1+k_1)\,\mathrm{J}_{3/2}\left(\frac{\pi e \mathrm{L}}{\lambda}\cos\gamma\right)$$

$$\mathrm{C}_y = \mathrm{C}_z \sin\gamma$$

$$\mathrm{C}_z = \frac{3\sqrt{2}}{2\pi e^{3/2}}\left(\frac{\lambda}{\mathrm{L}\cos\gamma}\right)^{3/2}$$

$$\left[1 + k_2 + \left(\frac{2\pi\mathrm{L}}{\lambda}\right)^{\frac{1}{2}} f\cos\gamma\,(k_2 - k_1)\right]\mathrm{J}_{3/2}\left(\frac{\pi e\mathrm{L}}{\lambda}\cos\gamma\right)$$

$$\mathrm{C}_{zz} = \mathrm{C}_{yy}\sin\gamma$$

$$\mathrm{C}_{yy} = \frac{3\sqrt{2}}{4\pi e^{1/2}}\left(\frac{\lambda}{\mathrm{L}\cos\gamma}\right)^{3/2}\left\{\left[1 + \left(\frac{\beta^2+1}{\beta^2-1}\right)^2 k'\right]\right.$$

$$\mathrm{J}_{5/2}\left(\frac{\pi e\mathrm{L}}{\lambda}\cos\gamma\right) - \left(\frac{2\pi\mathrm{L}}{\lambda}\right)^{\frac{1}{2}} f\cos\gamma$$

$$\left\{\left[1 + \left(\frac{\beta^2+1}{\beta^2-1}\right)^2 k' - \tfrac{2}{3}(1+k_1)(1+k_2)\right]\mathrm{J}_{5/2}\left(\frac{\pi e\mathrm{L}}{\lambda}\cos\gamma\right)\right.$$

$$\left.\left. + \tfrac{1}{3}(1+k_1)(1-2k_2)\,\mathrm{J}_{1/2}\left(\frac{\pi e\mathrm{L}}{\lambda}\cos\gamma\right)\right\}\right\}$$

$$. \quad . \quad . \quad . \quad . \quad (11)$$

3. We note that the surging force is independent of the speed of advance except, of course, for the alteration in the period. Except for the surging force, all the components can be derived from the expressions in I Section 5 by replacing the wavelength λ by $\lambda/\cos\gamma$ and the speed f by $f\cos\gamma$. Further, the coefficients C_y and C_{zz} differ from C_z and C_{yy} respectively only by a factor $\sin\gamma$. Putting 180 deg. $-\gamma$ for γ in the expressions (60), we see that at zero speed C_y and C_z are symmetrical about the middle position $\gamma = 90$ deg., while C_x, C_{yy} and C_{zz} are anti-symmetrical. Taking account of the speed of advance removes this element of symmetry, for the effect is different according as the waves are from ahead or from astern.

When $\gamma = 90$ deg. the solid is moving parallel to the wave crests and this case is of some interest. The results can be obtained by taking the limiting values of (11) as γ is made equal to 90 deg., or can be worked out independently.

We have for $\gamma = 90$ deg.

$$\mathrm{C}_x = 0; \quad \mathrm{C}_y = \mathrm{C}_z = 1 + k_2$$

$$\mathrm{C}_{zz} = \left(\frac{\lambda}{2\pi\mathrm{L}}\right)^{\frac{1}{2}} f\,(1+k_1)(1-2k_2) \quad . \quad (12)$$

In this position the forces are independent of the speed of advance. As might be expected, the moments are zero at zero speed; but it is specially interesting that pitching and yawing moments are developed when the solid is advancing parallel to the wave crests. No doubt

this arises because when account is taken of the speed, the pressure distribution on the solid is altered and is no longer symmetrical fore and aft.

The general character of the results in (11) can be shown best by diagrams, and for this purpose we take the case of a fairly long spheroid for which we assume the approximate values $e = 1$, $k_1 = 0$, $k_2 = k' = 1$. Figs. 4, 5, and 6 show curves of the coefficients of heaving force, pitching moment, and yawing moment for the waves at any angle to the direction of advance. In each case the results are shown for three different ratios of λ/L. The continuous curves are for zero speed; and, in order to bring out the difference, the dotted curves are for a speed of advance given by the Froude number $f = 0{\cdot}5$. The curves give some indication of the manner in which these quantities depend upon the ratio λ/L, upon the speed and direction of attack, and upon whether the waves are from ahead or from astern.

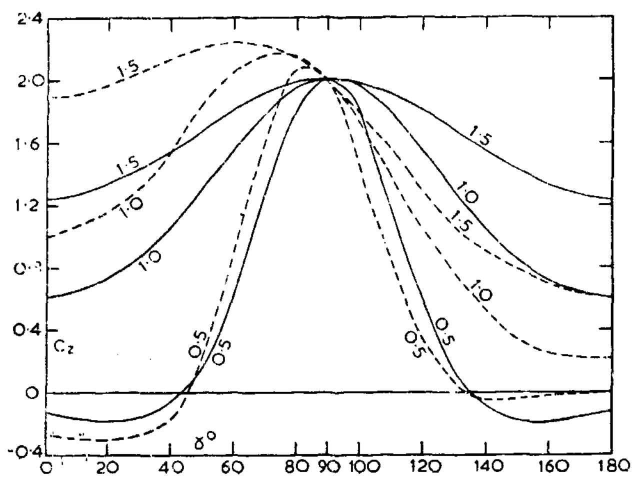

FIG. 4.—HEAVING FORCE COEFFICIENT FOR $\lambda/L = 0{\cdot}5, 1{\cdot}0, 1{\cdot}5$; ——— AT ZERO SPEED, – – – – AT SPEED $f = 0{\cdot}5$

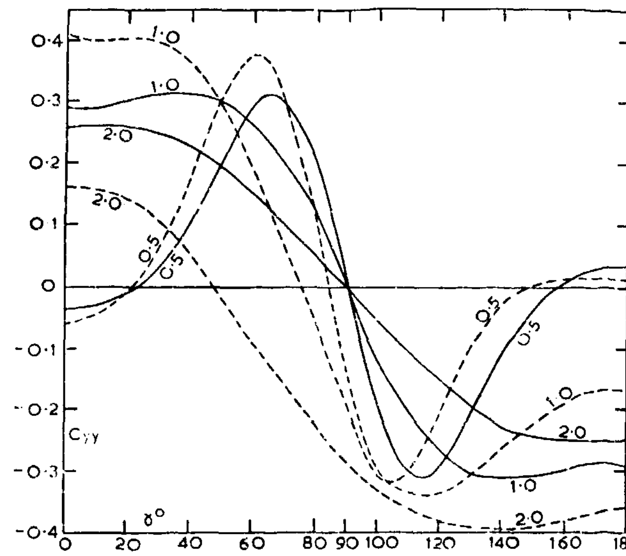

FIG. 5.—PITCHING MOMENT COEFFICIENT FOR $\lambda/L = 0{\cdot}5, 1{\cdot}0, 2{\cdot}0$; ——— AT ZERO SPEED, – – – – AT SPEED $f = 0{\cdot}5$

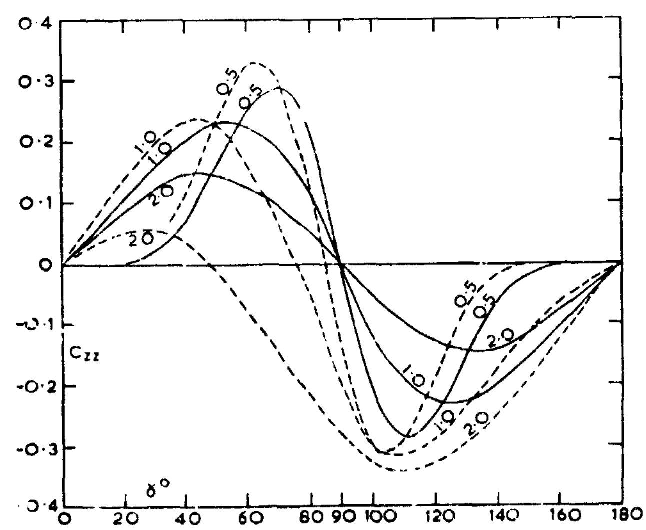

FIG. 6.—YAWING MOMENT COEFFICIENT FOR $\lambda/L = 0{\cdot}5, 1{\cdot}0, 2{\cdot}0$; ——— AT ZERO SPEED, – – – – AT SPEED $f = 0{\cdot}5$

ISSUED FOR WRITTEN DISCUSSION

Members are invited to submit written contributions to this paper, which should reach the Secretary not later than March 15, 1955: they should be typewritten.

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THE COUPLING OF HEAVE AND PITCH DUE TO SPEED OF ADVANCE

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Summary

The object of the note is to discuss a particular type of coupling and to estimate its probable magnitude. The coupling effect is isolated by considering a specially simplified problem, namely a spheroid floating half-submerged in a uniform stream with surface conditions which preclude wave formation and without damping. This problem is solved completely; numerical calculations indicate that the alteration in resonance frequencies due to the coupling is likely to be negligible in ship problems.

1. The chief cause of coupling between heave and pitch is lack of symmetry of the ship fore and aft, as for instance in the well-known hydrostatic coupling or in that due to damping. There is one type, which may be called hydrodynamic coupling due to speed of advance, which seems to exist even if the ship is symmetrical fore and aft. This effect was introduced into the equations of motion of the ship by Haskind.[1] In that work Haskind replaced the ship by the approximation used in wave resistance theory, namely a source distribution over the longitudinal vertical section; further, the expressions were left in a complicated form and no indication was given of the relative importance of the terms in the equations. Recently Stoker and Peters[2] have made a systematic study of the general problem of the motion of a ship in a seaway, developing the equations in terms of a single parameter, namely the ratio of beam to length. In the equations of motion to the first order, they do not obtain any coupling terms of the type in question for a symmetrical ship. This might be expected as in their work the ratio of beam to draught is also supposed small; in fact their model approximates to a thin flat disc. Haskind's work is also criticized as implying damped oscillations since the coupling terms occur as first order derivatives; but we shall see later that this criticism is unfounded as far as the coupling terms are concerned. This type of coupling has been the subject of discussion recently, for instance Weinblum,[3] and it seemed of sufficient interest to attempt to estimate its importance or otherwise. It is easy to see on general grounds that the coupling exists. If a floating solid, symmetrical fore and aft, is made to oscillate vertically in a uniform stream, the alteration in pressure is anti-symmetrical and so we get a couple acting on the solid; if it is given pitching oscillations the alteration in pressure is symmetrical and we get a heaving force. It also seems likely that the effect will be small, and that is confirmed by the present calculations.

In the theory of wave resistance for a ship advancing steadily in still water, a first approximation based on the linearized free surface condition is in general a good approximation for ships of small beam/length ratio, and this remark applies even when the beam/draught ratio is not also small. But in attempting further approximations it is difficult to know how far one may go without amending the free surface condition by including some approximation to finite wave theory.

On the other hand, consider heaving and pitching of a ship at zero speed of advance. Here we do not need to restrict the relative dimensions of the ship, either the beam/length or the beam/draught ratio; the linearized free surface condition is adequate for a good first approximation, except no doubt for exceptionally large motions.

Turning to heaving and pitching in waves with the ship advancing, one can see the difficulty of combining the general problem in a single calculation which will give useful results for the ship problem. The present unsatisfactory theory consists more or less in simply superposing the two sets of calculations; or if it is rather better than that, we are still left in doubt as to the validity and relative importance of the various terms in the equations of motion. On the other hand, if we limit ourselves to a rigorous development based on, say, a thin disc form, we may miss the important effects for the ship problem as regards heaving and pitching. However, the present note makes no attempt whatever on the general problem. The object is to isolate the particular type of coupling and if possible to estimate its importance. For this purpose we consider a specially simplified problem. It may be regarded as the opposite of the work just referred to; instead of taking a thin disc and including the wave motion, we consider a form more like a ship but we exclude the wave motion completely. The conditions may be visualized in this way. Imagine a solid floating in water and suppose the free water surface covered by a smooth rigid plane; the solid being assumed free to heave and pitch in a hole in this plane, the periods can be calculated. If there is a longitudinal uniform stream in the water, the oscillations are coupled and the periods can be obtained. The

results may be comparable with those for a ship advancing at slow speed; but in any case it seems likely that the coupling effects in the actual problem will be less than in this simplified case. The form of solid we consider is a prolate spheroid floating half-submerged; for this case the problem can be solved completely and the analysis is given in the Appendix.

2. If h is the heave and ψ the pitch, and U the stream velocity parallel to the axis of the spheroid, the equations of motion for free oscillations are obtained in the form (25),

$$(1+k_2)\,\mathrm{M}\,\ddot{h} - p\,\mathrm{M}\,\mathrm{U}\,\dot{\psi} + g\rho\,\mathrm{S}\,h = 0$$
$$(1+k')\,\mathrm{I}\,\ddot{\psi} + q\,\mathrm{M}\,\mathrm{U}\,\dot{h} + \mathrm{M}\,g\,m\,\psi = 0$$

The first and third terms are in the usual notation for uncoupled heave and pitch; the second terms give the coupling effect. k_2 and k' are the virtual inertia coefficients under the assumed water surface condition. k_2, k', p and q are positive numerical coefficients depending only upon the length/beam ratio of the spheroid; explicit expressions are given in (18), (20), (22), and (23), from which numerical values can be calculated.

If we write these equations in the form

$$\ddot{h} - \alpha\,\mathrm{U}\,\dot{\psi} + n^2{}_1\,h = 0$$
$$\ddot{\psi} + \beta\,\mathrm{U}\,\dot{h} + n^2{}_2\,\psi = 0$$

n_1 and n_2 are the frequencies for uncoupled heaving and pitching, taking into account the virtual inertia. If we assume a periodic coupled oscillation of frequency p, we have

$$p^4 - (n_1^2 + n_2^2 + \alpha\beta\,\mathrm{U}^2)\,p^2 + n_1^2 n_2^2 = 0$$

Both roots of this equation in p^2 are real and positive, and we have two simple undamped oscillations of, say, frequencies p_1 and p_2. In each mode the heave and pitch differ in phase by 90 deg., and the motion alternates between heaving and pitching. Further, if $n_1 < n_2$ and $p_1 < p_2$, then we have $p_1 < n_1$ and $p_2 > n_2$: thus the coupling increases the separation between the resonance frequencies. This is a general effect of coupling terms; incidentally it may be remarked that for the coupling caused by damping, Korvin-Kroukovsky and Lewis[4] observed that the resonance period for heaving was increased while that for pitching was diminished.

3. To estimate the magnitude of the effect we take a numerical example. We choose a spheroid of length/beam ratio equal to 10. The numerical values of the various coefficients were calculated with sufficient approximation for the present purpose. From (18) we find $k_2 = 2{\cdot}42$. This means an increase of about 80 per cent in the heaving period as found without the added mass; no doubt this is rather large, but we have taken the extreme condition of a rigid water surface. Similarly from (20) we find $k' = 1{\cdot}5$, giving a corresponding increase of about 60 per cent over the basic pitching period. From (22) and (23) we obtain, approximately, $p = 1{\cdot}16$, $q = 0{\cdot}57$. With f as the Froude number, we have $\mathrm{U} = f(2ga)^{\frac{1}{2}}$; and taking a 16-ft. model as a definite example, that is, $a = 8$ ft., the equations of motion are

$$\ddot{h} - 7{\cdot}7\,f\,\dot{\psi} + 17{\cdot}657\,h = 0$$
$$\ddot{\psi} + 0{\cdot}404\,f\,\dot{h} + 30{\cdot}187\,\psi = 0$$

and the frequency equation is

$$p^4 - (47{\cdot}844 + 3{\cdot}111\,f^2)\,p^2 + 533{\cdot}01 = 0$$

The periods of uncoupled heave and pitch are 1·495 sec. and 1·144 sec. respectively. For $f = 0{\cdot}2$, the coupled periods are 1·503 sec. and 1·138 sec. Even at a high speed $f = 0{\cdot}5$, the periods are only altered to 1·537 sec. and 1·112 sec. The curves in Fig. 1 show the variation in the coupled periods with increasing speed. At zero speed, the upper curve gives the period of uncoupled heave and the lower curve that of uncoupled pitch. The variation with increasing speed only becomes appreciable at very high speeds.

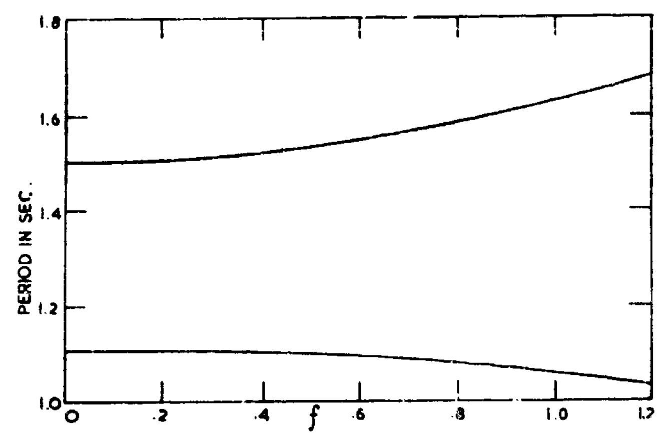

FIG. 1.—VARIATION OF RESONANCE PERIODS WITH SPEED

4. Although the surface condition $\partial\phi/\partial z = 0$ is a severe limitation as regards application to the ship problem, it was thought preferable to work out the simplified problem consistently on this basis. In the last section of the Appendix comparison is made with the work of Haskind. It appears that if we use tentatively rather mixed conditions with the oscillation potentials satisfying the condition $\phi = 0$ at the surface, then the coefficients of the coupling terms approach numerical equality for a long spheroid for which b/a is small; the coupling terms approximate to the values $\frac{1}{2}\,\mathrm{M}\,\mathrm{U}\,\dot{\psi}$ and $\frac{1}{2}\,\mathrm{M}\,\mathrm{U}\,\dot{h}$. For another numerical example, take k_2 and k' at their limiting values of unity for this surface condition and the equations (29) approximate to

$$2\,\mathrm{M}\,\ddot{h} - \tfrac{1}{2}\,\mathrm{M}\,\mathrm{U}\,\dot{\psi} + g\rho\,\mathrm{S}\,h = 0$$
$$2\,\mathrm{I}\,\ddot{\psi} + \tfrac{1}{2}\,\mathrm{M}\,\mathrm{U}\,\dot{h} + \mathrm{M}\,g\,m\,\psi = 0$$

For the 16-ft. model of the previous calculations these give a frequency equation

$$p^4 - (67{\cdot}92 + 2{\cdot}514\,f^2)\,p^2 + 1139 = 0$$

For $f = 0$, the uncoupled periods of heave and pitch

2

are 1·144 sec. and 1·023 sec. respectively. For $f = 0·2$, the coupled periods are 1·151 sec. and 1·017 sec.: while for $f = 0·5$, they are 1·181 sec. and 0·991 sec.

5. To sum up the discussion, it seems that the coupling terms are of the form $-p\,\mathrm{M\,U}\,\dot\psi$ and $+q\,\mathrm{M\,U}\,\dot h$, with p and q numerical coefficients approximately between unity and one-half. From the numerical examples, we may conclude that the alteration in resonance frequencies is negligible. It would be of interest to examine forced oscillations; for instance, with an impressed heaving force the response involves pitching as well as heaving, and similarly, with an impressed pitching moment. An effect of this sort seems to have been observed by Grim.[5] It is, of course, possible that even a small coupling effect might be magnified into something appreciable at or near resonance. However, any satisfactory examination of this would involve introducing suitable damping terms and that is beyond the scope of the present note, the purpose of which was to isolate the coupling effect in its simplest form, together with the consequent change in the resonance frequencies.

Appendix

6. We take the origin O at the centre of the spheroid, O x along the axis, O z vertically downwards and O y transversely. We use spheroidal co-ordinates given by

$$x = a e \mu \zeta;\ y = a e (1-\mu^2)^{\frac{1}{2}} (\zeta^2 - 1)^{\frac{1}{2}} \cos\omega;$$
$$z = a e (1-\mu^2)^{\frac{1}{2}} (\zeta^2 - 1)^{\frac{1}{2}} \sin\omega \quad . \quad (1)$$

The spheroid is given by $\zeta = \zeta_0 = 1/e$, and for the submerged half ω ranges from 0 to π. The spheroid floats half immersed in water, and there is a uniform stream U in the negative direction of O x; the solid makes small oscillations, in which the heaving velocity at any instant is $\dot h$ upwards, and the angular pitching velocity is $\dot\psi$ in the positive direction round O y. If ϕ is the velocity potential, we assume the condition $\partial\phi/\partial z = 0$ at the upper surface of the water. For small oscillations we assume the condition at the immersed surface of the solid to hold at the mean equilibrium position; thus, in the subsequent work, we neglect the square of the fluid velocity due to the oscillations.

We take for the velocity potential

$$\phi = \mathrm{U}\,x + a e \mathrm{U}\, \mathrm{P}_1(\mu)\,\mathrm{Q}_1(\zeta)/\dot{\mathrm{Q}}_1(\zeta_0)$$
$$- \dot h\, \mathrm{F}_1(\mu, \zeta, \omega) + \dot\psi\, \mathrm{F}_2(\mu, \zeta, \omega) \quad . \quad (2)$$

with
$$\left.\begin{aligned} \mathrm{F}_1 &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \mathrm{A}_n^s\, \mathrm{P}_n^s(\mu)\,\mathrm{Q}_n^s(\zeta)\cos s\omega \\ \mathrm{F}_2 &= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \mathrm{B}_n^s\, \mathrm{P}_n^s(\mu)\,\mathrm{Q}_n^s(\zeta)\cos s\omega \end{aligned}\right\} \quad . \quad . \quad (3)$$

The expression (2) satisfies the condition $\partial\phi/\partial z = 0$ at the upper surface of the water. The first two terms represent the spheroid in a uniform stream and give zero normal velocity over the immersed surface; hence, as for instance in Lamb's *Hydrodynamics*, p. 142, we must have

$$\left.\begin{aligned} \partial \mathrm{F}_1/\partial\zeta &= a e \zeta_0 (\zeta_0^2 - 1)^{-\frac{1}{2}}\, \mathrm{P}_1^1(\mu)\sin\omega \\ \partial \mathrm{F}_2/\partial\zeta &= \tfrac{1}{3} a^2 e^2 (\zeta_0^2 - 1)^{-\frac{1}{2}}\, \mathrm{P}_2^1(\mu)\sin\omega \end{aligned}\right\} \quad . \quad . \quad (4)$$

for $\zeta = \zeta_0$; $0 < \omega < \pi$.

Putting the expressions (3) in (4) and determining the coefficients in the usual way,

$$\mathrm{A}_n^s = \frac{2 a e \zeta_0}{\pi(\zeta_0^2 - 1)^{\frac{1}{2}}}\,\frac{2n+1}{s^2-1}\,\frac{(n-s)!}{(n+s)!}\,\frac{\mathrm{C}_n^s}{\dot{\mathrm{Q}}_n^s(\zeta_0)} \quad . \quad (5)$$

$$\mathrm{B}_n^s = \frac{2 a^2 e^2}{3\pi(\zeta_0^2 - 1)^{\frac{1}{2}}}\,\frac{2n+1}{s^2-1}\,\frac{(n-s)!}{(n+s)!}\,\frac{\mathrm{D}_n^s}{\dot{\mathrm{Q}}_n^s(\zeta_0)} \quad . \quad (6)$$

with the factors C, D given by

$$\mathrm{C}_n^s = \int_{-1}^{1} \mathrm{P}_1^1(\mu)\,\mathrm{P}_n^s(\mu)\,d\mu;\ \mathrm{D}_n^s = \int_{-1}^{1} \mathrm{P}_2^1(\mu)\,\mathrm{P}_n^s(\mu)\,d\mu \quad . \quad (7)$$

It should be noted that in the summations in (4) with (5) and (6) terms with $s = 0$ must be taken with a factor $\frac{1}{2}$. Further, s is even throughout, while n is even in (5) and is odd in (6); this follows from the fact that the integrals in (7) are only different from zero under these conditions.

7. The pressure is given by

$$p = g\rho z + \rho\,\partial\phi/\partial t + \tfrac{1}{2}\rho \mathrm{U}^2 - \tfrac{1}{2}\rho q^2 \quad . \quad (8)$$

with
$$q^2 = (\partial\phi/\partial s_\mu)^2 + (\partial\phi/\partial s_\zeta)^2 + (\partial\phi/\partial s_\omega)^2 \quad . \quad (9)$$

On the spheroid the last two terms in (9) come only from the last two terms of ϕ in (2) and are of the second order. Further, on the spheroid, we have the first two terms of ϕ in (2) given by

$$a \mathrm{U} \mu[1 - e\,\mathrm{Q}_1(\zeta_0)/\dot{\mathrm{Q}}_1(\zeta_0)] = a \mathrm{U}(1 + k_1)\mu \quad . \quad (10)$$

where k_1 is the virtual inertia coefficient for axial motion of the spheroid; also, on the spheroid,

$$\partial\phi/\partial s_\mu = [(1-\mu^2)^{\frac{1}{2}}/a e (\zeta_0^2 - \mu^2)^{\frac{1}{2}}]\,\partial\phi/\partial\mu \quad . \quad (11)$$

Hence, to first order terms in $\dot h$ and $\dot\psi$, we have on the spheroid

$$q^2 = \frac{1-\mu^2}{a^2 e^2 (\zeta_0^2 - \mu^2)}[a^2 \mathrm{U}^2 (1 + k_1)^2 + 2 a \mathrm{U}(1 + k_1)$$
$$(-\dot h\,\partial \mathrm{F}_1/\partial\mu + \dot\psi\,\partial \mathrm{F}_2/\partial\mu)] \quad . \quad (12)$$

If $-$Z is the upward resultant of the fluid pressures, and M is the moment about O y, we have

$$\mathrm{Z} = \iint p\, n\, d\mathrm{S}$$
$$= a^2 e^2 \zeta_0 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{\pi} \sin\omega\, d\omega \int_{-1}^{1} p\, \mathrm{P}_1^1(\mu)\, d\mu \quad . \quad (13)$$

$$\mathrm{M} = \iint (l z - n x)\, p\, d\mathrm{S}$$
$$= \tfrac{1}{3} a^3 e^3 (\zeta_0^2 - 1)^{\frac{1}{2}} \int_0^{\pi} \sin\omega\, d\omega \int_{-1}^{1} p\, \mathrm{P}_2^1(\mu)\, d\mu \quad . \quad (14)$$

8. We shall consider separately the contributions of the various terms in the pressure defined by (8) and (12). The term $g\rho z$ gives the hydrostatic vertical force, and moment, leading to the usual expressions for the restoring force proportional to the heaving displacement h, and restoring moment proportional to the pitching angle ψ. Then there is a steady vertical force arising from the terms in U^2, and corresponding to the bodily sinkage of a ship in motion. We obtain this by using for p in (13) the terms

$$\tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho U^2 (1+k_1)^2 (1-\mu^2)/e^2(\zeta_0^2-\mu^2) \quad . \quad (15)$$

The integrals can be evaluated, and the result for this vertical force can be reduced to the simple form

$$\tfrac{1}{2}\pi \rho a b U^2[(1+k_1)^2 a(a+2b)/(a+b)^2 - 1] \quad . \quad (16)$$

This result was given in an equivalent form in a previous paper,[6] which dealt with the sinkage of a general ellipsoidal form at low speeds. To estimate the magnitude of this effect we may equate (15) to $\pi g \rho a b s$, and call s the equivalent sinkage. If, for instance, $a/b = 10$, we find $s = 0{\cdot}0157\, U^2/g = 0{\cdot}314 f^2 b$, with f as the Froude number. The effect in the present problem means simply an alteration of the origin O; but, as it is small except for high speeds, we shall neglect it in what follows.

9. Turning now to the term $\rho\,\partial\phi/\partial t$ in the pressure, it is easily seen from the various expressions which have been given, that the term from F_2 gives no contribution to the vertical force; and we have for this part of the vertical force

$$a^2 e^2 \zeta_0 (\zeta_0^2-1)^{\frac{1}{2}} \rho \ddot{h} \int_0^{\pi} \sin\omega\, d\omega \int_{-1}^{1} P_1^1(\mu)\, F_1(\mu, \zeta_0, \omega)\, d\mu \quad . \quad . \quad . \quad . \quad (17)$$

Substituting for F_1 from (3) and carrying out the integrations we can express this vertical force upwards as $-k_2 M \ddot{h}$, where $M = \tfrac{4}{3}\pi\rho a b^2$ = mass of spheroid and

$$k_2 = -\frac{6}{\pi^2 e}\sum_n \sum_s \frac{2n+1}{(s^2-1)^2}\frac{(n-s)!}{(n+s)!}\frac{Q_n^s(\zeta_0)}{(\zeta_0^2-1)\dot{Q}_n^s(\zeta_0)}(C_n^s)^2 \quad . \quad . \quad . \quad . \quad (18)$$

This expression is essentially positive, and k_2 is the virtual inertia coefficient for broadside motion of the submerged spheroid under the assumed condition of a rigid water surface.

Similarly, putting the pressure term $\rho\,\partial\phi/\partial t$ in (14), we find that F_1 gives no contribution to the moment, and we have for this part of the moment

$$\tfrac{1}{3}\rho a^3 e^3 (\zeta_0^2-1)^{\frac{1}{2}} \ddot{\psi} \int_0^{\pi} \sin\omega\, d\omega \int_{-1}^{1} P_2^1(\mu)\, F_2(\mu, \zeta_0, \omega)\, d\mu \quad . \quad . \quad . \quad . \quad (19)$$

The moment of inertia of the spheroid about Oy is $I = \tfrac{4}{15}\pi\rho a b^2(a^2+b^2)$. We find that (19) can be expressed as $-k' I \ddot{\psi}$, with

$$k' = -\frac{10 a^2 e^3}{3\pi^2(a^2+b^2)}\sum_n \sum_s \frac{2n+1}{(s^2-1)^2}\frac{(n-s)!}{(n+s)!}\frac{Q_n^s(\zeta_0)}{(\zeta_0^2-1)\dot{Q}_n^s(\zeta_0)}(D_n^s)^2 \quad . \quad (20)$$

k' is the virtual inertia coefficient for rotation about Oy under the assumed surface condition.

10. Lastly, from (8) and (12), the remaining terms in the pressure are

$$-\rho a(1+k_1) U \frac{1-\mu^2}{a^2 e^2(\zeta_0^2-\mu^2)}(-\dot{h}\,\partial F_1/\partial\mu + \dot{\psi}\,\partial F_2/\partial\mu) \quad . \quad . \quad . \quad . \quad (21)$$

With this in (13) and (14), it is seen that the only contribution to Z comes from the term in F_2, and the only contribution to M from the term in F_1. Putting in the expressions for F_1 and F_2 and carrying out the integrations, the results can be expressed as an upward vertical force $p M U \dot{\psi}$ with

$$p = -\frac{2(1+k_1)}{\pi^2 e}\sum_{n=1}^{\infty}\sum_{s=0}^{n}\frac{2n+1}{(s^2-1)^2}\frac{(n-s)!}{(n+s)!}\frac{Q_n^s(\zeta_0)}{(\zeta_0^2-1)\dot{Q}_n^s(\zeta_0)} D_n^s E_n^s \quad . \quad (22)$$

and a moment $q M U \dot{h}$, with

$$q = \frac{2(1+k_1)}{\pi^2 e}\sum_{n=2}^{\infty}\sum_{s=0}^{n}\frac{2n+1}{(s^2-1)^2}\frac{(n-s)!}{(n+s)!}\frac{Q_n^s(\zeta_0)}{(\zeta_0^2-1)\dot{Q}_n^s(\zeta_0)} C_n^s F_n^s \quad . \quad (23)$$

where we have written

$$E_n^s = \int_{-1}^{1}\frac{1-\mu^2}{\zeta_0^2-\mu^2} P_1^1(\mu)\, \dot{P}_n^s(\mu)\, d\mu;$$

$$F_n^s = \int_{-1}^{1}\frac{1-\mu^2}{\zeta_0^2-\mu^2} P_2^1(\mu)\, \dot{P}_n^s(\mu)\, d\mu. \quad . \quad . \quad . \quad (24)$$

Summing up these results we get the equations of motion of the spheroid, with m as the metacentric height,

$$\left.\begin{array}{l}(1+k_2) M \ddot{h} - \rho M U \dot{\psi} + \pi g \rho a b h = 0 \\ (1+k') I \ddot{\psi} - q M U \dot{h} + M g m \psi = 0\end{array}\right\} \quad . \quad (25)$$

where k_2, k', p, q are positive coefficients given by (18), (20), (22), and (23). It should be noted again that in these expressions s is even, the terms in $s = 0$ having a factor $\tfrac{1}{2}$; further, in expressions involving the coefficients C and F, n is even, in those with D and E, n is odd.

11. In the coupling terms in (25) the coefficients p and q are, in general, of unequal value numerically, but in the corresponding terms in Haskind's equations they are equal. Haskind denotes these terms by $-c U \dot{\psi}$ and $+c U \dot{h}$, with c defined by a double surface integral. On examination it appears that this expression for c does not involve the wave motion, but involves only the velocity potential due to the oscillations determined as if the free surface condition were $\phi = 0$. Further, it is based on replacing the solid by a source distribution over the surface of density σ where $4\pi\sigma$ equals the normal surface velocity, and this is then contracted to a distribution over the vertical plane section; this is a simplification which is appropriate when the form approximates to a thin disc.

Turning to equation (2) the functions F_1 and F_2 were determined to satisfy the surface condition $\partial\phi/\partial z = 0$. Suppose, for a moment, that we determine F_1 and F_2 from the surface condition $\phi = 0$; then we should have

$$\begin{aligned}\phi = U x &- a e U P_1(\mu) Q_1(\zeta)/\dot{Q}_1(\zeta_0) \\ &- \frac{a e \zeta_0 \dot{h}}{(\zeta_0^2-1)^{\frac{1}{2}} \dot{Q}_1^1(\zeta_0)} P_1^1(\mu) Q_1^1(\zeta) \sin\omega \\ &- \frac{a^2 e^2 \dot{\psi}}{3(\zeta_0^2-1)^{\frac{1}{2}} \dot{Q}_2^1(\zeta_0)} P_2^1(\mu) Q_2^1(\zeta) \sin\omega \quad . \quad (26)\end{aligned}$$

To make this velocity potential consistent and satisfying,

say, the linearized free surface condition, there would be additional terms expressing the effect of the surface disturbance. Meantime we shall simply assume that these terms are small, or at least that they do not affect appreciably the coupling effect which is under consideration. Taking (26) as it stands and calculating Z and M as in the previous sections, the values of k_2 and k' will now be the usual values as if for motions in an infinite liquid. For the coupling terms, using (21) with (26), the vertical force upwards is

$$\tfrac{1}{6}\pi\rho a^3 e(1-k_1)U\dot{\psi}[Q_2^1(\zeta_0)/\dot{Q}_2^1(\zeta_0)]$$

$$\int_{-1}^{1}\frac{1-\mu^2}{\zeta_0^2-\mu^2}P_1^1(\mu)\dot{P}_2^1(\mu)\,d\mu$$

$$-\tfrac{1}{3}\pi\rho a^3 e^2(1-k_1)U\dot{\psi}(\zeta_0^2-1)^{\frac{1}{2}}Q_2^1(\zeta_0)\,. \quad (27)$$

Similarly for the additional moment we obtain the result

$$\tfrac{1}{6}\pi\rho a^3 e(1-k_1)U\dot{h}[Q_1^1(\zeta_0)/\dot{Q}_1^1(\zeta_0)]$$

$$\int_{-1}^{1}\frac{1-\mu^2}{\zeta_0^2-\mu^2}P_2^1(\mu)\dot{P}_1^1(\mu)\,d\mu$$

$$\tfrac{1}{3}\pi\rho a^3 e(1-k_1)U\dot{h}$$

$$[3(\zeta_0^2-1)^{\frac{1}{2}}Q_1^1(\zeta_0)-4Q_1^1(\zeta_0)/\dot{Q}_1^1(\zeta_0)]\,. \quad (28)$$

In general, the coefficients of $U\dot{\psi}$ and $U\dot{h}$ in (27) and (28) are not equal numerically. For the case $a/b = 10$, they are nearly equal. It can easily be shown that as $\zeta_0 \to 1$, we have

$$(\zeta_0^2-1)^{\frac{1}{2}}Q_2^1 \to (\zeta_0^2-1)$$

$$3(\zeta_0^2-1)^{\frac{1}{2}}Q_1^1(\zeta_0)-4Q_1^1(\zeta_0)/\dot{Q}_1^1(\zeta_0) \to (\zeta_0^2-1)$$

Thus for a long spheroid, with b/a small, the equations approximate to

$$\left.\begin{aligned}(1+k_2)M\ddot{h}-\tfrac{1}{3}(1-k_1)MU\dot{\psi}+g\rho Sh&=0\\(1+k')I\ddot{\psi}+\tfrac{1}{3}(1-k_1)MU\dot{h}+Mgm\psi&=0\end{aligned}\right\} \quad (29)$$

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Waves due to a floating sphere making periodic heaving oscillations

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The paper gives a discussion of the fluid motion due to a sphere, floating half immersed in water, which is made to describe small heaving oscillations. The velocity potential is obtained as a series for which the unknown coefficients are given by an infinite set of equations. These are solved approximately so as to obtain curves showing the variation with frequency of the virtual inertia coefficient and of the equivalent damping parameter.

1. When a floating solid is made to describe periodic oscillations wave motion is produced, and it is required to determine the resultant pressure on the solid and the energy radiated outwards in the wave motion. The problem has been studied in general form by John (1950), especially as regards the necessary conditions for the uniqueness of the solution of the potential problem. The only cases which, to my knowledge, have been worked out in any detail are two-dimensional problems. In particular, Ursell (1949, 1953) has examined fully the heaving motion of a circular cylinder half-immersed in water. Similar work has been carried out by Grim (1953) for cylinders with various forms of cross-section, more especially with a view to application to ship problems in estimating virtual inertia and damping coefficients for heaving motion. In all these cases the virtual inertia coefficient approaches an infinite value as the frequency becomes small; this is no doubt connected with the fact that the condition at the free-water surface then approximates to that for a rigid boundary, and the two-dimensional potential problem with that boundary condition is indeterminate. This does not arise for three-dimensional motion; the general case approximates to determinate potential problems in the two limits as the frequency approaches zero or infinity. The point of special interest is the variation of the virtual inertia coefficient with frequency between these limiting values. The general character of the variation has been surmised, but there do not seem to have been any actual calculations. In this paper we consider the simplest case, a sphere half-immersed and making small vertical oscillations. The calculations show that the virtual inertia coefficient rises to a maximum with increasing frequency, falls to a minimum and then presumably rises gradually to its final limiting value. The variation of the equivalent damping coefficient is also obtained. A solid of ship form would come between the two extremes of an infinite cylinder and a sphere, and could be represented better by, say, a spheroid. The limiting values of the virtual inertia coefficient for a spheroid can readily be calculated, but the general solution for any frequency leads to expressions too complicated for computation.

2. We take the origin O in the undisturbed water surface, with Oz vertically downwards. The water is assumed incompressible and frictionless and the motion is

symmetrical with respect to Oz. For periodic motion of frequency σ, the linearized condition for the velocity potential at the free surface is

$$\kappa_0\phi + \frac{\partial\phi}{\partial z} = 0; \quad z = 0, \tag{1}$$

with $\kappa_0 = \sigma^2/g$. If there is a periodic singularity of order n at the point $(0, 0, f)$ in the water, we have the known solution

$$\phi = \frac{P_n(\mu_1)}{r_1^{n+1}}\cos\sigma t - \frac{(-1)^n}{n!}\cos\sigma t\int_0^\infty \frac{\kappa+\kappa_0}{\kappa-\kappa_0}\kappa^n J_n(\kappa\varpi)\,e^{-\kappa(z+f)}\,d\kappa, \tag{2}$$

valid for $z+f>0$. The principal value of the integral in (2) is to be taken, and we have put $r_1^2 = x^2+y^2+(z-f)^2 = \varpi^2+(z-f)^2$, and $\mu_1 = (z-f)/r_1$. If r_2, μ_2 are polar co-ordinates referred to the image point $(0, 0, -f)$, the solution (2) can be expanded in the form

$$\frac{\phi}{\cos\sigma t} = \frac{P_n(\mu_1)}{r_1^{n+1}} + (-1)^n\frac{P_n(\mu_2)}{r_2^{n+1}} + 2(-1)^n\kappa_0\left\{\frac{1}{n}\frac{P_{n-1}(\mu_2)}{r_2^n} + \dots + \frac{\kappa_0^{n-1}}{n!}\frac{1}{r_2}\right\}$$
$$+ 2\frac{(-1)^n\kappa_0^{n+1}}{n!}\int_0^\infty \frac{J_0(\kappa\varpi)}{\kappa-\kappa_0}e^{-\kappa(z+f)}\,d\kappa. \tag{3}$$

The principal value of the integral in (3) is

$$-\pi Y_0(\kappa_0\varpi)\,e^{-\kappa_0(z+f)} - \frac{2}{\pi}\int_0^\infty \frac{\kappa_0\cos\kappa(z+f)+\kappa\sin\kappa(z+f)}{\kappa^2+\kappa_0^2}K_0(\kappa\varpi)\,d\kappa, \tag{4}$$

with the usual notation for the Bessel functions. We superpose on the motion given by (3) free symmetrical oscillations of frequency σ so that as $\varpi\to\infty$ the motion approximates to circular waves travelling outwards. For this purpose we add to (3) the term

$$2\frac{(-1)^n}{n!}\pi\kappa_0^{n+1}J_0(\kappa_0\varpi)\,e^{-\kappa_0(z+f)}\sin\sigma t. \tag{5}$$

The motion as $\varpi\to\infty$ then approximates to

$$\phi \to 2\pi\kappa_0^{n+1}\frac{(-1)^n}{n!}\left(\frac{2}{\pi\kappa_0\varpi}\right)^{\frac{1}{2}}\sin(\sigma t-\kappa_0\varpi+\tfrac{1}{4}\pi). \tag{6}$$

In general as $\varpi\to\infty$, ϕ is of order $\varpi^{-\frac{1}{2}}$; but from the expressions given in (2) and (3) it is possible to construct solutions in which ϕ is of order ϖ^{-2} or of higher order. These combinations of periodic singularities might be called wave-free singularities. They are given by

$$\phi = \left\{\frac{\kappa_0}{n+1}\frac{P_n(\mu_1)}{r_1^{n+1}} + \frac{P_{n+1}(\mu_1)}{r_1^{n+2}} - (-1)^n\frac{\kappa_0}{n+1}\frac{P_n(\mu_2)}{r_2^{n+1}} - (-1)^n\frac{P_{n+1}(\mu_2)}{r_2^{n+2}}\right\}\cos\sigma t. \tag{7}$$

For instance, taking $n = 1$, the singularity $\{\frac{1}{2}\kappa_0 r_1^{-2}P_1(\mu_1) + r_1^{-3}P_2(\mu_1)\}\cos\sigma t$ at the point $(0, 0, f)$ gives a surface elevation proportional to $(\varpi^2-2f^2)\sin\sigma t/(\varpi^2+f^2)^{\frac{5}{2}}$. For the particular application which is in view at present, we require the results when f is made zero. Thus from (7) we have wave-free solutions given by

$$\phi = \left\{\frac{\kappa_0}{2n}\frac{P_{2n-1}(\mu)}{r^{2n}} + \frac{P_{2n}(\mu)}{r^{2n+1}}\right\}\cos\sigma t, \tag{8}$$

with the origin O in the free surface.

3. We suppose a sphere, half-immersed in water, to be given small periodic oscillations, the velocity of the centre being $\cos\sigma t$. We take the boundary condition on the sphere to be satisfied at the mean position; thus, for all t,

$$-\frac{\partial\phi}{\partial r} = P_1(\mu)\cos\sigma t \quad (r = a;\ 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi). \tag{9}$$

We shall assume that the velocity potential can be expressed in terms of a series of functions (8) together with a suitable periodic source at the origin. Hence we take

$$\begin{aligned}\phi = a^2\left\{\frac{1}{r} - \pi\kappa_0 Y_0(\kappa_0\varpi)\,\mathrm{e}^{-\kappa_0 z} - \frac{2\kappa_0}{\pi}\int_0^\infty \frac{\kappa\sin\kappa z + \kappa_0\cos\kappa z}{\kappa^2+\kappa_0^2}K_0(\kappa\varpi)\,\mathrm{d}\kappa\right\}& \\ \times(C\cos\sigma t + D\sin\sigma t) + \pi\kappa_0 a^2 J_0(\kappa_0\varpi)\,\mathrm{e}^{-\kappa_0 z}(C\sin\sigma t - D\cos\sigma t)& \\ + \sum_1^\infty a^{2n+2}\left\{\frac{\kappa_0}{2n}\frac{P_{2n-1}(\mu)}{r^{2n}} + \frac{P_{2n}(\mu)}{r^{2n+1}}\right\}(A_n\cos\sigma t + B_n\sin\sigma t).&\end{aligned} \tag{10}$$

This expression satisfies the boundary condition (1) and also reduces to outward circular waves as $\varpi\to\infty$. After some reduction, we obtain (9) in the form

$$\begin{aligned}&L(C\cos\sigma t + D\sin\sigma t) + M(C\sin\sigma t - D\cos\sigma t)\\ &\quad + \sum_1^\infty\{\beta P_{2n-1}(\mu) + (2n+1)P_{2n}(\mu)\}(A_n\cos\sigma t + B_n\sin\sigma t) = P_1(\mu)\cos\sigma t,\end{aligned} \tag{11}$$

for all t and for $0\leqslant\theta\leqslant\frac{1}{2}\pi$. In (11) we have put $\beta = \kappa_0 a = \sigma^2 a/g$, and

$$\begin{aligned}L = 1 - \pi\beta^2\{\cos\theta Y_0(\beta\sin\theta) + \sin\theta Y_1(\beta\sin\theta)\}\,\mathrm{e}^{-\beta\cos\theta}&\\ - \frac{2\beta}{\pi}\int_0^\infty \frac{2u\sin(\beta u\cos\theta) + (1-u^2)\cos(\beta u\cos\theta)}{(1+u^2)^2}K_0(\beta u\sin\theta)\,\mathrm{d}u,&\end{aligned} \tag{12}$$

$$M = \pi\beta^2\{\cos\theta J_0(\beta\sin\theta) + \sin\theta J_1(\beta\sin\theta)\}\,\mathrm{e}^{-\beta\cos\theta}. \tag{13}$$

The coefficients C, D, A_n, B_n are to be determined from (11). The functions defined by (8) are not orthogonal, but it turns out to be convenient to follow the usual procedure with (11) to give an infinite set of equations for the coefficients. Thus we multiply both sides of (11) by $\beta P_{2m-1}(\mu) + (2m+1)P_{2m}(\mu)$ and integrate with respect to μ from 0 to 1; we take $P_0(\mu)$ for the case $m = 0$.

We use the notation, with L given by (12),

$$L_0 = \int_0^1 L\,\mathrm{d}\mu, \quad L_m = \int_0^1\{\beta P_{2m-1}(\mu) + (2m+1)P_{2m}(\mu)\}L\,\mathrm{d}\mu, \tag{14}$$

with a similar notation for M_m derived from (13). Taking the terms in $\cos\sigma t$ and $\sin\sigma t$ separately, we obtain in this way a set of equations of which the first eight are

$$\left.\begin{aligned}&L_0C - M_0D + \tfrac{1}{2}\beta A_1 - \tfrac{1}{8}\beta A_2 + \tfrac{1}{16}\beta A_3 + \ldots &&= \tfrac{1}{2},\\ &L_1C - M_1D + (\tfrac{9}{5} + \tfrac{3}{4}\beta + \tfrac{1}{3}\beta^2)A_1 + \tfrac{13}{48}\beta A_2 - \tfrac{1}{16}\beta A_3 + \ldots &&= \tfrac{3}{8} + \tfrac{1}{3}\beta,\\ &L_2C - M_2D + \tfrac{13}{48}\beta A_1 + (\tfrac{25}{9} + \tfrac{45}{64}\beta + \tfrac{1}{7}\beta^2)A_2 + \tfrac{31}{128}\beta A_3 + \ldots &&= -\tfrac{5}{48},\\ &L_3C - M_3D - \tfrac{1}{16}\beta A_1 + \tfrac{11}{128}\beta A_2 + (\tfrac{49}{13} + \tfrac{175}{256}\beta + \tfrac{1}{11}\beta^2)A_3 + \ldots &&= \tfrac{7}{128}.\end{aligned}\right\} \tag{15}$$

$$(\text{Similar equations in } D,\ -C,\ B_1,\ B_2,\ \ldots = 0.) \tag{16}$$

It may be noted that it is only the second equation in (15) which includes a term in β on the right.

4. For large values of the frequency parameter, that is $\beta \to \infty$, we may expect the solution to approach that appropriate to the free surface condition $\phi = 0$, namely,

$$\phi = \tfrac{1}{2}a^3 \frac{P_1(\mu)}{r^2} \cos \sigma t. \tag{17}$$

The equations (15) are consistent with this if C, D approximate to zero, $A_2, A_3, \ldots$ being of order β^{-2} and βA_1 approximating to unity. However, there are difficulties in evaluating some of the integrals involved for large values of β, due partly to having taken a concentrated point source at the origin instead of a distributed source. We shall therefore limit the calculations to moderate values of β. For large values the problem is better treated separately, possibly by the method used by Ursell (1953) for the similar two-dimensional case.

For small values of β, the free surface condition approximates to $\partial\phi/\partial z = 0$ and the solution is then

$$\phi = \left\{\frac{1}{2}\frac{a^2}{r} + \sum_1^{\infty} \frac{a^{2n+2}}{r^{2n+1}} A_n P_{2n}(\mu)\right\} \cos \sigma t, \tag{18}$$

where $$A_n = \frac{4n+1}{2n+1}\int_0^1 P_1(\mu) P_{2n}(\mu)\, \mathrm{d}\mu = \frac{(-1)^{n+1}(4n+1)(2n!)}{2^{2n}(2n-1)(2n+1)(2n+2)(n!)^2}. \tag{19}$$

This is given by (15) and (16) with $\beta = 0$, the coefficient L_0 being then unity and the other L and M coefficients being zero. The series in (18) is convergent. We shall assume, in the general case, the convergence of the solution in (10) with the unknown coefficients derived from (15) and (16).

5. The expression for I given in (12) may be put into a more suitable form for computation. If we write I for the integral in (12) we have

$$I = -\beta \frac{\partial}{\partial \beta}\int_0^{\infty} \frac{u \sin(\beta u \cos\theta) + \cos(\beta u \cos\theta)}{1+u^2} K_0(\beta u \sin\theta)\, \mathrm{d}u. \tag{20}$$

Further, if we put $$X = \int_0^{\infty} \frac{u \sin(pu) + \cos(pu)}{1+u^2} K_0(qu)\, \mathrm{d}u. \tag{21}$$

X reduces to $\frac{1}{4}\pi^2\{H_0(q) - Y_0(q)\}$ for $p = 0$, $q > 0$, where H is the Struve function. Also we have

$$\frac{\partial X}{\partial p} + X = \int_0^{\infty} K_0(qu) \cos(pu)\, \mathrm{d}u = \frac{\pi}{2}\frac{1}{(p^2+q^2)^{\frac{1}{2}}}. \tag{22}$$

From this we deduce for the integral in (20) the form

$$\tfrac{1}{4}\pi^2\{H_0(\beta\sin\theta) - Y_0(\beta\sin\theta)\} e^{-\beta\cos\theta} + \tfrac{1}{2}\pi e^{-\beta\cos\theta}\int_0^1 (\tan^2\theta + t^2)^{-\frac{1}{2}} e^{\beta t\cos\theta}\, \mathrm{d}t. \tag{23}$$

Using (23) and (20), we find, after some reduction,

$$L = 1 + \beta^2 - \tfrac{1}{2}\pi\beta^2 e^{-\beta\cos\theta}[\{H_0(\beta\sin\theta) + Y_0(\beta\sin\theta)\}\cos\theta$$
$$+ \{H_1(\beta\sin\theta) + Y_1(\beta\sin\theta)\}\sin\theta] - \beta^2\cos\theta\, e^{-\beta\cos\theta}(A + \beta B\cos\theta). \tag{24}$$

with the notation

$$A = \int_0^1 (\tan^2\theta + t^2)^{-\frac{1}{2}} e^{\beta t\cos\theta}\,dt, \quad B = \int_0^1 (\tan^2\theta + t^2)^{\frac{1}{2}} e^{\beta t\cos\theta}\,dt. \tag{25}$$

The integrals A and B can readily be computed, either by quadrature or by expansion in powers of β. For $\theta = 0$, L reduces to $1+\beta-\beta^2 e^{-\beta} Ei(\beta)$.

For any given β, values of L were computed from (24) at intervals of 18° from 0 to 90°; then by numerical or graphical interpolation intermediate values were obtained. These were then used to compute the quantities from (14) by numerical quadrature. Expressions suitable for small values of β can be obtained. We find from (24)

$$\begin{aligned} L = {} & 1+\beta+\beta^2(1-\cos\theta)+\beta^3(\tfrac{1}{2}\cos^2\theta-\tfrac{3}{4}\sin^2\theta+2\sin\theta\cos\theta) \\ & -(\beta^2\cos\theta-\beta^3\cos^2\theta)\log\{\tfrac{1}{2}\beta\gamma(1+\cos\theta)\}-\beta^2\sin^2\theta\log(\tfrac{1}{2}\beta\gamma\sin\theta)+\ldots, \end{aligned} \tag{26}$$

with $\ln\gamma = 0{\cdot}57712$. Using this in (14) we obtain expansions for the coefficients L_m.

The coefficients M_m were computed either by quadrature or from a power series in β which can be found from the expansion

$$M_m = -\sum_{n=1}^{\infty}(-1)^n \frac{\beta^{n-1}}{(n-1)!}\int_0^1 P_n(\mu)\{\beta P_{2m-1}(\mu)+(2m+1)P_{2m}(\mu)\}\,d\mu. \tag{27}$$

Returning to (15) and (16), once the L and M coefficients have been calculated, the equations are in suitable form for approximate solution to any required degree of accuracy, at least for moderate values of β.

Accurate computation has not been attempted, but a somewhat crude approximation is sufficient to bring out the general character of the results. Calculations were carried out for $\beta = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, $1{\cdot}0$, $2{\cdot}0$ and $3{\cdot}0$. As an example of the numerical values, we find for $\beta = 0{\cdot}4$,

$$L_0 = 1{\cdot}4707, \quad L_1 = 0{\cdot}2391, \quad L_2 = -0{\cdot}0582, \quad L_3 = -0{\cdot}0547,$$
$$M_0 = 0{\cdot}2464, \quad M_1 = 0{\cdot}1400, \quad M_2 = -0{\cdot}0428, \quad M_3 = 0{\cdot}0262.$$

With these values we solve the first four equations from (15) and from (16) for eight unknowns, neglecting the unknowns of higher order; this gives

$$C = 0{\cdot}3029, \quad D = -0{\cdot}0486, \quad A_1 = 0{\cdot}2012, \quad A_2 = -0{\cdot}0352,$$
$$A_3 = 0{\cdot}0193, \quad B_1 = -0{\cdot}0146, \quad B_2 = 0{\cdot}0039, \quad B_3 = -0{\cdot}0027.$$

These may be compared with the corresponding values for the limiting case $\beta = 0$, namely,

$$C = 0{\cdot}5, \quad D = 0, \quad A_1 = 0{\cdot}2083, \quad A_2 = -0{\cdot}0375, \quad A_3 = 0{\cdot}0145, \quad A_{2n} = 0, \quad B_n = 0.$$

6. The resultant hydrodynamic pressure on the sphere is given by

$$Z = -2\pi\rho a^2\int_0^{\frac{1}{2}\pi}\frac{\partial\phi}{\partial t}\sin\theta\cos\theta\,d\theta. \tag{28}$$

On the sphere we have

$$\phi = aL'(C\cos\sigma t + D\sin\sigma t) + \pi\beta a J_0(\beta\sin\theta)\,\mathrm{e}^{-\beta\cos\theta}(C\sin\sigma t - D\cos\sigma t)$$
$$+\sum_1^{\infty} a\left\{\frac{\beta}{2n}P_{2n-1}(\mu) + P_{2n}(\mu)\right\}(A_n\cos\sigma t + B_n\sin\sigma t), \quad (29)$$

where, after using (21), (23) and (25),

$$L' = 1 - \tfrac{1}{2}\pi\beta\,\mathrm{e}^{-\beta\cos\theta}\{H_0(\beta\sin\theta) + Y_0(\beta\sin\theta)\} - \beta A\,\mathrm{e}^{-\beta\cos\theta}. \quad (30)$$

We obtain Z in the form

$$Z = \tfrac{2}{3}\pi\rho a^3 k\sigma\sin\sigma t - \tfrac{2}{3}\pi\rho a^3 2h\sigma\cos\sigma t, \quad (31)$$

with

$$\tfrac{1}{3}k = L_1'C - \pi\beta M_1'D + (\tfrac{1}{6}\beta + \tfrac{1}{8})A_1 - \tfrac{1}{48}A_2 + \tfrac{1}{128}A_3 - \ldots, \quad (32)$$

$$\tfrac{2}{3}h = L_1'D + \pi\beta M_1'C + (\tfrac{1}{6}\beta + \tfrac{1}{8})B_1 - \tfrac{1}{48}B_2 + \tfrac{1}{128}B_3 - \ldots. \quad (33)$$

In (32) and (33) we have put

$$L_1' = \int_0^1 L'P_1(\mu)\,\mathrm{d}\mu, \quad M_1' = \int_0^1 J_0(\beta\sin\theta)\,\mathrm{e}^{-\beta\cos\theta}P_1(\mu)\,\mathrm{d}\mu. \quad (34)$$

The velocity of the sphere being $\cos\sigma t$, the first term in (31) represents an addition to the effective mass, the virtual inertia coefficient being k as given by (32). The

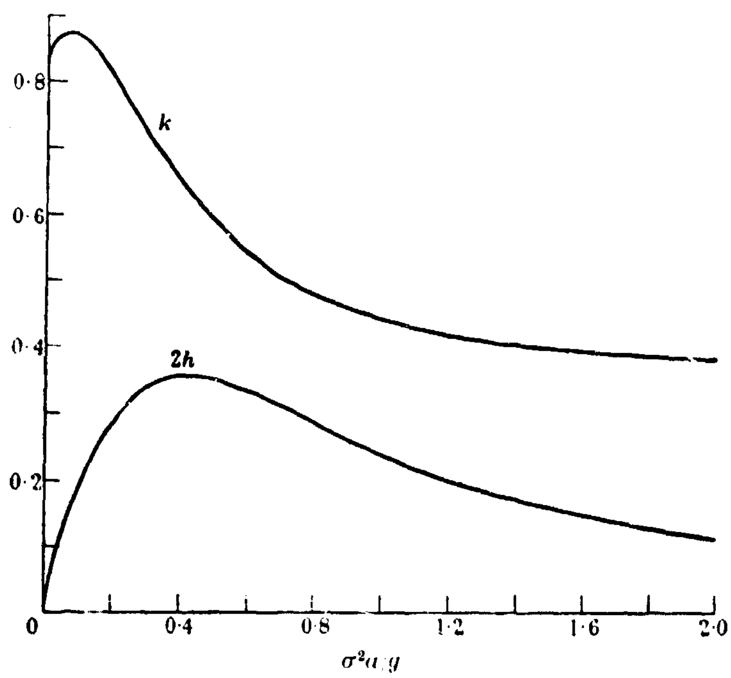

FIGURE 1. Variation of virtual inertia coefficient k and damping parameter $2h$ with frequency

second term in (30) being proportional to the velocity, the quantity h as given by (33) may be called a damping parameter; it gives some estimate of the damping factor if the motion were unforced damped periodic motion. We may obtain an

alternative expression for h from energy considerations. The motion as $\varpi \to \infty$ is given by

$$\phi \to \pi\kappa_0 a^2 \left(\frac{2}{\pi\kappa_0\varpi}\right)^{\frac{1}{2}} e^{-\kappa_0 z}\{C\sin(\sigma t - \kappa_0\varpi + \tfrac{1}{4}\pi) - D\cos(\sigma t - \kappa_0\varpi + \tfrac{1}{4}\pi)\} \quad (35)$$

The average rate of flow of energy outwards is $\pi^2\rho\sigma a^3(C^2+D^2)$; equating this to $\frac{2}{3}\pi\rho\sigma a^3 h$ we have

$$h = \tfrac{3}{2}\pi\beta(C^2+D^2). \quad (36)$$

For numerical evaluation, the L' and M' quantities were computed by methods similar to those used for the L and M quantities in § 5. As an example, from the values given above for $\beta = 0{\cdot}4$, we find $k = 0{\cdot}656$; from (33) we obtain $h = 0{\cdot}174$, while (36) gives $h = 0{\cdot}177$. It will be appreciated that the values for the B coefficients are more liable to error than for the A coefficients; however, the two values for h were in fair agreement. Although the numerical computations for k and h were only made approximately the results were sufficiently consistent to be represented by smooth curves; these are shown in figure 1. The virtual inertia coefficient k begins from a limiting value of 0·828, rises to a maximum of about 0·88, falls to a minimum of 0·38 and it then, presumably, rises slowly to the limiting value of 0·5. In order to use the same ordinate scale, the damping parameter $2h$ is shown in figure 1; this rises to a maximum of about 0·35, the largest values of the damping parameter occurring in the frequency range in which the virtual inertia coefficient varies most rapidly.

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A Note on Form Friction and Tank Boundary Effect

Sir Thomas Havelock

The following remarks are concerned with a suggestion made by Professor Horn many years ago for estimating form friction by means of the sinkage of the model, and with the possible application of this method to motion in restricted water.

The influence of the walls and bed of the tank can, in usual circumstances, be conveniently separated into wave effect and frictional effect. The underlying theory of the wave effect is well known; the bed contributes the so-called shallow water effect, while the walls may give rise to interference effects due to the waves reflected from them. It is true that the actual calculations are beset with difficulties, such as occur in wave theory generally; but at least it may be said that the fundamental causes can be specified reasonably. The theoretical aspect of the frictional effect seems to me to be less clear. The point in question is the difference between the ship form and a plank. A thorough analysis, theoretical and experimental, seems impracticable in general; though useful and important results are available for completely submerged solids of revolution. Assuming the form friction to be small, the usual practical method is to use the idea of effective equivalent velocity; that is, the actual frictional resistance of the ship at a given speed is taken as equal to that of a plank at some slightly higher speed. Failing a complete analysis of the actual flow, we can only make some reasonable assumption for defining this equivalent effective speed.

Horn [1] proposed to use the measured sinkage of the model for this purpose. If v is the velocity, and h is the sinkage, he gives for the required effective mean velocity v_m the expression

$$v_m = (v^2 + 2gh)^{1/2}, \qquad (1)$$

or if $v_m = v + \delta v$, the relative increase in velocity is

$$\delta v/v = (1 + 2gh/v^2)^{1/2} - 1. \qquad (2)$$

If the frictional resistance R is proportional to v^n, the relative increase in resistance, or the form friction, is given by $\delta R/R = n\,\delta v/v$. It was shown from model data that this gave reasonable values for the form friction, of the order of 8 per cent.

In a short paper a few years later [2], I examined the theoretical solution for a particular form, namely the general ellipsoid, including the case of a spheroid. The problem was treated as the motion of a double model, that is, a complete ellipsoid moving axially in an infinite liquid: a problem which can be solved exactly.

Taking the motion along a horizontal axis Ox with the transverse axis Oy horizontal and with Oz vertical, an expression was obtained for the resultant vertical fluid pressure on one-half of the surface of the ellipsoid with respect to the xy-plane. If we now suppose the ellipsoid to be floating half immersed and if the velocity is small so that we may neglect the surface disturbance of the water, we can define an equivalent sinkage. If Z is this defect of vertical pressure and S is the area of the water plane section, we take $h = Z/g\rho S$. The results were compared numerically with Horn's value and also with those obtained by Amtsberg [3] for totally submerged spheroids. The analytical expressions for the general ellipsoid were given in terms of ellipsoidal coordinates; I quote now the special case of a prolate spheroid, where the result can be put into a simple form.

The value of Z is given by

$$Z = \tfrac{1}{2}\pi\rho a b v^2 (1 + k_1)^2 \frac{a(a + 2b)}{(a + b)^2} - \tfrac{1}{2}\pi\rho a b v^2 \qquad (3)$$

and the sinkage, as defined, is $h = Z/\pi\rho a b g$.

In this, 2a is the length of the spheroid, 2b the equatorial diameter, and k, the virtual inertia coefficient of the spheroid for axial motion. If, for example, we take a length-beam ratio of 8, we find $h = 0.029\ v^2/g$; and assuming $n = 1.825$, we get an increase in frictional resistance of 5.3 per cent, agreeing fairly well with Amtsberg's values.

Comparing (3) with Horn's definition of the mean velocity we see that in this case

$$v_m = v\,(1 + k_1)\left[a\,(a + 2\,b)\right]^{1/2} / (a + b)\,. \qquad (4)$$

For most cases of interest, a/b is fairly large, say 8 or more, and we have approximately

$$v_m = v\,(1 + k_1)\,. \qquad (5)$$

Hence we have the simple, and interesting approximation

$$\delta v/v = k_1 \;;\; \delta R/R = n\,k_1\,. \qquad (6)$$

For example, the virtual inertia coefficient for a spheroid of length/beam ratio of 8 is 0.029, and $\delta R/R = 0.053$. It might be going too far to apply this to ship forms, where the inertia coefficient is itself subject to uncertainty; however, assuming an effective virtual coefficient of 5 per cent would give a form friction of about 9 per cent.

Of course for a spheroid the velocity distribution is known exactly and we might take some other suitable definition of the mean velocity. For instance, it might be obtained from the mean of the square of the tangential velocity per unit area of surface. It can easily be shown that this leads to the same approximation (5) when a/b is large. The point of Horn's definition is that the sinkage can be determined experimentally.

Coming now to the corresponding problem in restricted water, the tank boundary effect or the so-called blockage effect has become important in view of the need for greater accuracy and certainty in interpreting experimental model results. Reference may be made, for instance, to two recent papers: the B.S.R.A. experiments on the Lucy Ashton [4] (Conn, Lackenby and Walker), and the scale effect in Victory ships and models [5] (van Lammeren, van Manen and Lap). In the discussion on the former paper, Professor Horn referred to his method of using measured sinkages to estimate form friction and suggested that it might be used to determine the necessary correction due to the boundaries of the tank. However it seems that, at least for the Lucy Ashton, the differences in sinkage were too small to be determined experimentally with sufficent accuracy. It might be of interest to extend my previous calculations to the similar problem in restricted water. Consider a spheroid half-immersed and moving along a tank of breadth B and depth H. With the same limitations as for unrestricted water, we consider the motion of the complete spheroid in an enclosed rectangular channel filled with water, B being the distance between the side walls and 2 H that between the upper and lower walls. We require to calculate the quantity Z of (3), that is the resultant vertical force on the lower half of the spheroid. It is possible to obtain analytical expressions in a series of terms involving spheroidal harmonics; but they become very complicated and it is difficult to assess the degree of approximation numerically. The particular case of a sphere can be worked out in more detail, but the spheroid is complicated by the additional parameter of the length-beam ratio.

Taking only the first step in the approximation I give now the result obtained for the quantity Z; it is

$$Z = \tfrac{1}{2}\pi\varrho\,a\,b\,v^2\left[(1 + k_1)^2\,\frac{a\,(a + 2\,b)}{(a + b)^2}\left\{1 + (1 + k_1)\,\frac{a\,b^2}{(a^2 - b^2)^{1/2}}\,\alpha\right\}^2 - 1\right]. \qquad (7)$$

If we write

$$q^2 = (m^2B^2 + 4\,n^2H^2) / (a^2 - b^2)\;;$$

$$\beta = \tfrac{1}{2}\left[q + (q^2 + 4)^{1/2}\right],$$

the coefficient α is given by

$$\alpha = \sum_m \sum_n \left(\log\frac{\beta + 1}{\beta - 1} - \frac{2}{\beta} - \frac{1}{3}\,\frac{1}{\beta^3}\right) \qquad (8)$$

where the double summation is taken over all positive and negative integral values of m and n, excluding the pair $m = 0, n = 0$. This summation arises from the doubly infinite series of images involved in the solution. This result may be subject to correction if the analysis is carried to a further stage, and the range of applicability is uncertain on that account. As before, we may simplify the result if b/a is small; we have approximately,

$$v_m = v\,(1 + k_1)\left[1 + \frac{b^2}{a^2}\,(1 + k_1)\,\alpha\right] \qquad (9)$$

and, instead of (6) for unrestricted water, we have

$$\frac{\delta v}{v} = k_1 + \frac{b^2}{a^2}\,(1 + k_1)^2\,\alpha\,. \qquad (10)$$

Numerical computation has been made for a few cases for the spheroid with $a = 8\,b$. We have taken $B = 2\,H$ as a usual tank ratio and it also simplifies the computation. For $B/2\,b$ equal to 12, 8, $4\sqrt{2}$ the approximate values of the coefficient α are 0.065, 0.160, 0.392 respectively. If we define the blockage coefficient as the ratio of the maximum cross section of the half-spheriod to the sectional area of the tank, this coefficient is 0.005, 0.012, 0.024. From (10) the percentage form resistances at these values are 5.46, 5.84 and 6.93 respectively, the value for unrestricted water being 5.29.

The differences are negligible for small values of the blockage coefficient. It is not worth while attempting any direct comparison with model results meantime. The calculations were made for a spheroid under the limitations specified; moreover they refer only to the effect on form friction and take no account of surface disturbance or wave effects.

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THE DAMPING OF HEAVE AND PITCH: A COMPARISON OF TWO-DIMENSIONAL AND THREE-DIMENSIONAL CALCULATIONS

By Professor Sir Thomas H. Havelock, M.A., D.Sc., F.R.S. (*Honorary Member and Associate Member of Council*)

1. Damping coefficients for heave and pitch are usually derived by calculating the mean rate at which energy travels outwards in the wave motion produced by the oscillations. The calculation is based upon approximate solutions for the two-dimensional motion due to heaving oscillations of a long cylindrical floating solid; the application to heaving and pitching for a ship then proceeds by the so-called strip method. Each thin section of the ship is treated as part of an infinite cylinder of the cross-section at that point, sending out two-dimensional waves on either side. The coefficients for the ship are obtained by integrating along the length of the ship. Reference may be made to Weinblum and St. Denis(2) for a detailed exposition with calculations. In the work of those authors no allowance was made for the difference between the assumed flow and the actual three-dimensional flow; this may be justified to some extent in that results in practical cases seem to give reasonable agreement for heaving, but the application to pitching requires more consideration.

In discussing this point, Korvin-Kroukovsky and Lewis(3) remark that the damping coefficient for heaving may be assumed to be correctly represented by the two-dimensional calculation, but they adopt an empirical reduction factor of one-half for the similar calculations for pitching.

In a recent paper Korvin-Kroukovksy(4) discusses the matter in considerable detail, and expresses the opinion that an important effect of three-dimensional flow may exist. He estimated the validity of the two-dimensional calculations by comparing the data with results from towing-tank experiments on two models. It was found that, at the natural frequencies of the models, the results were in substantial agreement both for heaving and for pitching within the limits of experimental error, which were admittedly rather wide limits. However, for more extended ranges of frequencies, it was found necessary to introduce empirical correction factors, in one case, for instance, reducing the damping coefficient for pitching to 75 per cent of the calculated value. Korvin-Kroukovsky remarks: "In the case of damping in heave, most of the force comes from the middle part of the body where the flow hardly differs from the assumed two-dimensional one. The good agreement in regard to damping in heave was therefore not surprising. The close agreement in the damping in pitch was not expected, however, and in fact was later not confirmed in the application of the calculations to the entire set of model motions. Most of the contribution to the moment coefficient comes from the ends of the ship, where one logically should expect a large change from the assumed two-dimensional flow to the actual three-dimensional flow." It is clear that the matter is not in a very satisfactory state, especially as the use of an inclusive empirical factor may hinder recognition of the true cause of the discrepancy.

2. The present work is intended, not as a solution of the problem, but as a contribution towards elucidating the particular point of the difference between two- and three-dimensional calculations. Of course the only really satisfactory method would be to work out the problem for a floating solid. It is not difficult to formulate the mathematical equations; but even for a simple form, such as a spheroid half immersed, the expressions soon become very complicated and numerical computation of prohibitive length. In this paper we deal with the simpler problem of a solid which is wholly immersed in the water, and we obtain the damping coefficients by the two methods: strip-method and three-dimensional. Although the separate results would not be applicable to a surface ship, it is thought that the ratios of the coefficients obtained by the two methods should at least give a useful indication of the sort of difference that might be expected. The calculations are given in the Appendix, comprising the basic theory, application to a submerged spheroid, approximate expressions for any elongated solid of revolution, and some remarks on the general ellipsoid with unequal axes.

3. We consider now some numerical results for a spheroid submerged in water with its axis horizontal. The spheroid is made to describe (i) heaving oscillations, (ii) pitching oscillations. E_H is the rate of energy loss for heaving calculated from three-dimensional flow, E_{HS} from the strip method. The corresponding damping coefficients in the equations of motion of the solid are directly proportional to the energy loss; thus E_H/E_{HS} is the ratio of the coefficients by the two methods. Similarly, for pitching E_P/E_{PS} is the required ratio. The general formulae are given in (23) and (24). We take a spheroid with a length-beam ratio of 8, as a fair value for comparison with ship models; in this case $e = 0{\cdot}996$, $k_2 = 0{\cdot}945$, $k' = 0{\cdot}84$. With these values (23) and (24) were computed for integral values of $\kappa_0 a$, that is of $\sigma^2 a/g$, up to 10. The results are shown in Fig. 1 on a base

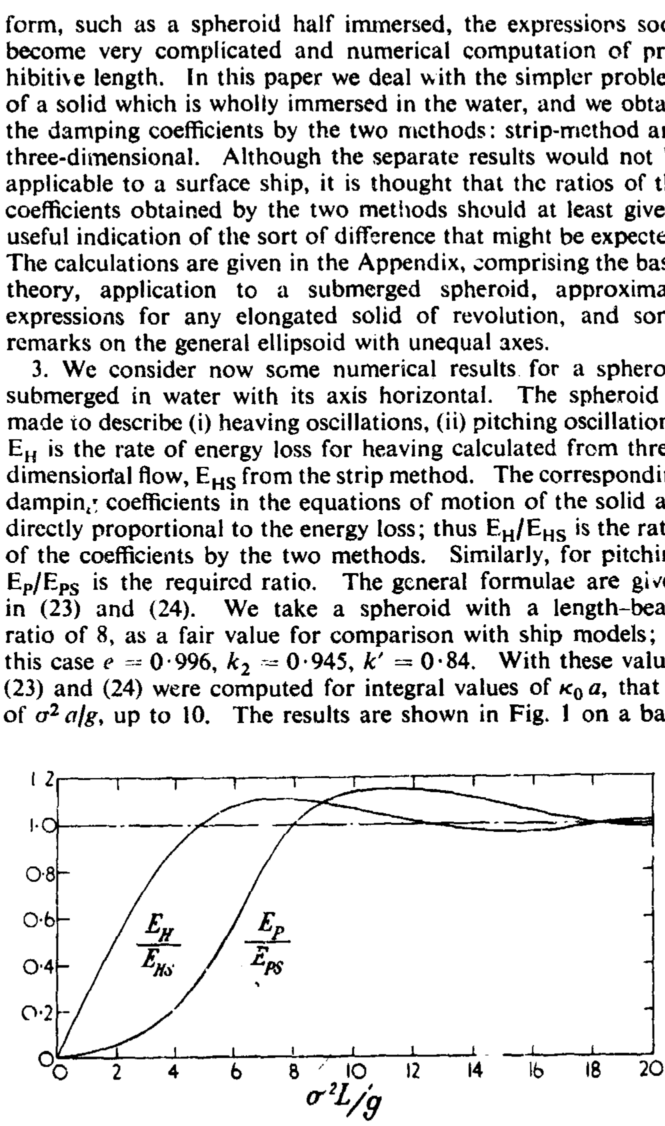

Fig. 1.—Ratios of damping coefficients for heaving and for pitching

of $\sigma^2 L/g$, where σ is the circular frequency of the oscillations and L the length of the model; this seems to be the suitable parameter for comparison with model results.

The ratio for heaving rises rapidly at first and attains a maximum of about $1{\cdot}1$, and then, with small alternations, it soon approaches unity. With the sort of accuracy attainable practically, the ratio may be taken as unity when $\sigma^2 L/g$ is greater than about 6. The ratio for pitching rises slowly at first and then very rapidly up to a maximum of about $1{\cdot}15$. In this case it

might, for practical purposes, be taken as unity if $\sigma^2 L/g$ exceeds about 8.

In addition to this approximation to unity above these respective values of $\sigma^2 L/g$, a specially interesting feature of the curves is the rapid fall in both ratios for smaller values of the parameter.

It has already been remarked that these results can only be taken as suggestive when applied to surface ships; however, it is of interest to see what are the relevant ranges of the parameter in such cases, referring in particular to work in which the damping coefficients have been calculated by the strip method over a range of frequencies.

For free oscillations at the natural frequencies, there are data in the paper by St. Denis[2] for a ship of length 600 ft. and beam 81 ft. The values of σ for free oscillations are given as 0·706 and 0·821 for heaving and pitching respectively. The corresponding values of $\sigma^2 L/g$ are 10 and 12·6 and these both lie within the ranges given above where no correction factor is needed. A similar remark would, no doubt, apply to the experiments with 5 ft. models used by Korvin-Kroukovsky,[4] although the natural frequencies do not seem to be given in the paper. However, the present work may be taken to confirm his experimental result that for natural oscillations the two-dimensional calculation does not require any appreciable correcting factor either for heaving or for pitching.

For forced oscillations we have a wider range of frequencies. In work for which data are available, the heaving and pitching are produced by driving the model at given speed through regular waves of given wavelength. For instance, from St. Denis,[2] for a 600-ft. ship moving against waves with $\lambda/L = 1\cdot25$, σ ranges from 0·518 at zero speed to 0·77 at 30 knots; thus $\sigma^2 L/g$ ranges from 5 to 11. From the curves in Fig. 1, heaving may be said to require no correcting factor; but for pitching, the lower values are well within the critical range where a large correction is needed and where it changes rapidly.

There are similar data from Korvin-Kroukovsky[4] for 5-ft. models. With one model and $\lambda/L = 1$ the parameter ranges from 6·3 to 20, and with another model and $\lambda/L = 1\cdot5$, it ranges from 4·3 to 12·5. Here again the pitching calculation seems to require considerable correction at the lower speeds.

It should be noted again that one can only expect general indications in applying the present results to surface ships. For one thing, a spheroid is not a normal ship form. A more important point is that the flow round a completely submerged solid may differ considerably from that round a floating body. However, it is possible that the strip method and the three-dimensional calculation might be affected in much the same way; if so, the ratios for the two methods may not be so far astray.

Finally, in all calculations for forced oscillations due to advancing through waves, it is assumed that the only effect of the speed is to alter the frequency of encounter. But a satisfactory theory of heaving and pitching including the effect of speed of advance, for anything like a normal ship form, is one of the main outstanding problems. The corresponding theory for a wholly submerged body might prove more tractable, and it may be possible later to extend the present work to a submerged spheroid which is moving forward while making heaving and pitching oscillations.

APPENDIX

1. The underlying theory was given in a previous paper[5] for a source distribution; it is convenient to give now explicit expressions for a distribution of vertical dipoles.

Take the origin O in the free surface of the water, with O x and O y horizontal and O z vertically downwards. If there is a vertical dipole of moment M $\cos \sigma t$ at the point (h, k, f) in the water, the velocity potential of the fluid motion is given by

$$\phi = \mathrm{M} \cos \sigma t \left[\frac{z-f}{r_1^3} - \frac{z+f}{r_2^3} + 2\kappa_0 \int_0^\infty \frac{J_0(\kappa \varpi')}{\kappa - \kappa_0} e^{-\kappa(z+f)} \kappa \, d\kappa \right] - 2\pi\kappa_0^2 \mathrm{M} \, J_0(\kappa_0 \varpi') \, e^{-\kappa_0(z+f)} \sin \sigma t \qquad (1)$$

where

$$r_1^2 = (x-h)^2 + (y-k)^2 + (z-f)^2 = \varpi'^2 + (z-f)^2$$
$$r_2^2 = \varpi'^2 + (z+f)^2; \quad \sigma^2 = g\kappa_0$$

The motion as $\omega' \to \infty$ is given by

$$\phi \to -2\pi\kappa_0^2 \mathrm{M} \, (2/\pi\kappa_0\varpi')^{1/2} e^{-\kappa_0(z+f)} \sin(\sigma t - \kappa_0 \varpi' + \pi/4) \qquad (2)$$

representing circular waves travelling outwards. For a given distribution of vertical dipoles all at the same depth f, we obtain the velocity potential from (1) or from (2) by integrating with respect to h and k over the given distribution. The rate of flow of energy outwards through a vertical cylindrical surface of radius ϖ is given by the rate of work of the fluid pressure over this surface, namely

$$-\int_0^\infty dz \int_0^{2\pi} \rho \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial \varpi} \varpi \, d\theta \qquad (3)$$

Taking the radius of the cylinder large, we only need ϕ to the order $\omega^{-1/2}$ as $\varpi \to \infty$.

If in (2) we put

$$x = \varpi \cos\theta; \quad y = \varpi \sin\theta;$$
$$\varpi'^2 = \varpi^2 - 2h\varpi\cos\theta - 2k\varpi\sin\theta + h^2 + k^2$$

then, to the required order, (2) gives

$$\phi \to -2\pi\kappa_0^2 \mathrm{M} \left(\frac{2}{\pi\kappa_0\varpi}\right)^{1/2} e^{-\kappa_0(z+f)} \sin\left(\sigma t - \kappa_0\varpi + \frac{\pi}{4} + \kappa_0 h\cos\theta + \kappa_0 k \sin\theta\right) \qquad (4)$$

Hence, for a given distribution, we shall have

$$\phi \to \varpi^{-1/2} e^{-\kappa_0(z+f)} \left[\mathrm{A} \sin\left(\sigma t - \kappa_0\varpi + \frac{\pi}{4}\right) + \mathrm{B} \cos\left(\sigma t - \kappa_0 \omega + \frac{\pi}{4}\right) \right] \qquad (5)$$

with A, B known functions of θ.

Putting this form into (3) and taking the mean value with respect to the time, we get for the mean rate of flow of energy outwards

$$\mathrm{E} = \tfrac{1}{4}\rho\sigma(\mathrm{A}^2 + \mathrm{B}^2) \qquad (6)$$

Finally, inserting the forms for A and B obtained by integrating (4) over the given distribution,

$$\mathrm{E} = 2\pi\rho\sigma\kappa_0^3 e^{-2\kappa_0 f} \int_0^{2\pi} (\mathrm{P}^2 + \mathrm{Q}^2) \, d\theta \qquad (7)$$

with $$\mathrm{P} + i\mathrm{Q} = \iint \mathrm{M}(h, k) \, e^{i\kappa_0(h\cos\theta + k\sin\theta)} \, dh \, dk \qquad (8)$$

In the present work, we shall not need any more general expressions, but an obvious extension would give similar results for any distribution of dipoles not necessarily in a horizontal plane.

2. Consider now a spheroid, of length $2a$ and equatorial diameter $2b$, immersed with its axis horizontal and at a depth f below the free surface. Suppose the spheroid made to describe small vertical oscillations, the velocity at any instant being V $\cos \sigma t$. It would be possible, theoretically, to proceed step-

by-step with successive approximations to a solution satisfying both the condition on the surface of the spheroid and that on the free surface of the water. We could then, at any stage, obtain the fluid pressure on the spheroid and hence the resultant vertical force. Of this force, the part in phase with the acceleration represents a change in virtual inertia, the part in phase with the velocity is connected directly with the loss of energy in the wave motion. At present we are concerned only with the latter part of the force, and we adopt the simpler procedure of obtaining directly the energy loss, taking only two terms in a successive approximation to the velocity potential.

For the first term we take the exact solution ϕ_0 for the motion of the spheroid in an infinite liquid. Taking, momentarily, the origin O at the centre of the spheroid, and the usual spheroidal co-ordinates

$$x = a e \mu \zeta; \quad y = a e (1 - \mu^2)^{1/2} (\zeta^2 - 1)^{1/2} \cos \omega;$$
$$z = a e (1 - \mu^2)^{1/2} (\zeta^2 - 1)^{1/2} \sin \omega$$

we have the known solution, as given in Lamb's *Hydrodynamics*, p. 142, with a slight change of notation.

$$\phi_0 = -\tfrac{1}{2} a \mathrm{V} (\zeta_0^2 - 1)(1 + k_2) \mathrm{P}_1^1(\mu) \mathrm{Q}_1^1(\zeta) \sin \omega \cos \sigma t \quad . \quad (9)$$

where $\zeta_0 = e^{-1} = (a^2 - b^2)^{-1/2}$, and k_2 is the virtual inertia coefficient for motion perpendicular to the axis. For the next step we add a potential ϕ_1 such that $\phi_0 + \phi_1$ satisfies the condition at the free surface. For this purpose it is convenient to express ϕ_0 as the potential of an equivalent dipole distribution.

Using the gener[illegible]

$$\mathrm{P}_n^s(\mu) \mathrm{Q}_n^s(\zeta) e^{is} \ \ [illegible] \ \ i\frac{\partial}{\partial z}\Big)^s \int_{-ae}^{ae} \frac{(a^2 e^2 - h^2)^{1/2 s} \mathrm{P}_n^s(h/ae)}{[(x-h)^2 + y^2 + Z^2]^{1/2}} \, dh \quad . \quad . \quad . \quad (10)$$

and taking the particular case in (9), we see that ϕ_0 is the potential of a line distribution of vertical dipoles along the axis of the spheroid between the two foci and of moment per unit length

$$\frac{1}{4} \frac{1 - e^2}{e^3} (1 + k_2)(a^2 e^2 - h^2) \mathrm{V} \cos \sigma t \quad . \quad . \quad (11)$$

We could now obtain ϕ_1 by integrating (1) with respect to h. For our purpose we proceed directly to the energy loss from (7) and (8). We require the integral

$$\int_{-ae}^{ae} (a^2 e^2 - h^2) e^{i \kappa_0 h \cos \theta} \, dh \quad . \quad . \quad . \quad (12)$$

and this has the value

$$4 a^3 e^3 \left(\frac{\pi}{2}\right)^{1/2} \frac{\mathrm{J}_{3/2}(\kappa_0 a e \cos \theta)}{(\kappa_0 a e \cos \theta)^{3/2}} \quad . \quad . \quad . \quad (13)$$

Collecting the various factors from (7), (8), (11), and (13), we obtain the energy loss for heaving, which we shall denote by a suffix H, namely

$$\mathrm{E_H} = 4 \pi^2 \rho \sigma \kappa_0^3 a^2 b^4 (1 + k_2)^2 \mathrm{V}^2 e^{-2\kappa_0 f} \int_0^{\pi/2} \frac{\mathrm{J}_{3/2}^2(\kappa_0 a e \cos \theta)}{(\kappa_0 a e \cos \theta)^3} \, d\theta \quad . \quad . \quad . \quad . \quad (14)$$

3. Suppose the spheroid to be making rotational oscillations about the transverse axis with angular velocity $\Omega \cos \sigma t$. The velocity potential for an infinite liquid is

$$\phi_0 = \tfrac{1}{6} a^2 e (\zeta_0^2 - 1) [1 + (2 \zeta_0^2 - 1)^2 k']$$
$$\Omega \mathrm{P}_2^1(\mu) \mathrm{Q}_2^1(\zeta) \sin \omega \cos \sigma t \quad . \quad (15)$$

in which k' is the virtual inertia coefficient for rotation. Using (10), we find that ϕ_0 is the potential of a line distribution of vertical dipoles along the axis between the foci and of moment per unit length

$$1/4 \, \zeta_0 (\zeta_0^2 - 1) [1 + (2 \zeta_0^2 - 1)^2 k'] \Omega h (a^2 e^2 - h^2) \cos \sigma t \ . \ (16)$$

For (8) we require the integral

$$\int_{-ae}^{ae} h (a^2 e^2 - h^2) e^{i \kappa_0 h \cos \theta} \, dh \quad . \quad . \quad . \quad (17)$$

and this has the value

$$4 a^4 e^4 \left(\frac{\pi}{2}\right)^{1/2} i \frac{\mathrm{J}_{5/2}(\kappa_0 a e \cos \theta)}{(\kappa_0 a e \cos \theta)^{3/2}} \quad . \quad . \quad . \quad (18)$$

From (7), (8), (16), and (18) we obtain the energy loss for pitching, which we denote by a suffix P,

$$\mathrm{E_P} = 4 \pi^2 \rho \sigma \kappa_0^3 a^4 b^4 e^2 [1 + (2 \zeta_0^2 - 1)^2 k']^2 e^{-2\kappa_0 f} \Omega^2 \int_0^{\pi/2} \frac{\mathrm{J}_{5/2}^2(\kappa_0 a e \cos \theta)}{(\kappa_0 a e \cos \theta)^3} \, d\theta \quad . \quad . \quad . \quad . \quad (19)$$

4. We obtain now the corresponding expressions by the two-dimensional strip method, denoting them by an additional suffix S.

For a circular cylinder of radius r with its axis at depth f, making heaving oscillations $\mathrm{V} \cos \sigma t$, we have the known expression for the energy loss to the same approximation,

$$\mathrm{E} = 2 \pi^2 \rho \sigma \kappa_0^2 r^4 \mathrm{V}^2 e^{-2\kappa_0 f} \quad . \quad . \quad . \quad (20)$$

per unit length of the cylinder.

For the spheroid, this result is assumed to hold for each thin disc of width dh; in fact, we might picture the method by assuming thin partitions transverse to the axis, separating the elementary discs and making the fluid motion purely two-dimensional for each disc. Integrating along the axis, we obtain by this method

$$\mathrm{E_{HS}} = 2 \pi^2 \rho \sigma \kappa_0^2 \mathrm{V}^2 e^{-2\kappa_0 f} \int_{-a}^{a} b^4 (1 - h^2/a^2)^2 \, dh$$
$$= \frac{32}{15} \pi^2 \rho \sigma \kappa_0^2 a b^4 \mathrm{V}^2 e^{-2\kappa_0 f} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

For pitching oscillations by this method, we simply substitute $h \Omega$ for V; hence

$$\mathrm{E_{PS}} = 2 \pi^2 \rho \sigma \kappa_0^2 \Omega^2 e^{-2\kappa_0 f} \int_{-a}^{a} b^4 h^2 (1 - h^2/a^2)^2 \, dh$$
$$= \frac{32}{105} \pi^2 \rho \sigma \kappa_0^2 a^3 b^4 \Omega^2 e^{-2\kappa_0 f} \quad . \quad . \quad . \quad . \quad (22)$$

5. The particular point in question is the ratio of the damping coefficients obtained by the two methods, which we take as equal to the ratio of the corresponding energy loss.

From (14) and (21) we have for heaving

$$\frac{\mathrm{E_H}}{\mathrm{E_{HS}}} = \frac{15}{8} \kappa_0 a (1 + k_2)^2 \int_0^{\pi/2} \frac{\mathrm{J}_{3/2}^2(\kappa_0 a \theta \cos \theta)}{(\kappa_0 a e \cos \theta)^3} \, d\theta \ . \ (23)$$

From (19) and (22) we have for pitching

$$\frac{\mathrm{E_P}}{\mathrm{E_{PS}}} = \frac{105}{8} \kappa_0 a e^2 [1 + (2 \zeta_0^2 - 1)^2 k']^2 \int_0^{\pi/2} \frac{\mathrm{J}_{5/2}^2(\kappa_0 a e \cos \theta)}{(\kappa_0 a e \cos \theta)^3} \, d\theta \quad . \quad . \quad . \quad . \quad (24)$$

As the length of the spheroid is increased, with a given breadth,

both e and k_2 approximate to unity, further, it can be shown that the asymptotic value of the integral in (23) is $2/15\,\kappa_0\, a\, e$. Hence, as one would expect, the ratio (23) approaches unity for a sufficiently long narrow spheroid. Under the same conditions, k' approaches unity and the asymptotic value of the integral in (24) is $2/105\,\kappa_0\, a\, e$; hence the ratio (24) approaches unity under these limiting conditions.

For numerical computation we can obtain power series for the integrals by substituting the known expression for the square of a Bessel function and integrating term-by-term: thus we have

$$\int_0^{\pi/2} J^2_{3/2}(\kappa_0\, a\, e \cos\theta)\sec^3\theta\, d\theta = 2\sum_{m=0}^{\infty}\frac{(-1)^m (m+1)!\,(\kappa_0\, a\, e)^{2m+3}}{(2m+1)(2m+3)(m!)^2(m+3)!} \quad . \quad (25)$$

$$\int_0^{\pi/2} J^2_{5/2}(\kappa_0\, a\, e \cos\theta)\sec^3\theta\, d\theta = 2\sum_{m=0}^{\infty}\frac{(-1)^m (m+2)\,(\kappa_0\, a\, e)^{2m+5}}{(2m+3)(2m+5)\,m!\,(m+5)!} \quad . \quad (26)$$

The series can be computed readily for values of $\kappa_0\, a\, e$ up to about 6. For higher values, the integrals were computed by direct quadrature, using intervals of 5 deg. throughout the range. Owing to the lack of suitable tables, the Bessel functions had to be evaluated separately in each case; however $\kappa_0\, a\, e$ was not taken larger than 10 as, with the degree of accuracy attempted, there was no appreciable difference then from the asymptotic value.

6. We may extend the method to give approximate formulae for any long solid of revolution which is completely immersed. There is a well-known approximate solution for the transverse motion of a long solid of revolution in an infinite liquid, in which the flow is treated as two-dimensional; it consists of taking a distribution along the axis of two-dimensional dipoles of moment S/π per unit length per unit velocity, where S is the cross-sectional area at any point.

We have seen in (11) that the transverse motion of a spheroid is given by a line distribution of three-dimensional dipoles along the axis from $-a\,e$ to $+a\,e$, of moment per unit length per unit velocity

$$\frac{1}{4}\frac{b^2}{a^2 e^3}(1+k_2)(c^2 e^2 - h^2) \quad . \quad . \quad . \quad (27)$$

For a long spheroid, for which e is nearly unity, (27) is approximately $(1+k_2)\, S/4\pi$; and to the same order we may take the distribution as extending over the whole of the axis. This suggests that for any elongated solid of revolution we might assume a distribution of three-dimensional dipoles along the axis of moment $(1+k)\, S/4\pi$ per unit length. Thus for heaving oscillations $V\cos\sigma t$ of such a solid with its axis at depth f, we may apply (7) and (8). If $2\,l$ is the length of the solid, and we take the origin at the centre of the axis, we have

$$E_H = \frac{1}{8\pi}\rho\,\sigma\,\kappa_0^3(1+k)^2 V^2 e^{-2\kappa_0 f}\int_0^{2\pi}(P^2+Q^2)\,d\theta \quad (28)$$

with

$$P + iQ = \int_{-e}^{l} S(h)\, e^{i\kappa_0 h\cos\theta}\, dh \quad . \quad . \quad . \quad (29)$$

Similarly, for pitching oscillations to the same approximation

$$E_P = \frac{1}{8\pi}\rho\,\sigma\,\kappa_0^3(1+k')^2\,\Omega^2 e^{-2\kappa_0 f}\int_0^{2\pi}(P^2+Q^2)\,d\theta \quad (30)$$

with

$$P + iQ = \int_{e}^{l} h\, S(h)\, e^{i\kappa_0 h\cos\theta}\, dh \quad . \quad . \quad . \quad (31)$$

It may be noted that k and k' are the virtual inertia coefficients for the solid as a whole; though, under the given condition they both approximate to unity.

7. All the foregoing calculations are for a solid of revolution. With a view to removing this limitation, expressions were obtained for a general ellipsoidal form.

For an ellipsoid with unequal axes, $a > b > c$, and with the a, b axes horizontal, the dipole distribution is in a horizontal plane and extends over the area enclosed by the elliptic focal conic. Application of (7) and (8) leads to expressions for the energy loss.

If the larger transverse axis is vertical, $c > b$, the distribution lies in a vertical plane, and within the elliptic focal conic; a simple modification of (7) and (8) gives the required results. It was decided eventually that it was not worth while carrying out computations; the expressions are of the same type as for a spheroid, though more complicated. It appeared that if the transverse axes b, c do not differ greatly, the main difference in the results as compared with a spheroid is a scale factor arising from the different values of the virtual inertia coefficients.

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A Note on Wave Resistance Theory: transverse and diverging waves

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I wish to associate myself with this tribute to Professor Weinblum for his distinguished work in Ship Hydrodynamics, and I should like to add also that I am greatly indebted to him personally. This is my excuse for a few remarks on a certain aspect of wave resistance theory, though I have nothing new to add; the particular point is no doubt chiefly of theoretical interest, but it happens to have come to my notice again recently.

Considering an ideal frictionless liquid, the only resistance to the motion of a solid is the wave resistance, and it is obviously the horizontal resultant of the fluid pressures on the solid. Another method is to calculate the propagation of energy outwards in the wave motion, and so deduce the corresponding resistance. These two methods give the same result, provided the calculations are made to the same degree of approximation in each case. It may be noted that, in general, this involves obtaining the velocity potential to a higher stage of approximation for the resultant pressure calculation than for the wave-energy method. The energy method was used at first only for two-dimensional problems, as for instance the motion of a submerged circular cylinder; this was because there was available the well-known connection between energy transfer and group velocity for straight-crested plane waves. For three-dimensional problems, such as a submerged sphere, the resistance was found at first by the resultant pressure method. Subsequently I gave a theorem for the energy transfer in a ship wave pattern and its application to the calculation of wave resistance (Proc. Roy. Soc. A, 1932). This was done by considering control planes at great distances before and behind the moving solid, and calculating the rate of work and the transfer of energy across these planes. If 0x is in the direction of motion of the solid, 0 being a moving origin, we assume that the surface elevation ζ at a great distance to the rear approximates to a form which can be expressed by

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\vartheta) \sin \{k_0 \sec^2 \vartheta (x \cos \vartheta + y \sin \vartheta)\} \, d\vartheta$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\vartheta) \cos \{k_0 \sec^2 \vartheta (x \cos \vartheta + y \sin \vartheta)\} \, d\vartheta ,$$

where c is the velocity of the solid, and $k_0 = g/c^2$. It was shown that the wave resistance is given by

$$R = \tfrac{1}{4} \pi \varrho c^2 \int_0^{\frac{\pi}{2}} \{f(\vartheta)^2 + f(-\vartheta)^2 + F(\vartheta)^2 + F(-\vartheta)^2\} \cos^3 \vartheta \, d\vartheta .$$

The wave pattern can be considered as made up of elementary plane waves travelling in all directions. From our knowledge of the ship wave pattern it appears that the transverse waves are made up of plane waves making angles with 0x ranging from zero to a certain angle β, while the remaining plane waves from β to 90° make up the diverging waves. The angle β is given by $\sin^2\beta = \frac{1}{3}$, and is about $35^\circ 16'$. With this in mind I suggested (I. N. A., 1934) that one might possibly divide up the wave resistance integral similarly; thus the value of the integral in the range 0 to β would represent the part due to the transverse waves, and the part from β to 90° that due to the diverging waves. Of course this, as it stands, is no more than a fairly plausible assumption. I have been examining the possibility of putting it on a better basis by a different analytical approach; however I leave that meantime with the remark that I think it can be justified as a fairly reasonable assumption, which can be used to give some interesting results. Taking some simple cases, consider a sphere with its centre at a depth f; the total resistance is

$$R = 4 \pi g \varrho k_0^3 a^6 \int_0^{\frac{\pi}{2}} \sec^5 \vartheta \, e^{-2 k_0 f \sec^2 \vartheta} \, d\vartheta .$$

We see by inspection that for low speeds the greater part of the integral comes from the range 0 to β, while for high speeds the greater part comes from the range β to 90°; a direct calculation shows that at $c / \sqrt{(g f)} = 2$, the diverging waves account for about 30 per cent. of the total resistance. From another point of view, this illustrates the fact that diminishing draft increases the relative importance of the diverging waves, and vice versa.

We may illustrate interference effects by taking a system of a source and sink each of numerical strength mc, at a depth f, and at a distance l apart. The total resistance is given by

$$R = 32 \pi \varrho m^2 k_0^2 c^2 \int_0^{\frac{\pi}{2}} \{1 - \cos(2 k_0 l \sec \vartheta)\} e^{-2 k_0 f \sec^2 \vartheta} \sec^3 \vartheta \, d\vartheta .$$

Consider the oscillating part of the integral due to the factor $\cos(2 k_0 l \sec \vartheta)$ in the two parts of the range of integration. Approximately, the last hump on the resistance curve in each case will be near a value of $k_0 l$ given by $2 k_0 l \sec \vartheta = \pi$. For the range 0 to β, $\sec \vartheta$ is not much different from unity; so the last hump on the transverse wave resistance curve will be near $k_0 l = \pi/2$, or a Froude number $F = 0.56$. On the other hand, on the range β to 90° we may take $\sec \vartheta$ as about 2 to give the maximum result; so the last hump on the diverging wave resistance curve will be near $k_0 l = \pi/4$, or $F = 0.78$. The interference effects due to the superposition of two sets of transverse waves is a familiar idea: it is not so well-known that we may have interference of the diverging waves of two systems.

In conclusion I may refer to some calculations which have been made for simple ship forms on this assumption for separating the contributions of the transverse and diverging waves.

Wigley (I.N.A. 1942) has given numerical results for a simple parabolic model with two ratios of length to draft and up to a Froude number of 0.6. Inui (Intl. Conf. Ship Hydro 1954) refers to some similar unpublished calculations by himself, and gives an interesting diagram for water of finite depth: in which case there are only diverging waves above a critical speed. Finally, I would refer in particular to Lunde (S.N.A.M.E., 1951) who gives a diagram of curves of transverse and diverging wave resistance for a parabolic model. These curves are very interesting, bringing out clearly the humps and hollows on the two curves; for instance, the last hump on the transverse wave curve is at about $F = 0.45$, while that for the diverging waves is at some value greater than $F = 0.6$, outside the range shown on the diagram. It might be of interest to have calculations for other models, to show how the various elements of form affect the relative importance of the transverse and diverging waves.

THE EFFECT OF SPEED OF ADVANCE UPON THE DAMPING OF HEAVE AND PITCH

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Summary

Calculations are made for the damping coefficient for a specially simple case which may be taken to correspond approximately to a long narrow plank moving forward with velocity c and making forced pitching oscillations of frequency p. Curves are given for the variation of the damping moment with frequency at various speeds, the chief aim being to illustrate the effect of the critical condition when the parameter $p\,c/g$ has the value $\frac{1}{4}$. The results are discussed in reference to recent experimental work and the possibility of a steep rise and fall in the curve of damping near this critical point.

The damping of the heave or pitch of a floating solid is mainly due to the energy lost in the wave motion produced by the oscillations. If the solid is at rest, apart from the oscillations, the problem can be formulated satisfactorily as a potential problem with the usual linearized condition at the free surface of the water. If the complete solution could be found in any given case, it could no doubt be also expressed in terms of some source distribution over the immersed surface of the solid. However, what is usually known as the source method of solution is an approximation which begins by assuming some simple source distribution and then adding the wave motion due to these pulsating sources; the method has obvious limitations on its application in general, but it has served to give interesting and useful results. If, in addition to the oscillations, the solid is moving forward with a constant speed of advance, the formulation as a potential problem with the linearized free surface condition is not satisfactory except in the limiting case when the solid is like a thin disc moving in its own plane. However, some progress has been made by the approximate method of assuming some source distribution, and the calculations then require the wave motion due to a pulsating source advancing at constant speed. This problem has been examined by various writers and reference may be made in particular to Haskind,(1) Brard,(2) and Hanaoka.(3) If p is the circular frequency of the pulsation and c the velocity of advance, it is known that the wave motion changes in character when the parameter $p\,c/g = \frac{1}{4}$. It does not seem to have been pointed out explicitly that in fact some of the terms in the solution become infinite at this particular point. The object of the present paper is to examine this matter in some detail for a special case so as to see the effect of this mathematical infinity upon the damping for lower and higher values of the parameter. Consider for a moment a two-dimensional case, for instance a submerged circular cylinder making heaving oscillations of frequency p and advancing with velocity c. At zero speed, there are two wave trains, one on each side of the cylinder. At speed c, if $p\,c/g < \frac{1}{4}$, it can be shown that there are four wave trains, one in advance and three to the rear, the wave train in advance being that for which the group velocity is greater than the speed of advance. If the speed is increased, the amplitudes of two of these trains become infinite at the critical point when $p\,c/g = \frac{1}{4}$; and for higher values of the speed these two trains disappear, leaving only two wave trains both to the rear of the cylinder. The behaviour at the critical point clearly arises from a special, and interesting, case of resonance; and, as usual, the infinity could only be removed from the solution by introducing some frictional or other kind of dissipation.

Turning to the three-dimensional case of a point source, one might hope that the infinity would disappear through integration, but this is not the case; the solution contains integrals which are finite in general, but they become infinite at the critical value of the parameter.

Calculations have been made by Haskind and by Hanaoka for the damping of a Michell-type of model with the source distribution assumed to be in the vertical longitudinal plane; this assumption is the well-known approximation for wave resistance, and although it is of doubtful validity in general as regards the heaving or pitching oscillations it gives useful indications for simplified forms. Although the integrals used by Haskind become divergent at the critical value of the parameter, his curves do not show any infinity; possibly the range does not include the critical point. Hanaoka also gives a curve for the damping at various speeds; but the whole curve is explicitly for the value $p\,c/g = 0{\cdot}6$ and so is well beyond the critical point.

Some recent experimental work by Golovato(4) is of special interest. A model was made to perform heaving oscillations of given frequency while moving forward at some constant speed, and the forces and moment on the model were measured. In Fig. 13 of that paper the damping moment is shown in curves on a base $p\,(B/g)^{\frac{1}{2}}$ for various values of the Froude number. A striking feature is the pronounced peaks at low values of the parameter. Golovato remarks: "The steep rise at low frequencies appears to coincide with a velocity-wave celerity ratio of $\frac{1}{4}$ where the character of the waves generated by the oscillating body is known to change markedly." This ratio is what we have denoted here by $p\,c/g$. It is curious that the curves for heaving do not seem to show the same effect, though one would expect the same cause to be operative for both heaving and pitching.

The present calculations are for a simple line distribution of pulsating sources, but we can relate them to a possible physical problem. Suppose a long narrow plank, in a vertical plane, moving forward and at the same time making small pitching oscillations. Such a form, with pointed ends, is the most suitable for comparing wave resistance theory with experiment, and it might also be used similarly to test the approximate linear theory of heaving and pitching. However, even if it is not a practicable method experimentally, it is an appropriate form for the present state of theory. We may separate out the effects of the forward motion and the pitching; and we may assume the latter to be due to a simple source distribution over the flat submerged base of the plank, or for small enough beam to a distribution along the central line of the base. As numerical computation is rather lengthy in any case, we omit the pointed ends and reduce the

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form to a long plank, of length L and beam B, submerged to a draught d, moving forward with velocity c and making small pitching oscillations with angular velocity $\Omega \sin p\, t$.

The theoretical work is given in the Appendix. It begins with a different derivation of the fluid motion due to a moving pulsating source. Then by integration of an assumed source distribution we obtain the velocity potential for the plank. The fluid pressure is obtained for any point of the base and hence the moment of this pressure. Dealing only with the moment due to the pitching motion, the periodic part will be of the form $M_1 \sin p\, t + M_2 \cos p\, t$. The second term is in phase with the angular acceleration and can be considered as giving a virtual addition to the moment of inertia. The first term is in phase with the angular velocity and gives the corresponding damping coefficient; this is the only term which is examined here, and expressions for M_1 are given in equations (13), (14), and (15).

For numerical computation we have taken $L/B = 20$ and $d/B = 2$. These ratios do not allow any direct comparison with the usual models; they were chosen partly to lessen computation and partly so as to bring out certain points. Fig. 1 shows curves

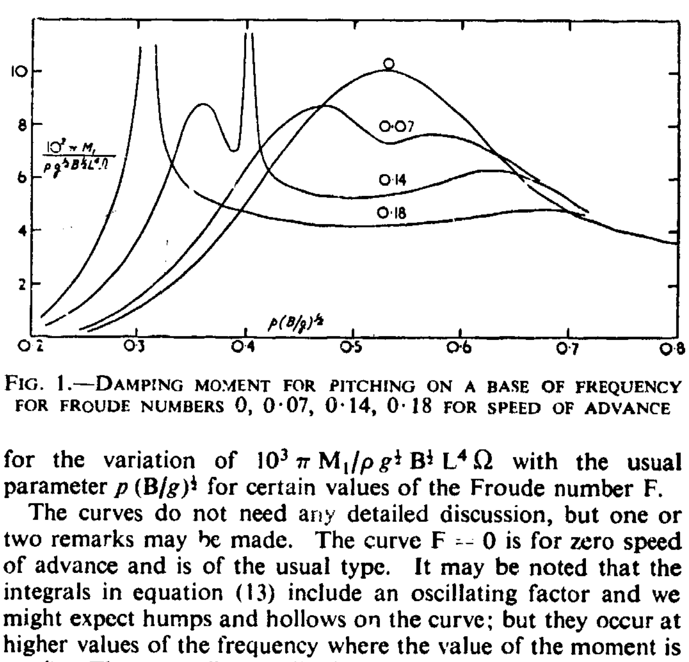

FIG. 1.—DAMPING MOMENT FOR PITCHING ON A BASE OF FREQUENCY FOR FROUDE NUMBERS 0, 0·07, 0·14, 0·18 FOR SPEED OF ADVANCE

for the variation of $10^3\, \pi\, M_1/\rho\, g^{\frac{1}{2}}\, B^{\frac{1}{2}}\, L^4\, \Omega$ with the usual parameter $p\,(B/g)^{\frac{1}{2}}$ for certain values of the Froude number F.

The curves do not need any detailed discussion, but one or two remarks may be made. The curve $F = 0$ is for zero speed of advance and is of the usual type. It may be noted that the integrals in equation (13) include an oscillating factor and we might expect humps and hollows on the curve; but they occur at higher values of the frequency where the value of the moment is small. These possible oscillations are not interference effects connected with the beam, such as have given rise to discussion in two-dimensional problems: the latter have been ruled out of the present calculation by the assumption of small enough beam. The interference effects here are in length, between bow and stern; no doubt the rectangular form of the base would tend to exaggerate any such effects.

Comparing the curve for zero speed with the other curves, a general effect is like moving the curve towards the lower frequencies with increasing speed, and we can see the interference effects coming into evidence. The other main point is the infinity at the critical value with a steep fall after this point followed by a small gradual rise. The critical point for $F = 0{\cdot}07$ is at $p\,(B/g)^{\frac{1}{2}} = 0{\cdot}8$; it is not shown in the diagram as the infinity is highly localized and computation would be tedious. The critical points for Froude numbers 0·14 and 0·18 are at 0·4 and 0·31 respectively. It should be stated that there are certain speeds for which M_1 does not become infinite at the critical point, though there are still peak values; these speeds are such that the Bessel Function in the integrals (13) has zero value for $\theta = 0$ when $p\, c/g = \frac{1}{4}$.

Naturally, in any experimental results the mathematical infinities would be smoothed down as in other resonance effects; and also they are likely to be highly localized and sensitive to small disturbances. Nevertheless they have their effect upon the rest of the curve; and with suitably devised experiments one might expect peaks on the damping curves in the region of the critical value of the parameter.

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Appendix

A point source of strength $m \sin p\, t$ is moving with velocity c at a depth d below the free surface of the water. We take moving axes with the origin O in the free surface immediately above the travelling source, O x in the direction of motion, O y transversely, and O z vertically upwards. We suppose the motion to have been started from rest and the solution we require is that to which the motion approximates at a sufficiently long time after the start. The result can be obtained by integrating the effect of infinitesimal steps in the motion from the start up to the time t. Suppose the motion started at a time u before the present instant, that is at a time $t - u$; then, using a general result for a variable source,[5] we have for the velocity potential

$$\phi = m\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \sin p\, t + \phi_1 \quad . \quad . \quad . \quad (1)$$

where

$$\phi_1 = \lim_{u\to\infty} \frac{4\, m\, g^{\frac{1}{2}}}{\pi} \, . \int_0^u \sin p\,(t-u)\, du$$
$$\int_0^{\pi/2} d\theta \int_0^\infty \cos\left[\kappa\,(x + c\,u)\cos\theta\right] \cos\,(\kappa\, y \sin\theta)$$
$$\times \sin\,(u\, g^{\frac{1}{2}}\, \kappa^{\frac{1}{2}})\kappa^{\frac{1}{2}}\, e^{-\kappa(d-z)}\, d\kappa \quad (2)$$

with $\quad r_1^2 = x^2 + y^2 + (z+d)^2; \quad r_2^2 = x^2 + y^2 + (z-d)^2$

Carrying out the integration with respect to u we obtain

$$\phi_1 = \lim_{u\to\infty} \frac{m\, g^{\frac{1}{2}}}{\pi} \int_0^{\pi/2} d\theta \int_0^\infty \kappa^{\frac{1}{2}}\, e^{-\kappa(d-z)} \cos\,(\kappa\, y \sin\theta)$$
$$\times \Bigg[[\sin\,(\kappa\, x\cos\theta + p\, t)\left\{\frac{\cos\left[\kappa\, c\cos\theta - p - (g\,\kappa)^{\frac{1}{2}}\right] u}{\kappa\, c\cos\theta - p - (g\,\kappa)^{\frac{1}{2}}}\right.$$
$$\left. - \frac{\cos\left[\kappa\, c\cos\theta - p + (g\,\kappa)^{\frac{1}{2}}\right] u}{\kappa\, c\cos\theta - p + (g\,\kappa)^{\frac{1}{2}}} - \frac{2\,(g\,\kappa)^{\frac{1}{2}}}{(\kappa\, c\cos\theta - p)^2 - g\,\kappa}\right\}$$
$$+ \cos\,(\kappa\, x\cos\theta + p\, t)\left\{\frac{\sin\left[\kappa\, c\cos\theta - p - (g\,\kappa)^{\frac{1}{2}}\right] u}{\kappa\, c\cos\theta - p - (g\,\kappa)^{\frac{1}{2}}}\right.$$
$$\left. - \frac{\sin\left[\kappa\, c\cos\theta - p + (g\,\kappa)^{\frac{1}{2}}\, u\right]}{\kappa\, c\cos\theta - p + (g\,\kappa)^{\frac{1}{2}}}\right\}$$
$$- \text{two similar terms with } -p \text{ for } p \Bigg] d\kappa \quad . \quad . \quad . \quad . \quad (3)$$

Considering the integration with respect to κ and the limiting value as $u \to \infty$, we require the positive values of κ for which the various denominators in equation (3) are zero, and the corre-

sponding positive square roots of these values. There are four such zeros in all and they are given by

$$\left.\begin{aligned}\kappa_1, \kappa_2 &= \tfrac{1}{2}\kappa_0 \sec^2\theta\,[1 + 2\beta\cos\theta \pm (1 + 4\beta\cos\theta)^{\frac{1}{2}}]\\ \kappa_3, \kappa_4 &= \tfrac{1}{2}\kappa_0 \sec^2\theta\,[1 - 2\beta\cos\theta \pm (1 - 4\beta\cos\theta)^{\frac{1}{2}}]\end{aligned}\right\} \quad (4)$$

where $\kappa_0 = g/c^2$, $\beta = p\,c/g$, and κ_3, κ_4 only exist if $\cos\theta < 1/4\,\beta$.

The integrals in equation (3) involving a factor of the form $\cos[u f(\kappa)]/f(\kappa)$ tend to zero as $u \to \infty$, interpreting the integrals where necessary as principal value integrals. For the integrals in equation (3) of the form

$$\int_0^\infty F(\kappa)\sin[u f(\kappa)]/f(\kappa)\,.\,d k \quad . \quad . \quad . \quad (5)$$

where $f(\kappa)$ has simple zeros, the contribution of each such zero, say κ_1, to the limiting value is $\pi\,F(\kappa_1)/|f'(\kappa_1)|$. All the relevant zeros are included in the four values given in equation (4).

After carrying out these operations, we obtain

$$\begin{aligned}\phi &= m\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\sin p\,t\\ &- \frac{2\,m\,g}{\pi}\int_0^{\pi/2} d\theta \int_0^\infty \left[\frac{\sin(\kappa\,x\cos\theta + p\,t)}{(\kappa\,c\cos\theta - p)^2 - g\,\kappa}\right.\\ &\left.- \frac{\sin(\kappa\,x\cos\theta - p\,t)}{(\kappa\,c\cos\theta + p)^2 - g\,\kappa}\right]\cos(\kappa\,y\sin\theta)\,\kappa\,e^{-\kappa(d-z)}\,d\kappa\\ &+ 2\,m\int_0^{\pi/2} \frac{\kappa_1\,e^{-\kappa_1(d-z)}}{(1+4\beta\cos\theta)^{\frac{1}{2}}}\cos(\kappa_1\,x\cos\theta + p\,t)\cos(\kappa_1\,y\sin\theta)\,d\theta\\ &- 2\,m\int_0^{\pi/2} \frac{\kappa_2\,e^{-\kappa_2(d-z)}}{(1+4\beta\cos\theta)^{\frac{1}{2}}}\cos(\kappa_2\,x\cos\theta + p\,t)\cos(\kappa_2\,y\sin\theta)\,d\theta\\ &- 2\,m\int_{\theta_1}^{\pi/2} \frac{\kappa_3\,e^{-\kappa_3(d-z)}}{(1-4\beta\cos\theta)^{\frac{1}{2}}}\cos(\kappa_3\,x\cos\theta - p\,t)\cos(\kappa_3\,y\sin\theta)\,d\theta\\ &- 2\,m\int_{\theta_1}^{\pi/2} \frac{\kappa_4\,e^{-\kappa_4 d - z}}{(1-4\beta\cos\theta)^{\frac{1}{2}}}\cos(\kappa_4\,x\cos\theta - p\,t)\cos(\kappa_4\,y\sin\theta)\,d\theta \quad . \quad . \quad . \quad . \quad (6)\end{aligned}$$

where

$$\begin{aligned}\theta_1 &= 0 \text{ if } 4\beta < 1\\ \theta_1 &= \cos^{-1}(1/4\beta) \text{ if } 4\beta > 1 \quad . \quad . \quad . \quad (7)\end{aligned}$$

In the last two integrals in equation (6) the integrand becomes infinite at the lower limit θ_1, but the integrals remain finite in general; however, they become divergent in the limiting case when $4\beta = 1$ and $\theta_1 = 0$.

The wave pattern at a great distance from the source need not be discussed here; it is obtained by combining the last four terms in equation (6) with the suitable contribution from the double integral in equation (6). Broadly speaking, the pattern at a great distance in advance is associated with the κ_4 value while at the rear it comes from the κ_1, κ_2 and κ_3 terms. Finally, it can be verified that for $c = 0$, the expressions reduce to the known form for a stationary pulsating source emitting circular waves at a great distance.

Consider now a long thin plank, of length L and beam B and with short pointed ends, floating vertically in water and immersed to a draught d. The plank moves forward with velocity c and makes small pitching oscillations with angular velocity $\Omega\sin p\,t$. We assume that the velocity potential due to the pitching motion can be derived from a source distribution over the flat base of the plank; further, we assume the source strength per unit area at a distance x from the mid-point is $1/4\,\pi$ times the normal velocity at that point, and for a sufficiently thin plank we take this as equivalent to a line distribution of amount $(B/4\,\pi)\,x\,\Omega\sin p\,t$. These are rather drastic simplifying assumptions, especially for pitching; but perhaps they are not too far amiss under the specified conditions for illustrating the particular point under consideration. To reduce the computation we extend the integration only to cover the rectangular part of the base, omitting the supposed short pointed ends. The velocity potential due to the forward motion could be obtained in the usual way by a source distribution over the curved sides at the two ends of the plank; as this does not enter into the present calculation we omit this part of the velocity potential.

Returning to equation (6) we obtain the required result by substituting $x - h$ for x, multiplying by $h\,B/4\,\pi$ and integrating between the limits $\pm l$ for h, where $L = 2\,l$. All the integrals can be evaluated explicitly, but to avoid lengthy expressions we write $F(x, y, z)$ for the contribution of the first term in equation (6). We obtain thus

$$\begin{aligned}\phi &= (B\,\Omega/4\,\pi)\,F(x, y, z)\sin p\,t\\ &+ g\,B\,\Omega\,(l^3/2\,\pi^3)^{\frac{1}{2}}\int_0^{\pi/2} d\theta\int_0^\infty (\kappa\sec\theta)^{\frac{1}{2}}\,J_{\frac{3}{2}}(\kappa\,l\cos\theta)\\ &\times\left[\frac{\cos(\kappa\,x\cos\theta + p\,t)}{(\kappa\,c\cos\theta - p)^2 - g\,\kappa}\right.\\ &\left.- \frac{\cos(\kappa\,x\cos\theta - p\,t)}{(\kappa\,c\cos\theta + p)^2 - g\,\kappa}\right]\cos(\kappa\,y\sin\theta)\,e^{-\kappa(d-z)}\,d\kappa\\ &+ B\,\Omega\left(\frac{l^3}{2\,\pi^3}\right)^{\frac{1}{2}}\int_0^{\pi/2}\frac{(\kappa_1\sec\theta)^{\frac{1}{2}}}{(1+4\beta\cos\theta)^{\frac{1}{2}}}\,J_{\frac{3}{2}}(\kappa_1\,l\cos\theta)\sin(\kappa_1\,x\cos\theta\\ &+ p\,t)\cos(\kappa_1\,y\sin\theta)\,e^{-\kappa_1(d-z)}\,d\theta\\ &+ \text{similar terms in } \kappa_2, \kappa_3, \kappa_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)\end{aligned}$$

where J denotes the ordinary Bessel Function.

The pressure on the base is given by

$$p = \rho\left(\frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x}\right) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the moment M of the pressure about the axis O y is given by

$$M = \iint p\,x\,d\,x\,d\,y \quad . \quad . \quad . \quad . \quad (10)$$

taken over the base. Or, to the present approximation,

$$M = \rho\,B\int_{-l}^{l}\left(\frac{\partial\phi}{\partial t} - c\frac{\partial\phi}{\partial x}\right)x\,d\,x \quad . \quad . \quad (11)$$

with $y = 0$ and $z = -d$ in equation (8).

On examination of the various terms in equation (8) it is easily seen that the only contribution to the terms in $\sin p\,t$ in equation (11) comes from the last four terms in equation (8). From the first of these terms, for instance, the contribution to this part of M is found to be

$$\rho\,B^2\,l^3\,\Omega\sin p\,t\int_0^{\pi/2}\frac{\kappa_1\,c\cos\theta - p}{(1 + 4\beta\cos\theta)^{\frac{1}{2}}}\,J^2_{\frac{3}{2}}(\kappa_1\,l\cos\theta)\,e^{-2\kappa_1 d}\sec\theta\,d\theta \quad . \quad . \quad . \quad . \quad (12)$$

For computation we change from the Bessel Function J to the Spherical Bessel Function given by $S(x) = (\pi/2\,x)^{\frac{1}{2}}\,J(x)$.

because extensive tables of these functions are available. With $F = c/(g\,L)^{\frac{1}{2}}$, and with $M_1 \sin p\,t$ being the required part of the moment, we obtain finally

$$\pi\, M_1/\rho\, B^{\frac{1}{2}}\, g^{\frac{1}{2}}\, L^4\, \Omega = \tfrac{1}{8}\,(B/L)^{\frac{1}{2}}\, F^{-3}$$

$$\times \left[\int_0^{\pi/2} \frac{A_1\, B_1 \sec^3\theta}{(1 + 4\,\beta\cos\theta)^{\frac{1}{2}}} S^2_{3/2}\left(\frac{A_1}{4\,F^2}\sec\theta\right) e^{-\frac{d\,A_1}{L\,F^2}\sec^2\theta}\, d\,\theta \right.$$

$$+ \int_0^{\pi/2} \frac{A_2\, B_2 \sec^3\theta}{(1 + 4\,\beta\cos\theta)^{\frac{1}{2}}} S^2_{3/2}\left(\frac{A_2}{4\,F^2}\sec\theta\right) e^{-\frac{d\,A_2}{4\,F^2}\sec^2\theta}\, d\,\theta$$

$$+ \int_{\theta_1}^{\pi/2} \frac{A_3\, B_3 \sec^3\theta}{(1 - 4\,\beta\cos\theta)^{\frac{1}{2}}} S^2_{3/2}\left(\frac{A_3}{4\,F^2}\sec\theta\right) e^{-\frac{d\,A_3}{4\,F^2}\sec^2\theta}\, d\,\theta$$

$$\left. + \int_{\theta_1}^{\pi/2} \frac{A_4\, B_4 \sec^3\theta}{(1 - 4\,\beta\cos\,)\theta^{\frac{1}{2}}} S^2_{3/2}\left(\frac{A_4}{4\,F^2}\sec\theta\right) e^{-\frac{d\,A_4}{4\,F^2}\sec^2\theta}\, d\,\theta \right] \quad (13)$$

with θ_1 given by equation (7), $\beta = p\,c/g$, and

$$\begin{aligned} A_1, A_2 &= 1 + 2\,\beta\cos\theta \pm (1 + 4\,\beta\cos\theta)^{\frac{1}{2}} \\ B_1, B_2 &= (1 + 4\,\beta\cos\theta)^{\frac{1}{2}} \pm 1 \\ A_3, A_4 &= 1 - 2\,\beta\cos\theta \pm (1 - 4\,\beta\cos\theta)^{\frac{1}{2}} \\ B_3, B_4 &= 1 \pm (1 - 4\,\beta\cos\theta)^{\frac{1}{2}} \quad . \;.\;.\;.\;.\;. \quad (14) \end{aligned}$$

If we write $q = p\,(B/g)^{\frac{1}{2}}$, it can be verified that when $c = 0$, equation (13) reduces to the result for this case which can be obtained directly, namely

$$\pi\, M_1/\rho\, g^{\frac{1}{2}}\, B^{\frac{1}{2}}\, L^4\, \Omega = q^3\, e^{-2\,q^2 d/B} \int_0^{\pi/2} S^2_{3/2}\left(\frac{1}{2}\,\frac{L}{B}\,q^2\cos\theta\right) d\,\theta \quad (15)$$

SUBJECT INDEX

All references are to pages in the collected edition.

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